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Probabilities

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Probabilities

ASNAT23

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Plan

- ① Basis
- ② Discrete Random Variables
- ③ Continuous Random Variables
- ④ Multivariate Probability Distributions
- ⑤ Gaussian Vectors
- ⑥ Convergence and Limit Theorems

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① Basis

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Review of Set Notations

Probability Triple

Conditional Probabilities

② Discrete Random Variables

③ Continuous Random Variables

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Introduction

Problem

We are 15 in this room. What is the probability that at least two of us share the same birth date?

Introduction

Solution

Complementary event: $\mathbb{P}(A) + \mathbb{P}(\bar{A}) = 1$

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With 35 persons : $\mathbb{P}(A) \approx 0.8$

Introduction

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- Exemple of die tossing



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Notations and definitions

Sets

A set is a collection of items or elements. If the set A contains the elements a_1, \dots, a_n , we write

$$A = \{a_1, \dots, a_n\}$$

Notations and definitions

Exemples of sets

- the set of colours in a deck of cards $\{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\} = \{\spadesuit, \diamondsuit, \clubsuit, \heartsuit\}$

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- the set of complex numbers \mathbb{C}

Notations and definitions

Sets with conditions

We can also define a set by mathematically stating the properties satisfied by its elements. In that case, we write

$$A = \{x \mid x \text{ satisfies some property}\}$$

where “ \mid ” is read as “such that”

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We can also use various properties using a comma

$$A = \{x \mid x \text{ satisfies property 1, } x \text{ satisfies property 2}\}$$

The comma is thus read as “and”

Notations and definitions

Exemples of sets with conditioning

- $\{x^2 | x \in \mathbb{N}\} = \{0, 1, 4, 9, \dots\}$

Notations and definitions

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- for $a < b$, $(a, b] = \{x \in \mathbb{R}|a < x \leq b\}$
- $\mathbb{C} = \{x + iy|x \in \mathbb{R}, y \in \mathbb{R}\}$

Notations and definitions

Inclusion

We say that A is a subset of B if every element of A is also an element of B . We note $A \subset B$

$$A \subset B \Leftrightarrow x \in A \Rightarrow x \in B$$

Equivalently, we say that B is a superset of A , and we note $B \supset A$

Notations and definitions

Exemples of inclusions

- $\{\heartsuit, \spadesuit\} \subset \{\spadesuit, \diamondsuit, \clubsuit, \heartsuit\}$

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- $\mathbb{N} \subset \mathbb{Z}$
- $\mathbb{R} \supset \mathbb{Q}$

Notations and definitions

Equality

Two sets are equal if they have the exact same elements

$$A = B \Leftrightarrow x \in A \Leftrightarrow x \in B \Leftrightarrow A \subset B \text{ and } B \subset A$$

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The null set

The set with no elements is called the null set or the empty set, and is denoted \emptyset

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Trivial inclusions

For any set A , we always have $A \subset A$ and $\emptyset \subset A$

Notations and definitions

Exemples of equal sets

- $\{1, 2, 3\} = \{2, 1, 3\}$

Notations and definitions

Exemples of equal sets

- $\{1, 2, 3\} = \{2, 1, 3\}$
- $\{x, x, y\} = \{x, y\}$

Notations and definitions

Universal set

The universal set is the set of all elements in a given context. Therefore, every set A is a subset of the universal set. The universal set is denoted Ω

Notations and definitions

Exemples of universal sets

- when considering the color of card, $\Omega = \{\spadesuit, \diamondsuit, \clubsuit, \heartsuit\}$

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- when considering the roll of a die, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- when considering the toss of a coin, $\Omega = \{\text{head}, \text{tail}\}$
- when considering the value of a bit of a telecommunication signal $\Omega = \{0, 1\}$

Notations and definitions

Power set

The power set of a set A is the set of all the subsets of A , including the empty set and A itself. It is denoted $\mathcal{P}(A)$

Notations and definitions

Examples of power set

- if $A = \{0, 1, 2\}$, we have

$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, A\}$$

Notations and definitions

Venn diagrams and set operations

- Useful to represent sets and their relations

Notations and definitions

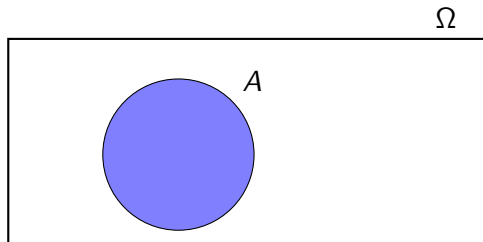
Venn diagrams and set operations

- Useful to represent sets and their relations
- A set is represented as a closed region

Notations and definitions

Venn diagrams and set operations

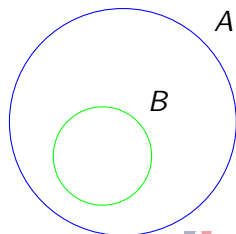
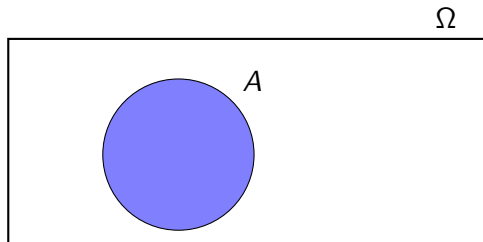
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Notations and definitions

Union of two sets

The union of two sets A and B is the set containing all the elements that are in A **or** B , eventually both. It is denoted $A \cup B$

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

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An example of the union of two sets

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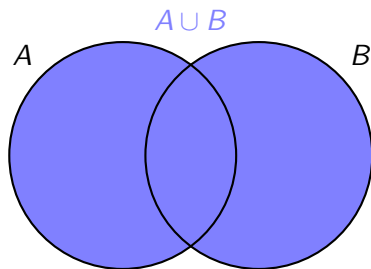
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Property

$$A \cup B = B \cup A$$

Notations and definitions

Venn diagram of an union



Notations and definitions

Union of n sets

$A_1 \cup \dots \cup A_n$ is the set containing all the elements that are at least in one of any of the sets A_1, \dots, A_n

$$A_1 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i$$

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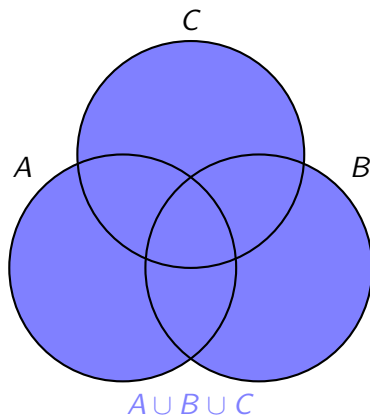
$$A_1 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i$$

An example of the union of three sets

If $A_1 = \{x, y, z\}$, $A_2 = \{v, y\}$ and $A_3 = \{x, u\}$, then
 $A_1 \cup A_2 \cup A_3 = \bigcup_{i=1}^3 A_i = \{x, y, z, u, v\}$

Notations and definitions

Venn diagram of an union



Notations and definitions

Intersection of two sets

The intersection of two sets A and B is the set containing all the elements that are in A **and** B . It is denoted $A \cap B$

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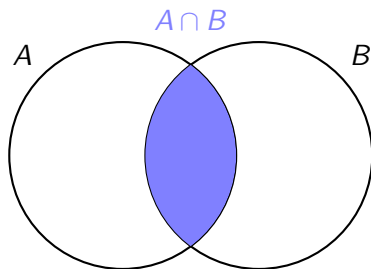
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Property

$$A \cap B = B \cap A$$

Notations and definitions

Venn diagram of an intersection



Notations and definitions

Intersection of n sets

$A_1 \cap \dots \cap A_n$ is the set containing all the elements that are in all sets A_1, \dots, A_n

$$A_1 \cap \dots \cap A_n = \bigcap_{i=1}^n A_i$$

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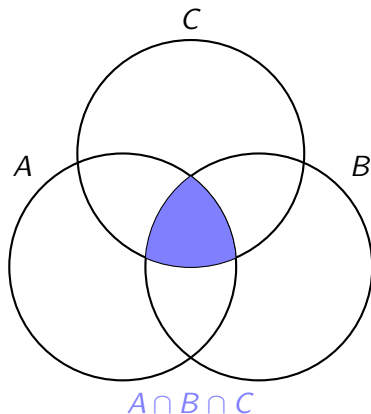
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An example of the intersection of three sets

If $A_1 = \{x, y, z\}$, $A_2 = \{x, y, w\}$ and $A_3 = \{v, x, y, w, z\}$, then $A_1 \cap A_2 \cap A_3 = \bigcap_{i=1}^3 A_i = \{x, y\}$.

Notations and definitions

Venn diagram of an union



Notations and definitions

Complement of a set

The complement of a set A is the set containing all the elements that are in the universal set Ω , but **not** in A . It is denoted \bar{A} (or sometimes A^c).

Notations and definitions

Complement of a set

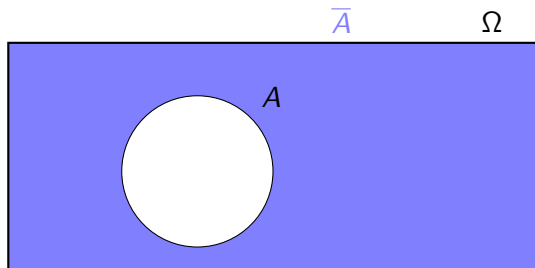
The complement of a set A is the set containing all the elements that are in the universal set Ω , but **not** in A . It is denoted \bar{A} (or sometimes A^c).

An example of a complementary set

In the universe of card colors $\overline{\{\heartsuit, \diamondsuit\}} = \{\clubsuit, \spadesuit\}$.

Notations and definitions

Venn diagram of a complementary set



Notations and definitions

Difference of sets

The difference of two sets A and B is defined as the elements that are in one of the two sets but not in the other. For instance, $A - B$ is constituted of the elements that are in A but not in B . In general, $A - B \neq B - A$

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In the universe of card colours $\{\heartsuit, \diamondsuit, \clubsuit\} - \{\clubsuit\} = \{\heartsuit, \diamondsuit\}$.

Notations and definitions

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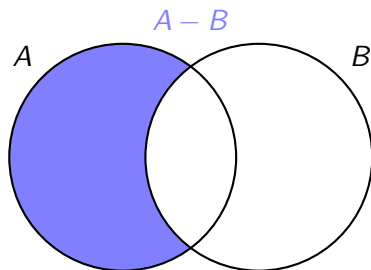
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Property

$$A - B = A \cap \overline{B}.$$

Notations and definitions

Venn diagram of a difference of two sets



Notations and definitions

Disjoint sets

Two sets A and B are said to be disjoint (or mutually exclusive) if they do not share any elements, i.e., if $A \cap B = \emptyset$. More generally, several sets are said to be disjoint if they are pairwise disjoint, i.e., none of them share a common element

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- $\{\heartsuit, \diamondsuit\}$ and $\{\clubsuit, \spadesuit\}$
- $(-\infty, 0)$ and $(0, +\infty)$

Notations and definitions

Disjoint sets

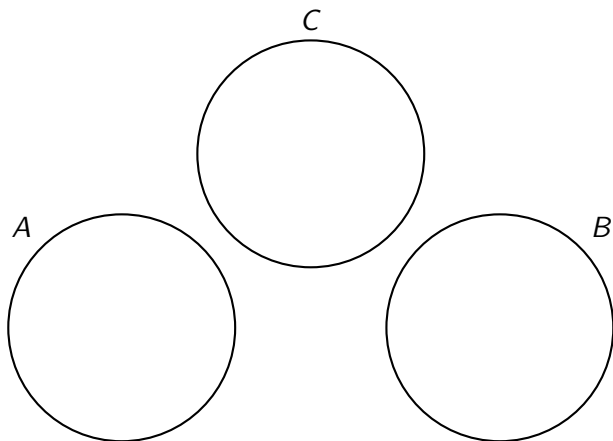
Two sets A and B are said to be disjoint (or mutually exclusive) if they do not share any elements, i.e., if $A \cap B = \emptyset$. More generally, several sets are said to be disjoint if they are pairwise disjoint, i.e., none of them share a common element

An example of a disjoint sets

- $\{\heartsuit, \diamondsuit\}$ and $\{\clubsuit, \spadesuit\}$
- $(-\infty, 0)$ and $(0, +\infty)$
- for all set A , A and \bar{A}

Notations and definitions

Venn diagram of a disjoint sets



Notations and definitions

Partitions

A partition of a set A is a collection of disjoint and non-empty sets A_1, \dots, A_n such that their union is A

$$\bigcup_{i=1}^n A_i = A, A_i \neq \emptyset, A_i \cap A_j = \emptyset, \forall i \neq j$$

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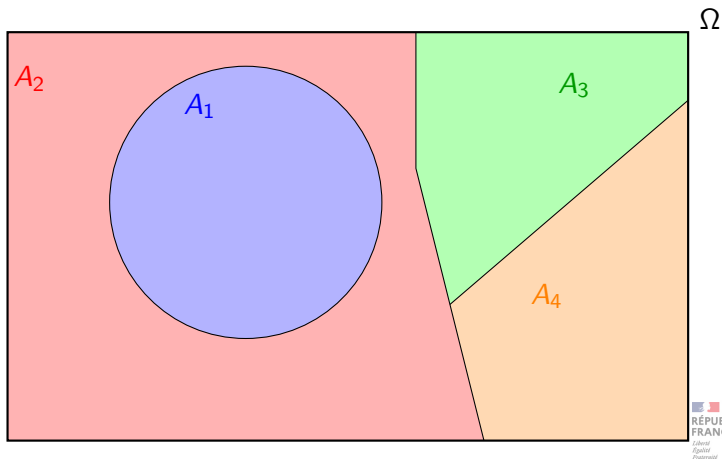
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An example of partitions

- the set of all the states of the US is a partition of the US
- red cards and black cards is a partition of playing cards
- for any set A of a universal set Ω , A and \bar{A} is a partition of Ω

Notations and definitions

Venn diagram of a partition



Properties of set operations

De Morgan's law

For any sets A_1, \dots, A_n we have

- $\overline{\bigcup_{i=1}^n A_i} = \bigcap_{i=1}^n \overline{A_i}$
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Distributive law

For any sets A, B and C we have

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Exercise

Consider the universal set $\Omega = \{1, 2, 3, 4, 5, 6\}$, and the the subsets $A = \{1, 6\}$, $B = \{2, 4, 6\}$, $C = \{1, 4, 5\}$ and $D = \{1, 3, 5\}$.

