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Deriving a dilution of precision indicator for GNSS factor graph optimization solutions[†]

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Abstract: Dilution of Precision (DOP) is routinely used in GNSS to assess the quality of the constellation geometry for the positioning algorithm. Those DOP factors are computed from the estimation covariance of a snapshot weighted least squares (WLS) estimate under certain hypotheses. This paper proposes to define DOP factors for GNSS solutions based on Factor Graph Optimization (FGO). FGO solutions have become popular in the GNSS domain. They allow to easily model probabilistic constraints, called factors, over a large time window, by mixing observations and motion constraints across consecutive epochs. The solution is solved by performing a batch WLS estimation for the states at all considered epochs, using all available factors. Due to the simple nature of the estimation algorithm – a WLS solution – it is possible to derive the theoretical estimation error covariance, which will indicate the accuracy of the computed solution. In this paper, a formula is proposed to approximate the DOP for the FGO solution. Then, the formula is validated in various scenarios involving fixed or changing satellite visibility.

Keywords: GNSS; Factor Graph Optimization; Precise Positioning

1. Introduction

The positioning algorithms implemented in GNSS receivers uses estimation techniques to combine GNSS observations, such as code pseudo-ranges or doppler offset measurements, to estimate the unknown position, velocity and clock bias of a GNSS receiver. Several algorithms are commonly used, such as the Weighted Least Squares estimation and the Kalman Filter [1],[2]. The Weighted Least Squares (WLS) estimation applied with code pseudo-ranges is called the Single Point Positioning (SPP) solution, and is the most basic solution provided by a GNSS receiver. Kalman filter-based techniques add a state transition model, which corresponds to a motion model for the position/velocity/clock states, in order to refine the estimate [3]. It can also be applied with phase pseudo-range observations, leading the well-known Precise Point Positioning [4].

Those algorithms provide both the estimate and the covariance matrix of the state vector. The covariance matrix gives an indication of the uncertainty of the estimate, and its validity depends on the matching of the assumed models with the real conditions of the data collect. Among the usual quality indicators, the DOP (Dilution of Precision) factors are often encountered [1]. The DOP factors are derived from the estimation covariance of the WLS solution, assuming that all code pseudo-ranges are affected by a Gaussian error that has the same variance and is independent from other pseudo-ranges. Given those assumptions, the DOP factors only depend on the relative satellite positions with regards to the receiver. They quantify the impact of the satellite constellation geometry on the uncertainty of the WLS solution. In the context of this paper, satellite constellation geometry refers both to the satellite position relative to the receiver, but also to the possible masking of satellites by surrounding obstacles. DOP is used for assessment of solution quality, filtering bad position estimates or planning data collections.

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More recently, Factor Graph Optimization (FGO) has been applied to the GNSS positioning problem with great success [5],[6]. FGO performs a batch WLS estimation over a large time window, as opposed to a snapshot algorithm as used in the SPP solution. In order to benefit from the optimization over a large time window, motion factors are introduced in addition to GNSS observations to provide constraints between states in adjacent epochs. Those motion factors are essentially similar to the state transition model in a Kalman filter. Therefore, all observations contribute to the accuracy of the solution at each epoch, through the correlation of the estimated states across epochs introduced by the motion factors.

While the impact of the constellation geometry on the solution accuracy is well described for snapshot WLS solutions, such as the SPP solution, it is more difficult to determine how the constellation geometry will impact the solution