- 1 Express the following sets: $A \cup B$, $A \cap B$, \bar{A} , \bar{B}
- 2 What is a partition of Ω ?
- 3 Check De Morgan's law for $\overline{A \cup B}$
- 4 Check the distributive law for $A \cap (B \cup C)$

Solution

Solution

① • $A \cup B = \{1, 2, 4, 6\}$

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Solution

- $A \cup B = \{1, 2, 4, 6\}$
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Notations and definitions

Cartesian product

The cartesian product of two sets A and B is the set containing ordered pairs of elements from A and B , and it is denoted $A \times B$

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Property: number of elements

If a finite set A has M elements and another finite set B has N elements, then $A \times B$ has $M \times N$ elements

Notations and definitions

Exemples of cartesian products

- $\{1, 2, 3\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

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- $\{1, 2, 3\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
- deck of cards = $\{1, 2, \dots, 10, J, Q, K\} \times \{\clubsuit, \spadesuit, \heartsuit, \diamondsuit\}$

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- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

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Cardinality of finite sets

If a set A is finite, its cardinality is its number of elements, and it is denoted $|A|$

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Property: inclusion-exclusion principle

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- $|\bigcup_{i=1}^n A_i| = \sum_{i=1}^n |A_i| - \sum_{i < j} |A_i \cap A_j| + \sum_{i < j < k} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n+1} |\bigcap_{i=1}^n A_i|$

Exercise

In a class, there are 12 students with black hair and 9 with blond hair. 6 students have a beard and black hair, and 2 students have a beard and blond hair. The total number of students with black hair or blond hair or a beard is 23. How many students have a beard?

Solution

We denote A the set of students with black hair, B the set of students with blond hair, and C the set of students with a beard. We have

- $|A| = 12$
- $|B| = 9$
- $|A \cap C| = 6$
- $|B \cap C| = 2$
- $|A \cup B \cup C| = 23$

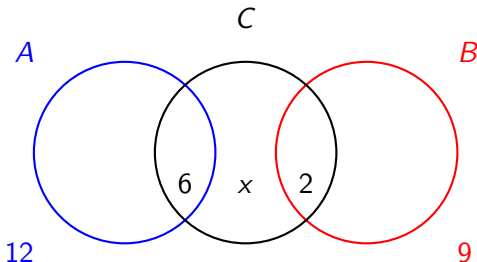
We want to find $|C|$. First, remark that we have A and B disjoint, i.e., $|A \cap B| = 0$ and thus $|A \cap B \cap C| = 0$. We apply the previous inclusion-exclusion principle to have

$$\begin{aligned} |A \cup B \cup C| &= 23 \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 12 + 9 + |C| - 0 - 6 - 2 + 0 \end{aligned}$$

Solution

Finally, we have $|C| = 10$.

Another way to solve this problem, is to use a Venn diagram representation, as represented hereafter.



Here we have $12 + x + 9 = 23$, where x represents $|C| - |A \cap C| - |B \cap C|$.

We therefore have $x = 2$. On the other hand, we have

$|C| = 6 + x + 2 = 10$, yielding the same results.

Notations and definitions

Infinite sets

If A has an infinite number of elements, there are two situations

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Countable set

A set A is said to be countable if one of the following conditions is satisfied

- A is a finite set ($|A| < +\infty$)
- A is an infinite set and there is a bijection between A and \mathbb{N} (namely, A can be put in one to one correspondence with \mathbb{N})

Notations and definitions

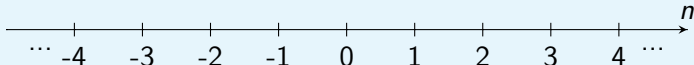
Exemples

- \mathbb{N} obviously countable, the bijection being the identity
- \mathbb{Z}

Notations and definitions

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- \mathbb{Z} is countable: we define φ such as $\varphi(n) = 2n$ for $n \in \mathbb{Z}, n \geq 0$ and $\varphi(n) = -2n - 1$ for $n \in \mathbb{Z}, n < 0$



Notations and definitions

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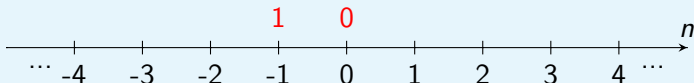
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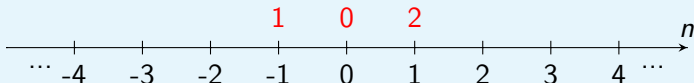
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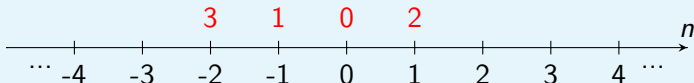
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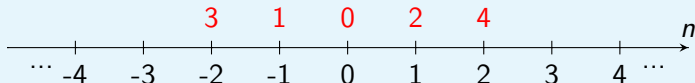
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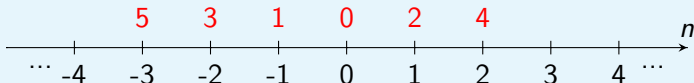
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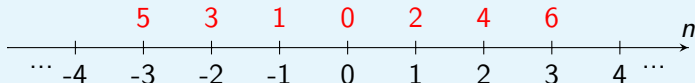
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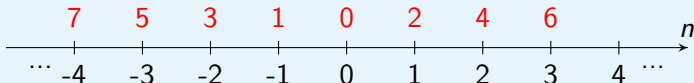
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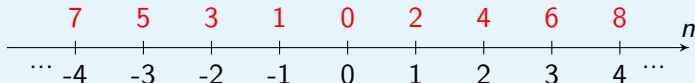
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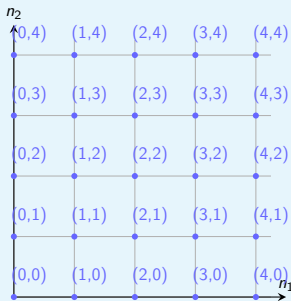
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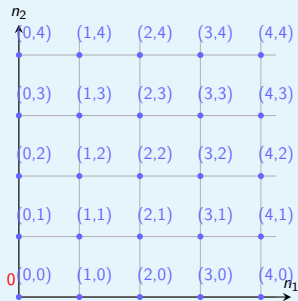
\mathbb{N}^2 is also countable. The mathematical expression of the bijection is $\varphi(n_1, n_2) = \frac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1) + n_2$.



Notations and definitions

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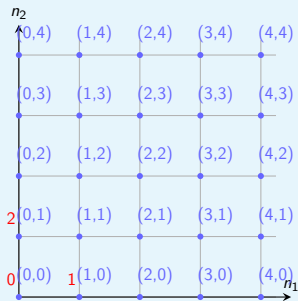
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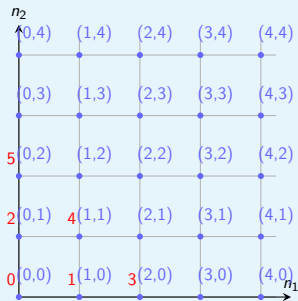
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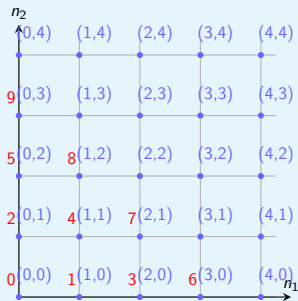
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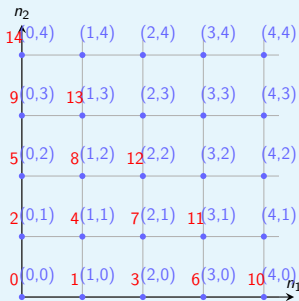
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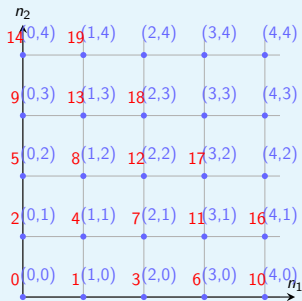
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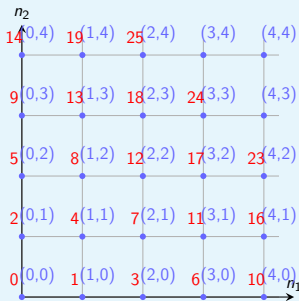
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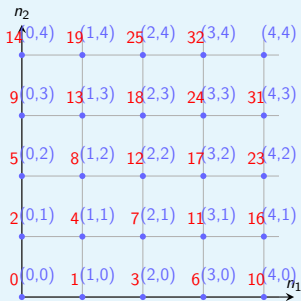
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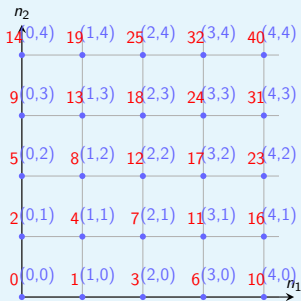
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Notations and definitions

Exemples

- \mathbb{Q}

Notations and definitions

Exemples

- \mathbb{Q} is countable: each element x of \mathbb{Q} can be expressed as $x = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$ and $\gcd(p, q) = 1$.

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We define $\varphi(x) : x \in \mathbb{Q} \mapsto (p, q) \in \mathbb{Z} \times \mathbb{N}^*$.

Notations and definitions

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This function is injective (two elements of \mathbb{Q} can't have the same image by φ), so it is bijective on its image, which is a subset of $\mathbb{Z} \times \mathbb{N}^*$. Using a same reasoning as before, we can show that $\mathbb{Z} \times \mathbb{N}^*$ is countable, hence a subset of it clearly is. Hence \mathbb{Q} is countable

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- Any interval of \mathbb{R} is uncountable. The proof is omitted, but you can check Cantor's diagonal argument^a.

^ahttps://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

Properties of countable and uncountable sets

Subsets and supersets

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Product of countable sets

If A and B are countable sets, then $A \times B$ is countable

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Exemples

- $2\mathbb{N} = \{2n | n \in \mathbb{N}\} \subset \mathbb{N}$ is countable

Properties of countable and uncountable sets

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Conditional Probabilities

② Discrete Random Variables

③ Continuous Random Variables

④ Multivariate Probability Distributions

⑤ Gaussian Vectors

⑥ Convergence and Limit Theorems

Introduction

In this section, theoretical definition of probabilities

- **random experiment**: the result is unknown

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Exemples of sample spaces

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 - $\Omega = \{0, 1, 2, \dots\}$

Events

The goal is to assign each event a probability

Rolling a die

Event “getting an even number”

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Correspondance between the events and the outcomes: an event A is a subset of the sample space

$$A \subset \Omega$$

Set of events

Definition

The set of events \mathcal{F} must verifies

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Such \mathcal{F} is called a σ -algebra. We will assume in the rest of this class that we always deal with such event sets.

Set of events

Property

The set of events \mathcal{F} is stable by intersection and union: let $A \in \mathcal{F}$ and $B \in \mathcal{F}$ be two events. Then $A \cup B$ and $A \cap B$ are also events

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Remember: “or” and “at least” go with unions, and “and” and “all of” go with intersection

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Remember: “or” and “at least” go with unions, and “and” and “all of” go with intersection

Note: $\emptyset \subset \mathcal{F}$

Elementary and compound events

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If $a \in \Omega$, then $\{a\}$ is called an elementary event. It is thus an event which contains only a single outcome of the sample space

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- the event “obtain 1” corresponds to the outcome 1 and is an elementary event
- the event “obtain an even number” corresponds to the outcomes $\{2, 4, 6\}$ and is a compound event

Probability measure

Kolmogorov axioms

A probability measure P is an application that has the following properties

- $\forall A \in \mathcal{F}, P(A) \geq 0$

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A probability measure P is an application that has the following properties

- $\forall A \in \mathcal{F}, P(A) \geq 0$
- $P(\Omega) = 1$
- if A_1, \dots, A_n are disjoint events, then $P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$

Exercise

Using the probability axioms, what is the probability of 1 or 6 when rolling a fair die?

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Events $\{1\}, \{2\}, \dots, \{6\}$ are disjoint:

$$P(\{1\} \cup \{2\} \cup \dots \cup \{6\}) = P(\{1\}) + P(\{2\}) + \dots + P(\{6\}) = 6P(\{1\})$$

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Finally, the event we are looking for corresponds to the outcomes $\{1, 6\} = \{1\} \cup \{6\}$, which are two disjoint events, so

$$P\{1, 6\} = P(\{1\}) + P(\{6\}) = \frac{1}{3}$$

Probability measures

Properties

If P is a probability measure on \mathcal{F} , and A and B are two elements of \mathcal{F}

¹When you are referring to events, no problem if you use \cap , but later, for random variables, please use the comma

Probability measures

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If P is a probability measure on \mathcal{F} , and A and B are two elements of \mathcal{F}

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For probabilities, we denote

- probability of the intersection¹: $P(A \cap B) = P(A, B)$
- probability of an elementary event: $P(\{a\}) = P(a)$

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Exercise

Suppose that you want to travel without stopovers from Toulouse to London. The companies proposing this trip are Ryanair, EasyJet, and British Airways. Suppose that there is a 42% probability that you take Ryanair and 30% that you take British Airways. Suppose that someone booked your flight without telling you which company you're travelling with. What is the probability that you fly with a low cost company?

Solution

We denote

- $R =$ “flying with Ryanair” $P(R) = 0.42$
- $E =$ “flying with EasyJet”
- $B =$ “British Airways” $P(B) = 0.3$
- $L =$ “flying with a low cost company”

Solution

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Sample space $\Omega = \{R, E, B\}$

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Low cost companies: Ryanair and EasyJet: $L = R \cup E$

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R and E are disjoint events: $P(L) = P(R) + P(E)$

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$\bar{E} = \{R, B\} = R \cup B$, so $P(\bar{E}) = P(R) + P(B)$

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$P(L) = P(R) + 1 - (P(R) + P(B)) = 1 - P(B) = 1 - 0.3 = 0.7$

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R and E are disjoint events: $P(L) = P(R) + P(E)$

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$\bar{E} = \{R, B\} = R \cup B$, so $P(\bar{E}) = P(R) + P(B)$

$P(L) = P(R) + 1 - (P(R) + P(B)) = 1 - P(B) = 1 - 0.3 = 0.7$

Another way to solve this is to compute $P(\bar{L}) = P(B)$, which leads

$P(L) = 1 - P(B)$, and we find the same result.

Definition

Probability triple

The triple (Ω, \mathcal{F}, P) is called a probability triple or a probability space

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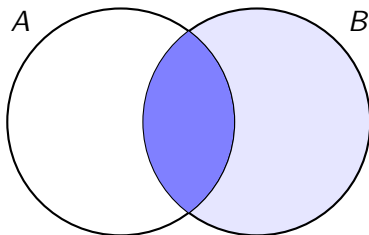
⑥ Convergence and Limit Theorems

Definition

Conditional probability

For two events A and B , we define

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A, B)}{P(B)}$$



Exercice

Suppose that having a girl has the same probability as having a boy.
Someone has two children

- 1 what is the probability that both are girls given that the first one is a girl?
- 2 what is the probability that both children are girls given that we know at least one of them is a girl?

Solution

The sample space is $\{(g, g), (g, b), (b, g), (g, g)\}$. We denote

$$A = \{\text{"both children are girls"}\} = \{(g, g)\}$$

$$B = \{\text{"the first child is a girl"}\} = \{(g, g), (g, b)\}$$

$$C = \{\text{"at least one child is a girl"}\} = \{(g, g), (g, b), (b, g)\}$$

$$\textcircled{1} P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$\textcircled{2} P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)}{P(C)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Independence

Consider $A = \{\text{"it will rain tomorrow"}\}$ and $B = \{\text{"the result of a coin flip"}\}$. What about $P(A|B)$?

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Property

If A and B are two independent events

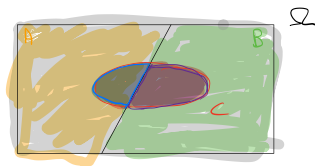
$$P(A|B) = P(A)$$

Law of Total Probability

Theorem

If A_1, A_2, \dots is a partition of Ω , then for any event B

$$P(B) = \sum_i P(B|A_i)P(A_i)$$



$$P(A) = \frac{\text{yellow}}{\text{total}}$$

$$P(B) = \frac{\text{green}}{\text{total}}$$

$$P(C) = \frac{\text{red} + \text{purple}}{\text{total}}$$

$$P(C|A) = \frac{\text{red}}{\text{yellow}}$$

$$P(C|B) = \frac{\text{purple}}{\text{green}}$$

$$P(C) = \frac{\text{red}}{\text{yellow}} + \frac{\text{purple}}{\text{green}}$$

$$= \frac{\text{red}}{\text{yellow}} + \frac{\text{purple}}{\text{green}}$$

$$= P(C|A)P(A) + P(C|B)P(B)$$

Exercise

I have three bags that each contain 100 marbles

- bag 1 has 75 red and 25 blue marbles
- bag 2 has 60 red and 40 blue marbles
- bag 3 has 45 red and 55 blue marbles

I choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

Solution

We have

$$P(R|B_1) = \frac{75}{100} = 0.75$$

$$P(R|B_2) = \frac{60}{100} = 0.60$$

$$P(R|B_3) = \frac{45}{100} = 0.45$$

$\{B_1, B_2, B_3\}$ is a partition: disjoint, union is sample space, and $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$

$$\begin{aligned} P(R) &= P(R|B_1)P(B_1) + P(R|B_2)P(B_2) + P(R|B_3)P(B_3) \\ &= 0.75\frac{1}{3} + 0.60\frac{1}{3} + 0.45\frac{1}{3} \\ &= 0.60 \end{aligned}$$

Brainstorming

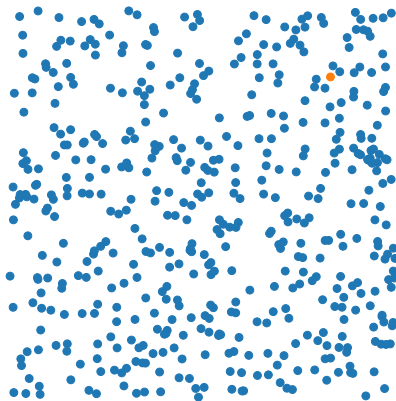
Medical test for a disease affecting 2‰ of the population

- Positive for 100% of ill people
- Negative for 95% of non ill people

You get tested positive. What is the probability you are indeed ill?

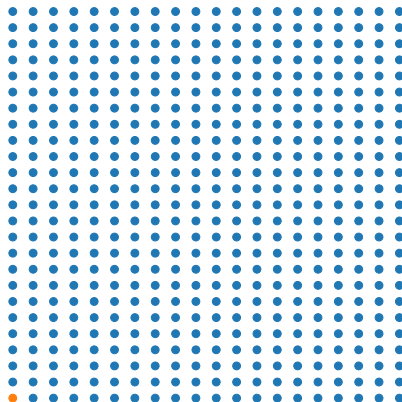
- 1 0-4.99%
- 2 5-49.99%
- 3 50-94.99%
- 4 95-100%

Brainstorming



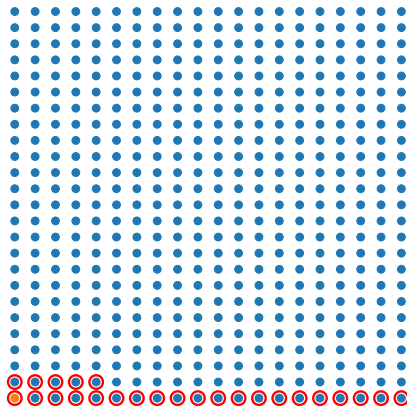
2‰: 500 people \rightarrow 1 sick

Brainstorming



2‰: 500 people \rightarrow 1 sick

Brainstorming



- 100% efficient for sick people: 1 tested positive
- 5% false alarm for healthy people: $(500-1) \times 0.05 \approx 24$ tested positive
- probability of being sick $\frac{1}{24+1} = 4\%$

Bayes rule

Theorem

For any two events A and B , we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Bayes rule

Theorem

For any two events A and B , we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Combination with law of total probability

For any A and partition $\{B_1, B_2, \dots\}$, we have

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_i P(A|B_i)P(B_i)}$$

Exercise

Find the previous probability of being sick using Bayes rule

Solution

- $P(\text{sick}) = 0.002$
- $P(+|\text{sick}) = 1$
- $P(+|\text{healthy}) = 0.05$
- we want $P(\text{sick}|+)$

$$\begin{aligned} P(\text{sick}|+) &= \frac{P(+|\text{sick}) P(\text{sick})}{P(+)} \\ &= \frac{P(+|\text{sick}) P(\text{sick})}{P(+|\text{sick}) P(\text{sick}) + P(+|\text{healthy}) P(\text{healthy})} \\ &= \frac{1 \times 0.002}{1 \times 0.002 + 0.05 \times (1 - 0.002)} \\ &= 0.038 \end{aligned}$$

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Random variable

Definition

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$$X : \Omega \rightarrow \mathbb{R}$$

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Usually: capital letter

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R_X is the set of all possible values of X

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Translate the problem to numbers

Examples

- For coin tossing: $X(\text{head}) = 0$ and $X(\text{tail}) = 1$, $R_X = \{0, 1\}$

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Examples

- For coin tossing: $X(\text{head}) = 0$ and $X(\text{tail}) = 1$, $R_X = \{0, 1\}$
- For die rolling: Y parity of the results $Y(1) = Y(3) = Y(5) = 0$ and $Y(2) = Y(4) = Y(6) = 1$, $R_Y = \{0, 1\}$
- T is the time before the next bus arrives: $R_T = [0, \infty)$

Exercise

X is the sum of tails from two coin toss

- 1 what is the range of X ?
- 2 what is the probability of each value in that range?

Solution

X is the sum of tails from two coin toss

① $R_X = 0, 1, 2$

Solution

X is the sum of tails from two coin toss

- 1 $R_X = 0, 1, 2$
- 2 4 possible outcomes: $\{HH, HT, TH, TT\}$
 - $X = 0$ corresponds to $HH \Rightarrow P(X = 0) = \frac{1}{4}$
 - $X = 1$ corresponds to HT and $TH \Rightarrow P(X = 1) = \frac{2}{4} = \frac{1}{2}$
 - $X = 2$ corresponds to $TT \Rightarrow P(X = 2) = \frac{1}{4}$

Probability Mass Function

X a discrete random variable, $R_X = \{x_1, x_2, \dots\}$

Definition

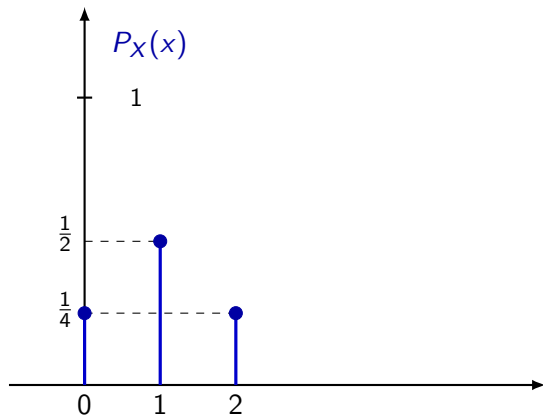
The Probability Mass Function (pmf) of X is given by

$$P_X(x_k) = P(X = x_k), \quad k = 1, 2, \dots$$

We can extend $P_X(x) = 0 \quad \forall x \notin R_X$

Example

From the previous exercise



Probability Mass Function

Properties

It is a probability measure

- $0 \leq P_X(x) \leq 1$
- $\sum_{x \in R_X} P_X(x) = 1$
- $\forall A \subset R_X, P(X \in A) = \sum_{x \in A} P_X(x)$

It can also be referred to as *distribution*

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Independent discrete random variables

Independent events: $P(A \cap B) = P(A, B) = P(A)P(B)$

Definition

Two random variables X and Y are independent if and only if

$$\forall(x, y) \quad P(X = x, Y = y) = P(X = x)P(Y = y).$$

More generally

$$\forall(A, B) \quad P(X \in A, Y \in B) = P(X \in A)P(X \in B)$$

When independence $P(X = x|Y = y) = P(X = x)$

Exercise

I toss a coin twice, X is the number of heads. Then I toss again the coin two times, and Y is the number of heads observed this time. What is $P((X < 2) \text{ and } (Y > 1))$

Solution

X and Y are independent

$$\begin{aligned}P((X < 2) \text{ and } (Y > 1)) &= P(X < 2)P(Y > 1) \\ &= (P_X(0) + P_X(1))P_Y(2) \\ &= \left(\frac{1}{4} + \frac{1}{2}\right) \frac{1}{4} \\ &= \frac{3}{16}\end{aligned}$$

Generalisation

Definition

X_1, \dots, X_n discrete random variables are independent if and only if

$$\forall (x_1, \dots, x_n)$$

$$P(X_1 = x_1, \dots, X_n = x_n) = P(X_1 = x_1) \cdots P(X_n = x_n)$$

Exercise

I toss an unfair coin where $P(H) = p$, $0 < p < 1$. I toss it repeatedly until I obtain a head. Let Y the total number of coin tosses. What is the range of Y ? What is the probability mass function of Y ?

Solution

We have $R_Y = \mathbb{N}^*$. For the PMF, we need $P_Y(x_k)$ for $x_k \in \mathbb{N}^*$. The events are independent

$$P(Y = 1) = P(H) = p$$

$$P(Y = 2) = P(TH) = P(T)P(H) = (1 - p)p$$

$$P(Y = 3) = P(TTH) = P(T)^2P(H) = (1 - p)^2p$$

$$\vdots$$

$$P(Y = k) = P(T \dots TH) = (1 - p)^{k-1}p$$

So

$$P_Y(y) = (1 - p)^{y-1}p$$

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Bernoulli distribution

Simplest discrete distribution: success (1) or failure (0)

Definition

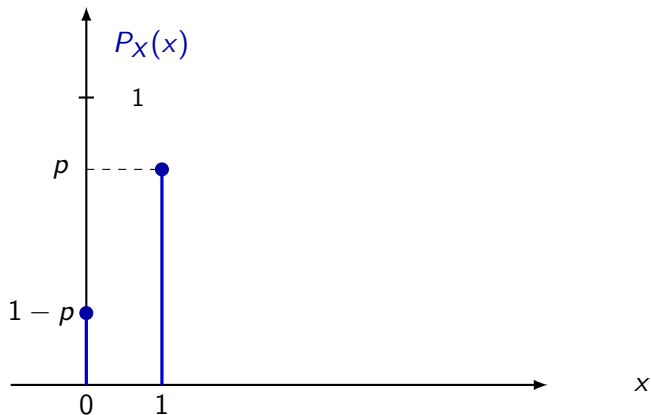
X is a Bernoulli random variable with parameter p ($X \sim \text{Bernoulli}(p)$) if its PMF is

$$P_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \\ 0 & \text{else} \end{cases}$$

Examples

- pass-fail an exam
- heads or tail
- a satellite has verification before launch

Bernoulli distribution



Geometric distribution

Succession of independent Bernoulli trials before success

Definition

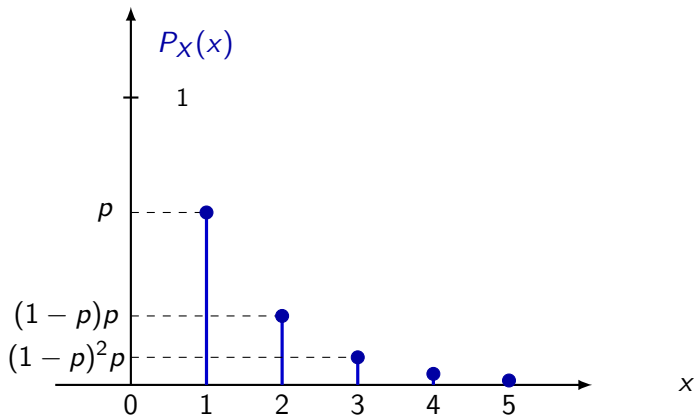
X is a geometric random variable with parameter p ($X \sim \text{Geometric}(p)$) if its PMF is

$$P_X(x) = \begin{cases} (1-p)^{x-1}p & \text{if } x \in \mathbb{N}^* \\ 0 & \text{else} \end{cases}$$

Examples

- how many time will I try driving license test
- how many heads before tail

Geometric distribution



Binomial distribution

Number of success for n independent Bernoulli trials

Definition

X is a binomial random variable with parameter n and p ($X \sim \text{Binomial}(n, p)$) if its PMF is

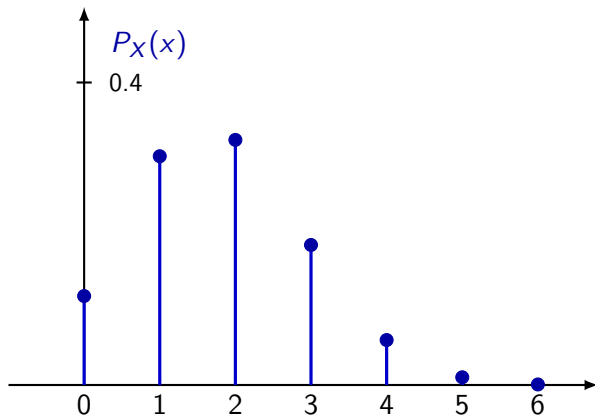
$$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if } x \in \{0, 1, \dots, n\} \\ 0 & \text{else} \end{cases}$$

Examples

- number of tails in a series of tossing
- how many students passed the exam

Binomial distribution

$$n = 6, p = 0.3$$



Binomial distribution

Number of success for n independent Bernoulli trials

Lemma

If X_1, X_2, \dots, X_n are **independent** *Bernoulli*(p) random variables, then $X = X_1 + \dots + X_n$ has a *Binomial*(n, p) distribution

Exercise: $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$ independent.
What is the distribution of $Z = X + Y$?

Poisson distribution

Definition

X is a Poisson random variable with parameter λ ($X \sim \text{Poisson}(\lambda)$) if its PMF is

$$P_X(x) = \begin{cases} e^{-\lambda} \frac{\lambda^x}{x!} & \text{if } x \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

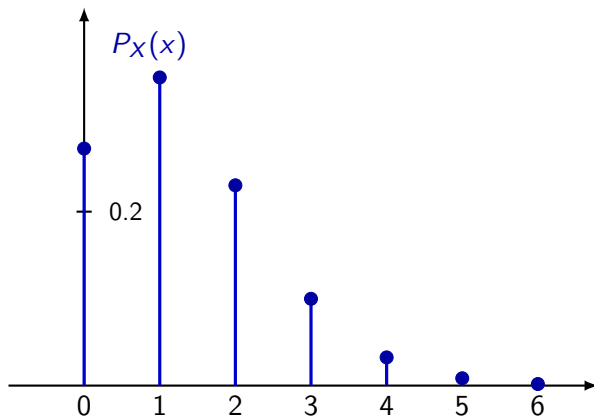
Examples

- counting events during intervals

Exercise: check it is a valid PMF

Poisson distribution

$$\lambda = 1.3$$



Cumulative distribution function

Definition

The cumulative distribution function of a random variable X is defined as

$$\forall x \in \mathbb{R} \quad F_X(x) = P(X \leq x)$$

For a discret random variable with range $R_X = \{x_1, x_2, \dots\}$

$$F_X(x) = \sum_{x_k \leq x} P_X(x_k)$$

Property

$$\forall a \leq b \quad P(a < X \leq b) = F_X(b) - F_X(a)$$

Example

At the airport security check, at a given time, we assume that the probability that there are k people at a single check is $\frac{1}{2^{k+1}}$. We consider the PMF of the random variable X corresponding to the number of people in the line

- Is it a valid PMF?
- Draw the corresponding CDF.
- What is the probability that there is at least 3 and max 5 people waiting?

When there are too many people at a single check, a new one is opened. We suppose that a new line is opened when there are 5 or more people at a single line.

- What is the probability that a new line is opened?

Solution

Is the probability measure is well defined?

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- $\Omega = \mathbb{N}$ countable

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- $P(n) = \frac{1}{2^{n+1}} \geq 0$ and depends only on n

Solution

Is the probability measure is well defined?

- $\Omega = \mathbb{N}$ countable
- $P(n) = \frac{1}{2^{n+1}} \geq 0$ and depends only on n

$$\begin{aligned} P(\Omega) &= \sum_{n=0}^{\infty} P(n) \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &= \frac{1}{2} \frac{1}{1 - \frac{1}{2}} \\ &= 1 \end{aligned}$$

Solution

We have $R_X = \{0, 1, \dots\}$ and

$$\begin{aligned}F_X(x) &= P(X \leq x) \\&= \sum_{x_k \leq x} P_X(x_k) \\&= \sum_{x_k=0}^{\lfloor x \rfloor} P_X(x_k) \\&= \sum_{x_k=0}^{\lfloor x \rfloor} \frac{1}{2^{x_k+1}} \\&= \frac{1}{2} \sum_{x_k=0}^{\lfloor x \rfloor} \frac{1}{2^{x_k}} \\&= \frac{1}{2} \frac{1 - \frac{1}{2^{\lfloor x \rfloor + 1}}}{1 - \frac{1}{2}} = 1 - \frac{1}{2^{\lfloor x \rfloor + 1}}\end{aligned}$$

Solution

So $P(3 \leq X \leq 5) = P(2 < X \leq 5) = F_X(5) - F_X(2)$ using the property of CDF. So

$$\begin{aligned}P(3 \leq X \leq 5) &= 1 - \frac{1}{2^{5+1}} - \left(1 - \frac{1}{2^{2+1}}\right) \\&= 1 - \frac{1}{2^6} - 1 + \frac{1}{2^3} \\&= \frac{2^3 - 1}{2^6} \\&\approx 0.109\end{aligned}$$

Solution

$$P(3) = \frac{1}{2^4} = 0.0625 = 6.25\%$$

Solution

$$P(3) = \frac{1}{2^4} = 0.0625 = 6.25\% L = \{ \text{"A new line is opened"} \} = P(n \geq 5) = \bigcup_{n=5}^{+\infty} P(n)$$

Solution

$$P(3) = \frac{1}{2^4} = 0.0625 = 6.25\% L = \{ \text{"A new line is opened"} \} = P(n \geq 5) = \bigcup_{n=5}^{+\infty} P(n)$$

$$\begin{aligned} P(L) &= \sum_{n=5}^{\infty} P(n) \\ &= \sum_{n=5}^{\infty} \frac{1}{2^{n+1}} \\ &= \sum_{m=0}^{\infty} \frac{1}{2^{m+6}} \\ &= \frac{1}{2^6} \sum_{m=0}^{\infty} \frac{1}{2^m} \\ &= \frac{1}{2^6} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^5} = 0.03125 \end{aligned}$$

Solution

3.125% chances that a new line is opened

Solution

3.125% chances that a new line is opened

Another way to compute this is

$$\begin{aligned}P(L) &= 1 - P(\bar{L}) \\&= 1 - \sum_{n=0}^4 P(n) \\&= 1 - \sum_{n=0}^4 \frac{1}{2^{n+1}} \\&= 1 - \frac{1}{2} \sum_{n=0}^4 \frac{1}{2^n} \\&= 1 - \frac{1}{2} \frac{1 - \frac{1}{2^5}}{1 - \frac{1}{2}} = 1 - \left(1 - \frac{1}{2^5}\right) = \frac{1}{2^5}\end{aligned}$$

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Expectation

Statistical mean or expected value

Definition

Let X be a discrete random variable with range $R_X = \{x_1, x_2, \dots\}$. The expectation of X is given by

$$E[X] = \sum_{x_k \in R_X} x_k P(X = x_k) = \sum_{x_k \in R_X} x_k P_X(x_k)$$

Exercise: find the expectation of a *Bernoulli*(p) and a *Poisson*(λ) distribution

Expectation

Theorem

The expectation is linear

$$\forall a, b \in \mathbb{R} \quad E[aX + b] = aE[X] + b$$

For random variables X_1, \dots, X_n

$$E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n]$$

Exercise: X is *Poisson*(λ) and Y is *Bernoulli*(p). What is the expectation of $Z = 3X + 2Y + 6$?

Function of random variable

Property

Let X a random variable and $Y = g(X)$

$$R_Y = \{g(x) | x \in R_X\}$$

and

$$P_Y(y) = P(Y = y) = P(g(X) = y) = \sum_{\{x \in R_X | g(x) = y\}} P_X(x)$$

Exercise $P_X(k) = \frac{1}{5}$ for $k = -1, 0, 1, 2, 4$. Find the range and PMF of $Y = 2|X|$

Function of random variable

Law of the unconscious statistician (LOTUS)

Let X a random variable with $R_X = \{x_1, x_2, \dots\}$

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k)$$

Exercise Let X be a discrete random variable with range

$R_X = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$, such that

$P_X(0) = P_X(\frac{\pi}{4}) = P_X(\frac{\pi}{2}) = P_X(\frac{3\pi}{4}) = P_X(\pi) = \frac{1}{5}$. Find $E[\sin(X)]$

Variance

The “spread” of the distribution

Definition

The variance of a random variable X is defined as

$$\text{Var}(X) = E[(X - E(X))^2] (= \sigma_X^2)$$

We also define the standard deviation $\sigma_X = \sqrt{\text{Var}(X)}$

Property

$$\sigma_X^2 = E[X^2] - E[X]^2$$

Exercise: prove it

Exercice

Variance of *Bernoulli*(p) and *Poisson*(λ)

Variance

Theorem

If X is a random variable and a and b are two real number

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

Exercise: prove it

Theorem

If X_1, \dots, X_n are **independent** random variables and $X = X_1 + \dots + X_n$

$$\text{Var}(X) = \text{Var}(X_1) + \dots + \text{Var}(X_n)$$

Solution

With the definition

$$\begin{aligned}\text{Var}(aX + b) &= E[(aX + b - E[aX + b])]^2 \\ &= E[(aX + b - aE[X] + b)^2] \quad \text{prop. of } E[\cdot] \\ &= E[(a(X - E[X]))^2] \\ &= E[a^2(X - E[X])^2] \\ &= a^2 E[(X - E[X])^2] \quad \text{prop. of } E[\cdot] \\ &= a^2 \text{Var}(X)\end{aligned}$$

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Definitions

Continuous random variable

A random variable X with CDF $F_X(x)$ if its range is continuous

Property

A continuous random variable X has a continuous CDF

Uniform distribution

Simplest continuous random variable

Definition

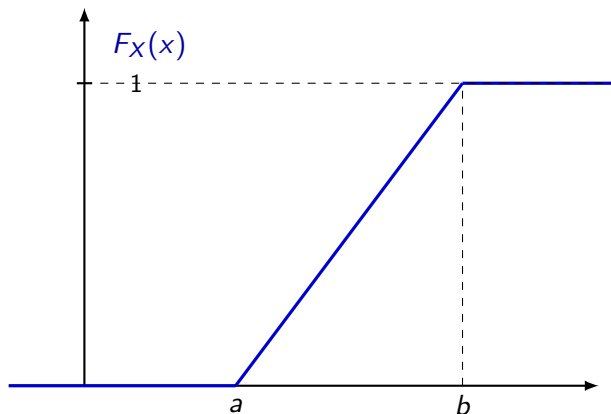
A continuous random variable X follows a uniform distribution over the interval $[a, b]$ ($a \leq b$) if

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x \geq b \end{cases}$$

We note $X \sim \mathcal{U}([a, b])$

Uniform distribution

Plot of CDF $X \sim \mathcal{U}([a, b])$



WARNING

For a continuous random variable

$$P(X = x) = 0$$

We rather have $P(X \in [x_1, x_2]) = P(x_1 \leq X \leq x_2)$.

Remark: for continuous random variables, we have

$$P(X \in [x_1, x_2]) = P(X \in (x_1, x_2]) = P(X \in [x_1, x_2)) = P(X \in (x_1, x_2))$$

Probability density function

PMF is not defined for continuous random variables

Definition

For a continuous random variable X with an absolutely continuous CDF $F_X(x)$, the probability density function (PDF) of X is defined as

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x)$$

Remark

$$R_X = \{x | f_X(x) > 0\}$$

Example

For $X \sim \mathcal{U}([a, b])$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{else} \end{cases}$$

Remark

For a given CDF $F_X(x)$, we have a unique PDF $f_X(x) = F'_X(x)$. So

$$\int_{-\infty}^x f_X(t)dt = F_X(x) - \underbrace{F_X(-\infty)}_0 = F_X(x)$$

\Rightarrow for a given PDF, there is a unique CDF and

$$P(X \leq x) = \int_{-\infty}^x f_X(t)dt$$

Can be generalized

$$P(a < X \leq b) = \int_a^b f_X(t)dt$$

Probability density function

Properties

- $\forall x \in \mathbb{R}, f_X(x) \geq 0$
- $\int_{-\infty}^{+\infty} f_X(u) du = 1$
- $F_X(x) = \int_{-\infty}^x f_X(t) dt$
- $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u) du$
- $P(X \in A) = \int_A f_X(u) du$

For instance $A = [a, b] \cup [c, d]$ with disjoint intervals

Exercise

We consider a continuous random variable X with PDF

$$f_X(x) = \begin{cases} ce^{-x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

- 1 find c
- 2 find the CDF of X
- 3 find $P(1 < X < 3)$

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Expectation of a continuous random variable

Reminder: X discrete, $E[X] = \sum_{x_k \in \mathcal{R}_X} x_k P_X(x_k)$

Definition

For a continuous random variable X , its expectation is given by

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

“sum” \leftrightarrow “int” and “PMF” \leftrightarrow “PDF”

Property

Expectation is linear, as in the discrete case

Example: expectation of a uniform distribution

Expectation of a function of a continuous random variable

The LOTUS generalizes to continuous random variables

LOTUS

For a continuous random variable X , and any function g we have

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

Example: X continuous random variable with PDF

$$f_X(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases} . \text{ Find } E[X^n]$$

Variance of a continuous random variable

Same as discrete

Variance of a continuous random variable

For a continuous random variable X ,

$$\sigma_X^2 = E[(X - E[X])^2] = E[X^2] - E[X]^2$$

Ans we still have $\text{Var}(aX + b) = a^2\text{Var}(X)$ for any real number a and b ,
and if X_1, \dots, X_n are independent random variables,

$$\text{Var}(X_1 + \dots + X_n) = \text{Var}(X_1) + \dots + \text{Var}(X_n)$$

Exercise

Let X a continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{3}{x^4} & \text{if } x \geq 1 \\ 0 & \text{else} \end{cases}$$

Find the mean and variance of X

Solution

$$\begin{aligned} E[X] &= \int_{-\infty}^{+\infty} x f_X(x) dx \\ &= \int_1^{+\infty} \frac{3}{x^3} dx \\ &= \left[-\frac{3}{2x^2} \right]_{x=1}^{+\infty} \\ &= 0 - \left(-\frac{3}{2} \right) \\ &= \frac{3}{2} \end{aligned}$$

Solution

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{+\infty} x^2 f_X(x) dx \\ &= \int_1^{+\infty} \frac{3}{x^2} dx \\ &= \left[-\frac{3}{x} \right]_{x=1}^{+\infty} \\ &= 0 - (-3) \\ &= 3 \end{aligned}$$

And

$$\sigma_X^2 = E[X^2] - E[X]^2 = 3 - \frac{9}{4} = \frac{3}{4}$$

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Uniform distribution

Definition

A continuous random variable X is said to have a uniform distribution over $[a, b]$ (denoted $X \sim \mathcal{U}([a, b])$) if its PDF is

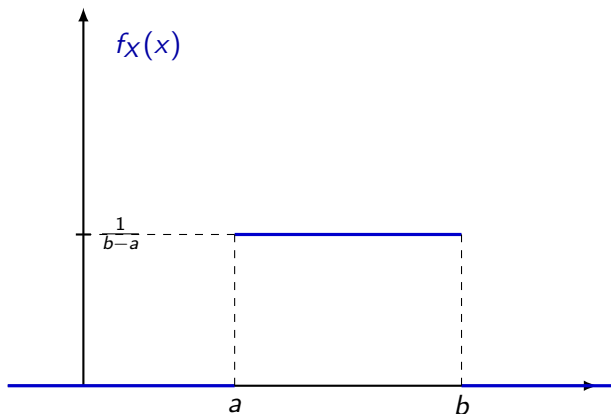
$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{else} \end{cases}$$

Property

$$E[X] = \frac{a+b}{2} \quad \text{Var}(X) = \frac{(b-a)^2}{12}$$

Uniform distribution

Plot of PDF $X \sim \mathcal{U}([a, b])$



Exponential distribution

Definition

A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$ (denoted $X \sim \mathcal{E}(\lambda)$) if its PDF is

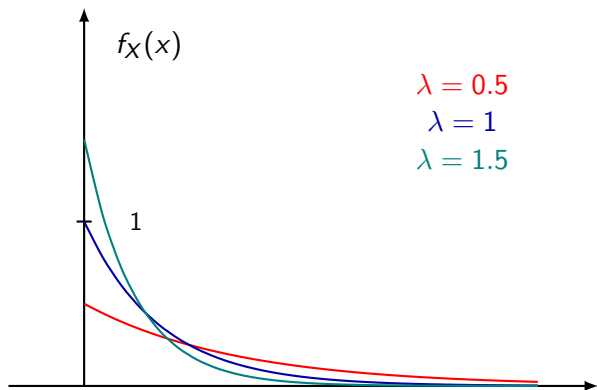
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$$

Property

$$E[X] = \frac{1}{\lambda} \quad \text{Var}(X) = \frac{1}{\lambda^2}$$

Exponential distribution

Plot of PDF $X \sim \mathcal{E}(\lambda)$



Exercise

Let $X \sim \mathcal{E}(\lambda)$

- find the CDF of X
- show that X is a memoryless random variable, i.e.,

$$\forall a, x > 0 \quad P(X > x + a | X > a) = P(X > x)$$

Solution

For $x < 0$

$$F_X(x) = 0$$

For $x \geq 0$

$$\begin{aligned} F_X(x) &= \int_{-\infty}^x f_X(u) du \\ &= \int_0^x \lambda e^{-\lambda u} du \\ &= \left[-e^{-\lambda u} \right]_{u=0}^x \\ &= 1 - e^{-\lambda x} \end{aligned}$$

Solution

$$\begin{aligned}P(X > x + a | x > a) &= \frac{P(X > x + a, X > a)}{P(X > a)} \\&= \frac{P(X > x + a)}{P(X > a)} \\&= \frac{1 - F_X(x + a)}{1 - F_X(a)} \\&= \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} \\&= e^{-\lambda(x+a)} \\&= P(X > x)\end{aligned}$$

Normal (or Gaussian) distribution

The most important distribution!

Definition

A continuous random variable X is said to have a **standard** normal distribution (denoted $X \sim \mathcal{N}(0, 1)$) if its PDF is

$$\forall x \in \mathbb{R} \quad f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Property

$$E[X] = 0 \quad \text{Var}(X) = 1$$

Normal (or Gaussian) distribution

General case

Definition

X is said to have a normal distribution with mean μ and variance σ^2 ($X \sim \mathcal{N}(\mu, \sigma^2)$) if its PDF is

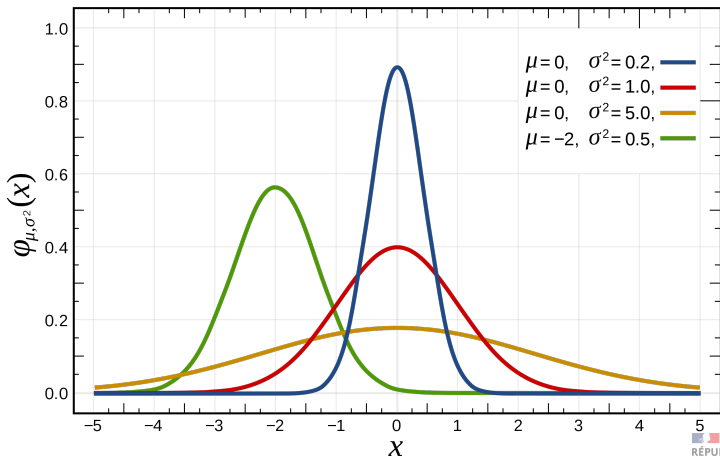
$$\forall x \in \mathbb{R} \quad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\}$$

Properties

$$E[X] = \mu \quad \text{Var}(X) = \sigma^2$$

If $Z \sim \mathcal{N}(0, 1)$, then $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$

Normal (or Gaussian) distribution



Gamma distribution

Definition

A continuous random variable X is said to have a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ ($X \sim \mathcal{G}(\alpha, \lambda)$) if its PDF is

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$$

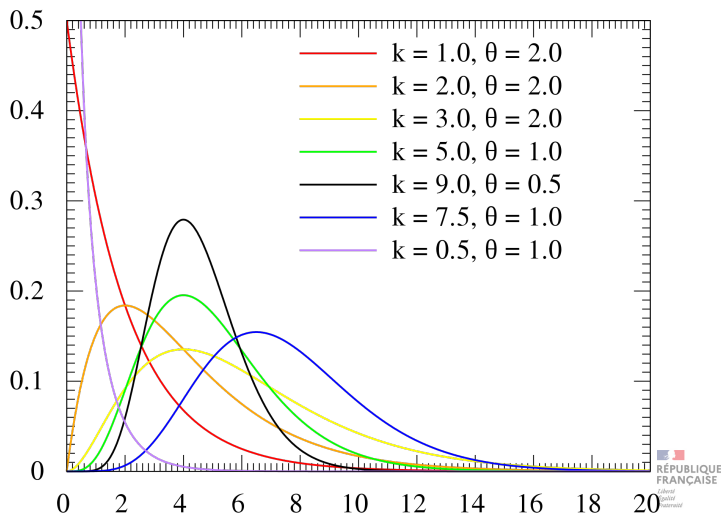
where $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$

Properties

$$E[X] = \frac{\alpha}{\lambda} \quad \text{Var}(X) = \frac{\alpha}{\lambda^2}$$

If $X \sim \mathcal{G}(1, \lambda)$, then $X \sim \mathcal{E}(\lambda)$

Gamma distribution



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Direct calculus

If X is a random variable, $Y = g(X)$ is also a random variable. How to find the PDF?

First method: compute CDF then PDF

Example: $X \sim \mathcal{U}([0, 1])$ and $Y = e^X$

Solution

For $X \sim \mathcal{U}([0, 1])$, we have

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{else} \end{cases}$$

On the other side

$$\begin{aligned} P(Y \leq y) &= P(e^X \leq y) \\ &= P(X \leq \ln(y)) \end{aligned}$$

$R_X = [0, 1]$, so if $\ln(y) < 0$, $F_Y(y) = 0$, i.e., for $y < 1$. More over, $F_X(x) = 1$ for $x > 1$, i.e., for $y > e$. For $y \in [1, e]$, we have

$$P(Y \leq y) = F_X(\ln(y))$$

$$P(Y \leq y) = \ln(y)$$

Solution

Finally

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 1 \\ \ln(y) & \text{if } 1 \leq y \leq e \\ 1 & \text{else} \end{cases}$$

so

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{1}{y} & \text{if } 1 \leq y \leq e \\ 0 & \text{else} \end{cases}$$

Method of transformations

Theorem

X is a continuous random variable and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a bijective differentiable function with g^{-1} also bijective. Let $Y = g(X)$, then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

Example: $X \sim \mathcal{E}(1)$ and $Y = \frac{1}{X}$

$g : x \mapsto \frac{1}{x}$, so $g^{-1} : y \mapsto \frac{1}{y}$. $x = \frac{1}{y} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2}$. We deduce from the theorem

$$\begin{aligned} f_Y(y) &= f_X\left(\frac{1}{y}\right) \left| -\frac{1}{y^2} \right| \\ &= \begin{cases} \frac{1}{y^2} e^{-\frac{1}{y}} & \text{if } \frac{1}{y} \geq 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

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Motivation

In real life application : several random variables are jointly involves

- throw of two dice
- evolution of temperature and pressure

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Joint probability mass function

Definition

For two discrete random variables X and Y , the joint PMF is defined as

$$P_{XY}(x, y) = P(X = x, Y = y)$$

Reminder: comma means “and”

Joint Range

Definition

The joint range is defined as

$$R_{XY} = \{(x, y) | P_{XY}(x, y) > 0\}$$

Property

If $R_X = \{x_1, x_2, \dots\}$ and $R_Y = \{y_1, y_2, \dots\}$, then

$$R_{XY} \subset R_X \times R_Y = \{(x_i, y_j) | x_i \in R_X, y_j \in R_Y\}$$

Sometimes to simplify: $R_{XY} = R_X \times R_Y$ (some elements might have a probability of 0)

Joint probability mass function

Properties

For two discrete random variables X and Y , we have

- $\sum_{(x_i, y_j) \in R_{XY}} P_{XY}(x_i, y_j) = 1$
- $\forall A \subset \mathbb{R}^2, P((X, Y) \in A) = \sum_{(x_i, y_j) \in (A \cap R_{XY})} P_{XY}(x_i, y_j)$

Marginal probability mass function

Definition

For two discrete random variables X and Y , we have

$$\begin{aligned} P_X(x) &= \sum_{y_j \in R_Y} P(X = x, Y = y_j) \\ &= \sum_{y_j \in R_Y} P_{XY}(x, y_j) \end{aligned}$$

Similarly $P_Y(y) = \sum_{x_i \in R_X} P_{XY}(x_i, y)$

Example

We define two discrete random variables X and Y with PMF

| | $Y = 0$ | $Y = 1$ | $Y = 2$ |
|---------|---------------|---------------|---------------|
| $X = 0$ | $\frac{1}{6}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |
| $X = 1$ | $\frac{1}{8}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

- Find $P(X = 0, Y \leq 1)$
- Find the marginal PMF of X and Y
- Find $P(Y = 1|X = 0)$
- Are X and Y independent?

Solution

- $P(X = 0, Y \leq 1) = P_{XY}(0, 0) + P_{XY}(0, 1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$
- $R_X = \{0, 1\}$ and $R_Y = \{0, 1, 2\}$. We have

$$\begin{aligned}P_X(0) &= P_{XY}(0, 0) + P_{XY}(0, 1) + P_{XY}(0, 2) \\ &= \frac{1}{6} + \frac{1}{4} + \frac{1}{8} \\ &= \frac{13}{24}\end{aligned}$$

$$\begin{aligned}P_X(1) &= P_{XY}(1, 0) + P_{XY}(1, 1) + P_{XY}(1, 2) \\ &= \frac{11}{24}\end{aligned}$$

Solution

$$\begin{aligned}P_Y(0) &= P_{XY}(0, 0) + P_{XY}(1, 0) \\ &= \frac{7}{24}\end{aligned}$$

$$\begin{aligned}P_Y(1) &= P_{XY}(0, 1) + P_{XY}(1, 1) \\ &= \frac{5}{12}\end{aligned}$$

$$\begin{aligned}P_Y(2) &= P_{XY}(0, 2) + P_{XY}(1, 2) \\ &= \frac{7}{24}\end{aligned}$$

Solution

$$\begin{aligned} P(Y = 1|X = 0) &= \frac{P(X = 0, Y = 1)}{P(X = 0)} \\ &= \frac{P_{XY}(0, 1)}{P(X = 0)} \\ &= \frac{\frac{1}{4}}{\frac{13}{24}} \\ &= \frac{6}{13} \end{aligned}$$

Solution

X and Y independent $\Leftrightarrow \forall (x_i, y_j) \in R_{XY}, P(Y = y_j | X = x_i) = P(Y = y_j)$.
Here, $P(Y = 1 | X = 0) = \frac{6}{13}$ and

$$P(Y = 1) = \frac{5}{12} \\ \neq \frac{6}{13}$$

Joint cumulative distribution function

Definition

The joint CDF of two random variables is defined as

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

Marginal CDF

The marginal CDF are defined as

$$F_X(x) = F_{XY}(x, +\infty) \quad F_Y(y) = F_{XY}(+\infty, y)$$

Example: $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Bernoulli}(q)$ independent. Find the joint PMF and CDF

Solution

We have

$$R_{XY} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

Independence

$$P_{XY}(i, j) = P_X(i)P_Y(j)$$

PMF

$$P_{XY}(0, 0) = P_X(0)P_Y(0) = (1 - p)(1 - q)$$

$$P_{XY}(0, 1) = P_X(0)P_Y(1) = (1 - p)q$$

$$P_{XY}(1, 0) = P_X(1)P_Y(0) = p(1 - q)$$

$$P_{XY}(1, 1) = P_X(1)P_Y(1) = pq$$

Solution

CDF

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

We have $0 \leq X, Y \leq 1$ so

$$F_{XY}(x, y) = 0 \text{ if } x < 0$$

$$F_{XY}(x, y) = 0 \text{ if } y < 0$$

$$F_{XY}(x, y) = 1 \text{ if } x \geq 1, y \geq 1$$

If $0 \leq x < 1$ and $y \geq 1$

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

$$= P(X = 0, y \leq 1)$$

$$= P(X = 0)$$

$$= 1 - p$$

Solution

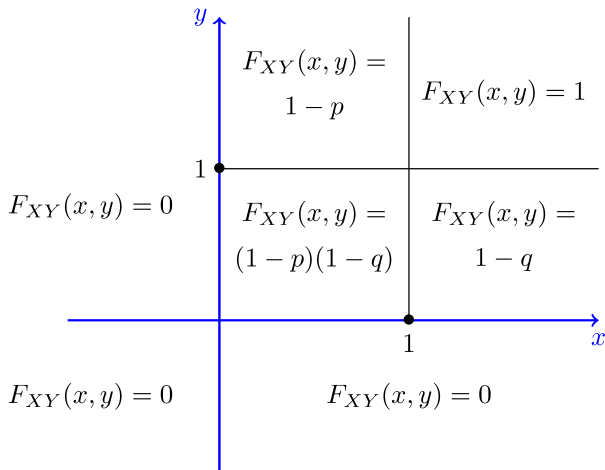
If $0 \leq y < 1$ and $x \geq 1$

$$\begin{aligned}F_{XY}(x, y) &= P(X \leq x, Y \leq y) \\&= P(X \leq 1, Y \leq y) \\&= P(Y = 0) \\&= 1 - q\end{aligned}$$

If $0 \leq x < 1$ and $0 \leq y < 1$

$$\begin{aligned}F_{XY}(x, y) &= P(X \leq x, Y \leq y) \\&= P(X = 0, Y = 0) \\&= P(X = 0)P(Y = 0) \\&= (1 - p)(1 - q)\end{aligned}$$

Solution



Conditional PMF and CDF

Conditional PMF

For a discrete random variable X and an event A , we define the conditional PMF of X given A as

$$\forall x_i \in R_X, P_{X|A}(x_i) = P(X = x_i|A)$$

Conditional CDF

Similarly, we define the conditional CDF of X given A

$$F_{X|A}(x) = P(X \leq x|A)$$

Conditional PMF

Conditional PMF

For discrete random variables X and Y , we define the conditional PMF of X given Y as

$$\forall (x_i, y_j) \in R_{XY}, P_{X|Y}(x_i|y_j) = P(X = x_i|Y = y_j)$$
$$P_{Y|X}(y_j|x_i) = P(Y = y_j|X = x_i)$$

Independent random variables

Properties

Two discrete random variables X and Y are independent if and only if

$$\forall(x, y) \quad P_{XY}(x, y) = P_X(x)P_Y(y)$$

or

$$\forall(x, y) \quad F_{XY}(x, y) = F_X(x)F_Y(y)$$

Conditional expectation

Definition

For two discrete random variables X , Y , and an event A , we can define the two conditional expectations

$$E[X|A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i)$$

$$E[X|Y = y_j] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y_j)$$

Exercise

Let $G = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| \leq 2\}$, with uniform distribution

- find the joint and marginal PMF of X and Y
- find the PMF of X given $Y = 1$
- are X and Y independent?
- find $E[X|Y = 1]$

Law of total probability

Law of total probability

For two discrete random variables X Y , and any set A , we have

$$P(X \in A) = \sum_{y_j \in R_Y} P(X \in A | Y = y_j) P_Y(y_j)$$

Law of total expectation

Law of total expectation

- if B_1, B_2, \dots is a partition of Ω

$$E[X] = \sum_i E[X|B_i]P(B_i)$$

- for two discrete random variables

$$E[X] = \sum_{y_j \in R_Y} E[X|Y = y_j]P_Y(y_j)$$

Example: $X \sim \text{Geometric}(p)$, find $E[X]$ by conditioning on the first toss

Functions of two random variables

Definition

For two discrete random variables X Y , and a function g , we define $Z = g(X, Y)$. Then

$$P_Z(z) = \sum_{(x_i, y_j) \in A_z} P_{XY}(x_i, y_j)$$

where $A_z = \{(x_i, y_j) \in R_{XY}, g(x_i, y_j) = z\}$

LOTUS

For two discrete random variables X Y , and a function g , we have

$$E[g(X, Y)] = \sum_{(x_i, y_j) \in R_{XY}} g(x_i, y_j) P_{XY}(x_i, y_j)$$

Conditional expectation

Property

For any $y \in R_Y$, we have defined $E[X|Y = y]$. We can then define

$$E[X|Y] = g(Y)$$

which is a function of the random variable Y

Example: $X = aY + b$ then $E[X|Y = y] = ay + b = g(y)$ so
 $E[X|Y = y] = aY + b$

Law of total expectation

$$E[X] = E[E[X|Y]]$$

Expectation of independent random variables

Properties

For two independent random variables X and Y we have

- $E[X|Y] = E[X]$
- $E[g(X)|Y] = E[g(X)]$
- $E[XY] = E[X]E[Y]$
- $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

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Joint probability density function

Definition

Two random variables X and Y are jointly continuous if there exists a function $f_{XY}: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for any set $A \subset \mathbb{R}^2$ we have

$$P((X, Y) \in A) = \int \int_A f_{XY}(x, y) dx dy$$

Property

If f_{XY} is a joint PDF

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x, y) dx dy = 1$$

Example

$$f_{XY}(x, y) = \begin{cases} x + cy^2 & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

- 1 find c
- 2 find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$

Marginal PDF

Definition

We define the marginal distributions of two continuous random variables X and Y as

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dx$$

Example: compute the marginal PDF of the previous example

Joint cumulative distribution function

Definition

The joint CDF of two continuous random variables X and Y is defined as

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

Properties

- marginal CDF of x : $F_X(x) = F_{XY}(x, \infty)$
- marginal CDF of y : $F_Y(y) = F_{XY}(\infty, y)$
- $F_{XY}(\infty, \infty) = 1$
- $F_{XY}(x, -\infty) = F_{XY}(-\infty, y) = 0$

Example: X and Y independent and uniform on $[0, 1]$. Find F_{XY}

Joint cumulative distribution function

Link with the joint PDF

For two continuous random variables X and Y we have

$$F_{XY}(x, y) = \int_{-\infty}^y \int_{-\infty}^x f_{XY}(u, v) du dv$$
$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$

Conditioning

Definition

If X is a continuous random variable and A is the event $a < X < b$, then

$$F_{X|A}(x) = \begin{cases} 1 & \text{if } x \geq b \\ \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & \text{if } a \leq x < b \\ 0 & \text{if } x < a \end{cases}$$
$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)} & \text{if } a \leq x < b \\ 0 & \text{else} \end{cases}$$

N.B., $P(A) = P(a < X < b) = F_X(b) - F_X(a)$

Conditioning

Conditional expectation

If X is a continuous random variable and A is the event $a < X < b$, then

$$E[X|A] = \int_{-\infty}^{+\infty} x f_{X|A}(x) dx$$

$$E[g(X)|A] = \int_{-\infty}^{+\infty} g(x) f_{X|A}(x) dx$$

Conditioning on other random variables

Definitions

For jointly continuous random variables X and Y , we define

- conditional PDF of X given $Y = y$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

- leading

$$f_{XY}(x, y) = f_{X|Y}(x|y)f_Y(y)$$

Conditioning on other random variables

Definitions

For jointly continuous random variables X and Y , we define

- conditional probability that $X \in A$ given $Y = y$

$$P(X \in A | Y = y) = \int_A f_{X|Y}(x|y) dx$$

- conditional CDF of X given $Y = y$

$$F_{X|Y}(x|y) = P(X \leq x | Y = y) = \int_{-\infty}^x f_{X|Y}(x|y) dx$$

Conditioning on other random variables

Conditional expectation

For jointly continuous random variables X and Y , we define

- expected value of X given $Y = y$

$$E[X|Y = y] = \int_{-\infty}^{+\infty} xf_{X|Y}(x|y)dx$$

- conditional LOTUS

$$E[g(X)|Y = y] = \int_{-\infty}^{+\infty} g(x)f_{X|Y}(x|y)dx$$

Independent random variables

Properties

Two continuous random variables X and Y are independent if

$$\forall(x, y) \quad f_{XY}(x, y) = f_X(x)f_Y(y)$$

or equivalently

$$\forall(x, y) \quad F_{XY}(x, y) = F_X(x)F_Y(y).$$

If X and Y are two continuous independent random variables then

$$E[XY] = E[X]E[Y]$$

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Law of total probability

Law of total probability

For two continuous random variables X and Y and any set A , we have

$$P(X \in A) = \int_{-\infty}^{+\infty} P(X \in A | Y = y) f_Y(y) dy$$

Law of total expectation

$$E[X] = E[E[X|Y]]$$

Example: $Y \sim \mathcal{U}([1, 2])$ and $X|Y = y \sim \mathcal{E}(y)$. Find $E[X]$

Covariance matrix

Covariance

For two random variables X and Y we define their covariance as

$$\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$$

Covariance matrix

For two random variables X and Y we define their covariance as

$$\Sigma_{XY}^2 = \begin{bmatrix} \text{Var}(X) & \text{cov}(X, Y) \\ \text{cov}(X, Y) & \text{Var}(Y) \end{bmatrix}$$

For independent random variables $\text{cov}(X, Y) = 0$

Functions of two continuous random variables

LOTUS

If X and Y are two continuous random variables

$$E[g(X, Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) f_{XY}(x, y) dx dy$$

Example: $f_{XY} = \begin{cases} x + y & \text{if } 0 \leq x, y \leq 1 \\ 0 & \text{else} \end{cases}$. Find $E[XY^2]$

Functions of two continuous random variables

Method of transformations

If X and Y are two jointly continuous random variables and $(Z, W) = g(X, Y) = (g_1(X, Y), g_2(X, Y))$ with $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ continuous and invertible function with continuous partial derivatives. Let $h = g^{-1}$ such as $(X, Y) = h(Z, W) = (h_1(Z, W), h_2(Z, W))$. Then Z and W are jointly continuous with joint PDF

$$f_{ZW}(z, w) = f_{XY}(h_1(z, w), h_2(z, w))|J|$$

where $J = \det \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix} = \frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial w} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial w}$

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Gaussian vector

Definition

A vector $\mathbf{X} = (X_1, \dots, X_n)^T$ follows a Gaussian distribution in dimension n , denoted $\mathbf{X} \sim \mathcal{N}_n(\mathbf{m}, \Sigma)$ if the joint PDF is

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \Sigma^{-1}(\mathbf{x} - \mathbf{m})\right)$$

where

- $\mathbf{m} \in \mathbb{R}^n$ is the mean: $E[\mathbf{X}] = \mathbf{m}$
- $\Sigma \in \mathbb{R}^{n \times n}$ symmetric definite positive is the covariance: $\text{Cov}(\mathbf{X}) = \Sigma$

Examples:

- $n = 1$
- Σ diagonal

Exercise

Write the general pdf in dimension 2.

Let X and Y with PDF

$$f_{XY}(x, y) \propto \exp\left(-x^2 - \frac{1}{2}y^2 - xy + 4x + 6y\right)$$

What is the distribution (X, Y) ?

Solution

We have

$$\begin{aligned}f_{XY}(x, y) &\propto \exp\left(-x^2 - \frac{1}{2}y^2 - xy + 4x + 6y\right) \\ &\propto \exp\left(-\frac{1}{2}[2x^2 + y^2 + 2xy - 8x - 12y]\right)\end{aligned}$$

and in the general case

$$f_{XY}(x, y) \propto \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \Sigma^{-1}(\mathbf{x} - \mathbf{m})\right)$$

We denote

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mathbf{m} = \begin{bmatrix} m_x \\ m_y \end{bmatrix} \quad \Sigma^{-1} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

(for Σ^{-1} : Σ is symmetric, so its inverse is)

Solution

Therefore

$$\begin{aligned} f_{XY}(x, y) &\propto \exp \left(-\frac{1}{2} \begin{bmatrix} x - m_x & y - m_y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x - m_x \\ y - m_y \end{bmatrix} \right) \\ &\propto \exp \left(-\frac{1}{2} \begin{bmatrix} x - m_x & y - m_y \end{bmatrix} \begin{bmatrix} a(x - m_x) + b(y - m_y) \\ b(x - m_x) + c(y - m_y) \end{bmatrix} \right) \\ &\propto \exp \left(-\frac{1}{2} \left[(x - m_x)^2 a + (y - m_y)^2 c \right. \right. \\ &\quad \left. \left. + 2b(x - m_x)(y - m_y) \right] \right) \end{aligned}$$

Solution

We develop the previous expression and forget the terms that do not depend on x or y in the exponential as we are interested in the PDF up to a multiplicative constant

$$\begin{aligned}
 p(x, y) &\propto \exp\left(-\frac{1}{2} [x^2 a - 2m_x x a + y^2 c - 2y m_y c \right. \\
 &\quad \left. + 2bxy - 2b x m_y - 2b y m_x]\right) \\
 &\propto \exp\left(-\frac{1}{2} [a x^2 + c y + 2bxy \right. \\
 &\quad \left. - x(2m_x a + 2b m_y) - y(2m_y c + 2b m_x)]\right) \\
 &\propto \exp\left(-\frac{1}{2} [2x^2 + 1y^2 + 2xy - 8x - 12y]\right)
 \end{aligned}$$

Solution

$$\begin{aligned}
 & \left\{ \begin{array}{l} a = 2 \\ c = 1 \\ 2b = 2 \\ (2m_x a + 2b m_y) = 8 \\ (2m_y c + 2b m_x) = 12 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} a = 2 \\ c = 1 \\ b = 1 \\ 4m_x + 2m_y = 8 \\ 2m_y + 2m_x = 12 \end{array} \right. \\
 & \Leftrightarrow \left\{ \begin{array}{l} a = 2 \\ c = 1 \\ b = 1 \\ 2m_x = -4 \\ 2m_y + 2m_x = 12 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} a = 2 \\ c = 1 \\ b = 1 \\ m_x = -2 \\ m_y = 8 \end{array} \right.
 \end{aligned}$$

Solution

So

$$\mathbf{m} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

and

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \boldsymbol{\Sigma} = \frac{1}{2-1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

And we conclude

$$(x, y) \sim \mathcal{N}_2 \left(\begin{bmatrix} -2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \right)$$

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Gaussian vector

Property: affine transformation

Let $\mathbf{X} \sim \mathcal{N}_n(\mathbf{m}, \Sigma)$, $\mathbf{A} \in \mathbb{R}^{p \times n}$ ($p \leq n$) with full rank and $\mathbf{b} \in \mathbb{R}^p$, and let $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$. Then

$$\mathbf{Y} \sim \mathcal{N}_p(\mathbf{A}\mathbf{m} + \mathbf{b}, \mathbf{A}\Sigma\mathbf{A}^T)$$

Gaussian vector

Property: marginal distributions

Let $\mathbf{X} \sim \mathcal{N}_n(\mathbf{m}, \Sigma)$, such as

- $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$, $\mathbf{m} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \Sigma_1 & \mathbf{M} \\ \mathbf{M}^T & \Sigma_2 \end{bmatrix}$
- $\mathbf{X}_1, \mathbf{m}_1 \in \mathbb{R}^p$ and $\mathbf{X}_2, \mathbf{m}_2 \in \mathbb{R}^q$, $\Sigma_1 \in \mathbb{R}^{p \times p}$, $\Sigma_2 \in \mathbb{R}^{q \times q}$, $\mathbf{M} \in \mathbb{R}^{p \times q}$, and $p + q = n$

Then

$$\mathbf{X}_1 \sim \mathcal{N}_p(\mathbf{m}_1, \Sigma_1) \text{ and } \mathbf{X}_2 \sim \mathcal{N}_q(\mathbf{m}_2, \Sigma_2)$$

Gaussian vector

Property: independence

Let $\mathbf{X} \sim \mathcal{N}_n(\mathbf{m}, \Sigma)$, such as

- $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$, $\mathbf{m} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \Sigma_1 & \mathbf{M} \\ \mathbf{M}^T & \Sigma_2 \end{bmatrix}$
- $\mathbf{X}_1, \mathbf{m}_1 \in \mathbb{R}^p$ and $\mathbf{X}_2, \mathbf{m}_2 \in \mathbb{R}^q$, $\Sigma_1 \in \mathbb{R}^{p \times p}$, $\Sigma_2 \in \mathbb{R}^{q \times q}$, $\mathbf{M} \in \mathbb{R}^{p \times q}$, and $p + q = n$

Then

\mathbf{X}_1 and \mathbf{X}_2 are independent if and only if $\mathbf{M} = \mathbf{0}$

The χ^2 distribution

Definition

Let X_1, \dots, X_n be n independent random variables with distribution $\mathcal{N}(0, 1)$. Then $Y = \sum_{i=1}^n X_i^2$ has a χ^2 distribution with n degrees of freedom ($Y \sim \chi_n^2$)

Properties

- PDF: $f_Y(y) = \frac{y^{\frac{n}{2}-1} e^{-\frac{y}{2}}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})}$ if $y \geq 0$, and $f_Y(y) = 0$ else
- mean and variance: $E[Y] = n$ and $\text{Var}(Y) = 2n$
- additivity: if Y, Z independent with $Y \sim \chi_n^2$ and $Z \sim \chi_m^2$, then $Y + Z \sim \chi_{n+m}^2$

The Student distribution

Definition

Let X, Y independent random variables with distributions $X \sim \mathcal{N}(0, 1)$ and $Y \sim \chi_n^2$. Then

$$Z = \frac{X}{\sqrt{\frac{Y}{n}}} \sim t_n$$

Properties

- PDF: $f_Z(z) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} \left(1 + \frac{z^2}{n}\right)^{-\frac{n+1}{2}}$
- mean and variance: $E[Z] = 0$ and $\text{Var}(Z) = \frac{n}{n-2}$

Gaussian samples

iid: independent identically distributed

Definition

Let $\mathbf{X} = (X_1, \dots, X_n)^T$ iid with $\mathcal{N}(m, \sigma^2)$

We define the two random variables

- the sample mean: $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$
- the sample variance: $S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$

Properties

One can show that \bar{X} and S^2 are independent and

- $\bar{X}_n \sim \mathcal{N}\left(m, \frac{\sigma^2}{n}\right)$
- $\frac{n-1}{\sigma^2} S_n^2 \sim \chi_{n-1}^2$

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Convergence in law

Definition

A sequence of random variables X_1, \dots, X_n converges in law to a random variable X if and only if

$$F_n(x) \xrightarrow{n \rightarrow +\infty} F(x) \text{ at each } x \in \mathbb{R} \text{ where } F \text{ is continuous}$$

We denote

$$X_n \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} X$$

N.B., the limit X can be deterministic

Example: $P_{X_n}(1) = \frac{1}{n}$ and $P_{X_n}(0) = 1 - \frac{1}{n}$

Solution

Let $X = 0$. Then

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{else} \end{cases}$$

On the other hand

$$F_{X_n}(x) = \begin{cases} 1 - \frac{1}{n} & \text{if } x < 0 \\ 1 & \text{else} \end{cases}$$

Therefore

$$F_{X_n}(x) \xrightarrow{n \rightarrow +\infty} F_X(x)$$

Convergence in law

Property

If $X_n \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} X$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous

$$g(X_n) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} g(X)$$

Convergence in probability

Definition

A sequence of random variables X_1, \dots, X_n converges in probability to a random variable X if and only if

$$\forall \varepsilon > 0, \quad P(|X_n - X| > \varepsilon) \xrightarrow{n \rightarrow +\infty} 0$$

We denote

$$X_n \xrightarrow[n \rightarrow +\infty]{\mathcal{P}} X$$

Example: show that X_n with PDF $f_{X_n}(x) = \frac{ne^{-nx}}{(1+e^{-nx})^2}$ for $x \in \mathbb{R}$ converges in probability to 0

Solution

We have

$$\begin{aligned}
 P(|X_n| > \varepsilon) &= 1 - P(|X_n| \leq \varepsilon) \\
 &= 1 - \int_{-\varepsilon}^{\varepsilon} f_{X_n}(t) dt \\
 &= 1 - \int_{-\varepsilon}^{\varepsilon} \frac{ne^{-nt}}{(1 + e^{-nt})^2} dt \\
 &= 1 - \left[\frac{1}{1 + e^{-nt}} \right]_{t=-\varepsilon}^{\varepsilon} \\
 &= 1 - \frac{1}{1 + e^{-n\varepsilon}} + \frac{1}{1 + e^{n\varepsilon}} \\
 &\xrightarrow[n \rightarrow +\infty]{} 0 \\
 X_n &\xrightarrow[n \rightarrow +\infty]{\mathcal{P}} 0
 \end{aligned}$$

Convergence in probability

Property

If $X_n \xrightarrow[n \rightarrow +\infty]{\mathcal{P}} X$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous

$$g(X_n) \xrightarrow[n \rightarrow +\infty]{\mathcal{P}} g(X)$$

Convergence in mean square

Definition

A sequence of random variables X_1, \dots, X_n converges in mean square to a random variable X if and only if

$$E[(X_n - X)^2] \xrightarrow{n \rightarrow +\infty} 0$$

We denote

$$X_n \xrightarrow[n \rightarrow +\infty]{\text{m.s.}} X$$

Example: $P_{X_n}(n) = \frac{1}{n^p}$ and $P_{X_n}(0) = 1 - \frac{1}{n^p}$ with $p = 2$ and $p = 3$

Solution

$$\begin{aligned} E[X_n^2] &= \sum_{k \in R_{X_n}} k^2 P_{X_n}(k) \\ &= n^2 \frac{1}{n^p} \\ &= \frac{1}{n^{p-2}} \end{aligned}$$

So

- $p = 2 \Rightarrow E[X_n^2] = 1$ so X_n does not converge in mean square to 0
- $p = 3 \Rightarrow E[X_n^2] = \frac{1}{n} \xrightarrow{n \rightarrow +\infty} 0$ so $X_n \xrightarrow[n \rightarrow +\infty]{\text{m.s.}} 0$

Almost sure convergence

Definition

A sequence of random variables X_1, \dots, X_n converges almost surely to a random variable X if and only if

$$\forall \omega \in \Omega, \quad X_n(\omega) \xrightarrow[n \rightarrow +\infty]{} X(\omega)$$

We denote

$$X_n \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} X$$

Relation

$$\left. \begin{array}{l} \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \\ \xrightarrow[n \rightarrow +\infty]{\text{m.s.}} \end{array} \right\} \Rightarrow \xrightarrow[n \rightarrow +\infty]{\mathcal{P}} \Rightarrow \xrightarrow[n \rightarrow +\infty]{\mathcal{L}}$$

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The weak law of large numbers

Definition

Let X_1, \dots, X_n iid. The sample mean is defined as

$$\bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k$$

Law

Let X_1, \dots, X_n iid random variables with $E[X_k] = m < \infty$. Then

$$\bar{X}_n \xrightarrow[n \rightarrow +\infty]{\mathcal{P}} m \left(\Leftrightarrow P(|\bar{X}_n - m| \geq \varepsilon) \xrightarrow[n \rightarrow +\infty]{} 0 \right)$$

The strong law of large numbers

Law

Let X_1, \dots, X_n iid random variables with $E[X_k] = m < \infty$ and $\text{Var}(X_k) = \sigma^2 < \infty$. Then

$$\bar{X}_n \xrightarrow[n \rightarrow +\infty]{\text{m.s.}} m \left(\Leftrightarrow E \left[(\bar{X}_n - m)^2 \right] \xrightarrow[n \rightarrow +\infty]{} 0 \right)$$

The central limit theorem

Theorem (CLT)

Let X_1, \dots, X_n iid random variables with $E[X_k] = m < \infty$ and $\text{Var}(X_k) = \sigma^2 < \infty$. Then

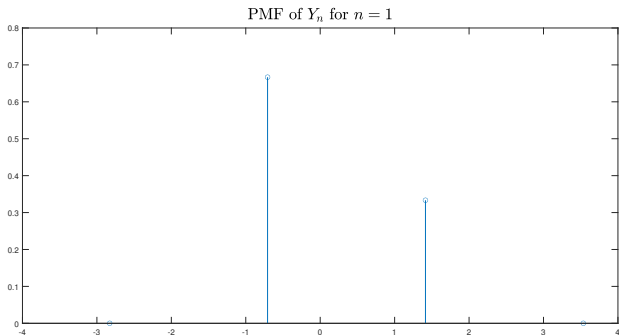
$$Y_n = \frac{\bar{X}_n - m}{\sqrt{\frac{\sigma^2}{n}}} = \frac{\sum_{k=1}^n X_k - nm}{\sqrt{n\sigma^2}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 1)$$

Example

Let X_k iid with *Bernoulli* ($\frac{1}{3}$). Then $E[X_i] = p$ and $\text{Var}(X_i) = p(1 - p)$.

The CLT ensures

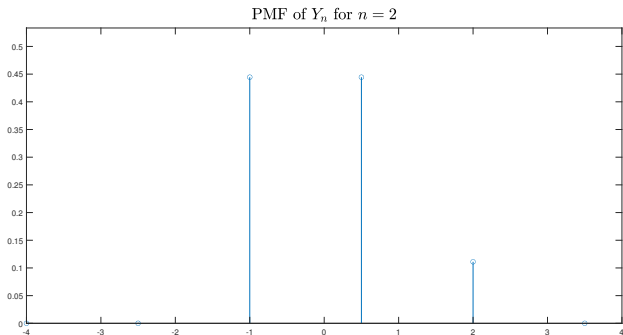
$$Y_n = \frac{\sum_{k=1}^n X_k - np}{\sqrt{np(1-p)}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 1)$$



Example

Let X_k iid with *Bernoulli* ($\frac{1}{3}$). Then $E[X_i] = p$ and $\text{Var}(X_i) = p(1 - p)$.
The CLT ensures

$$Y_n = \frac{\sum_{k=1}^n X_k - np}{\sqrt{np(1-p)}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 1)$$

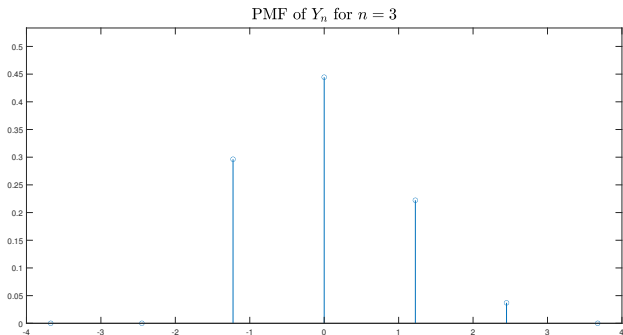


Example

Let X_k iid with *Bernoulli* ($\frac{1}{3}$). Then $E[X_i] = p$ and $\text{Var}(X_i) = p(1 - p)$.

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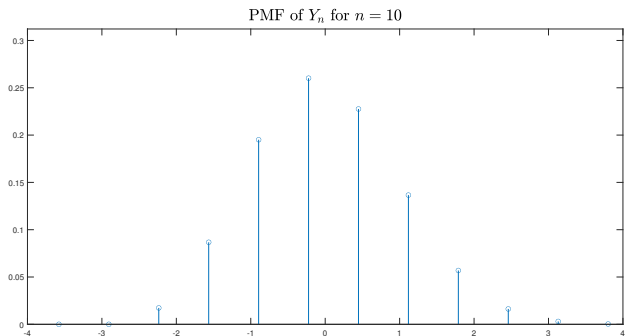


Example

Let X_k iid with *Bernoulli* ($\frac{1}{3}$). Then $E[X_i] = p$ and $\text{Var}(X_i) = p(1 - p)$.

The CLT ensures

$$Y_n = \frac{\sum_{k=1}^n X_k - np}{\sqrt{np(1-p)}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 1)$$

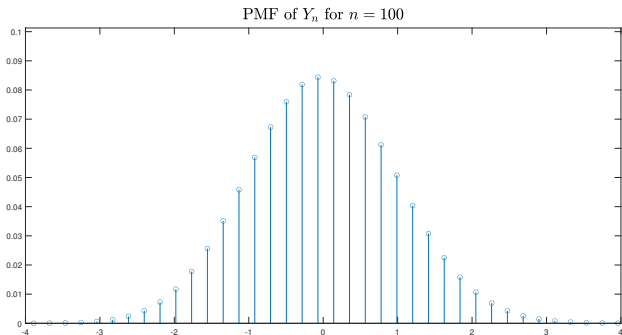


Example

Let X_k iid with *Bernoulli* ($\frac{1}{3}$). Then $E[X_i] = p$ and $\text{Var}(X_i) = p(1 - p)$.

The CLT ensures

$$Y_n = \frac{\sum_{k=1}^n X_k - np}{\sqrt{np(1-p)}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 1)$$



Example

Let X_k iid with *Bernoulli* ($\frac{1}{3}$). Then $E[X_i] = p$ and $\text{Var}(X_i) = p(1 - p)$.
The CLT ensures

$$Y_n = \frac{\sum_{k=1}^n X_k - np}{\sqrt{np(1-p)}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 1)$$

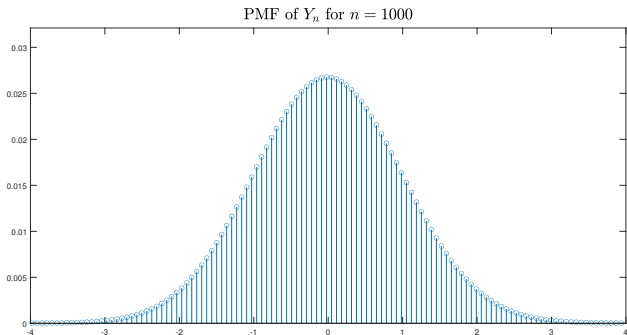


Table of discrete distributions

| Name | Parameters | Range | PMF | Mean | Variance |
|-----------|----------------------------------|-------------------|--|-----------------|--------------------|
| Uniform | $n \in \mathbb{N}^*$ | $\{1, \dots, n\}$ | $P_X(x) = \frac{1}{n}$ | $\frac{n+1}{2}$ | $\frac{n^2-1}{12}$ |
| Bernoulli | $p \in (0, 1)$ | $\{0, 1\}$ | $P_X(x) = p^x(1-p)^{1-x}$ | p | $p(1-p)$ |
| Geometric | $p \in (0, 1)$ | \mathbb{N}^* | $P_X(x) = p(1-p)^{x-1}$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ |
| Binomial | $p \in (0, 1), n \in \mathbb{N}$ | $\{0, \dots, n\}$ | $P_X(x) = \binom{n}{x} p^x(1-p)^{n-x}$ | np | $np(1-p)$ |
| Poisson | $\lambda > 0$ | \mathbb{N} | $P_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}$ | λ | λ |

Notes:

- the PMF is given for x inside the range. Outside, $P_X(x) = 0$
- $\binom{n}{x} = \frac{n!}{x!(n-x)!}$

Table of continuous distributions

| Name | Parameters | Range | PDF | Mean | Variance |
|---------------------|---|----------------|---|----------------------|------------------------|
| Uniform | (a, b) $a < b$ | (a, b) | $f_X(x) = \frac{1}{b-a}$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ |
| Gamma | $\nu > 0$ $\theta > 0$ | \mathbb{R}^+ | $f_X(x) = \frac{\theta^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\theta x}$ | $\frac{\nu}{\theta}$ | $\frac{\nu}{\theta^2}$ |
| Univariate Normal | $m \in \mathbb{R}$ $\sigma^2 > 0$ | \mathbb{R} | $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$ | m | σ^2 |
| Multivariate Normal | $\mathbf{m} \in \mathbb{R}^p$ $\Sigma \in \mathbb{R}^{p \times p}$ | \mathbb{R}^p | $f_{\mathbf{X}}(\mathbf{x}) = K e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \Sigma^{-1}(\mathbf{x}-\mathbf{m})}$ $K = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}}$ | \mathbf{m} | Σ |
| Chi-square | $\nu \in \mathbb{N}^*$ | \mathbb{R}^+ | $f_X(x) = \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}$ | ν | 2ν |

Notes:

- the exponential distribution with parameter θ is equivalent to a Gamma distribution with $\nu = 1$ and θ
- the PDF is given for x inside the range. Outside, $f_X(x) = 0$
- $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ and $\forall n \in \mathbb{N}, \Gamma(n+1) = n!$
- for multivariate, Σ must be symmetric definite positive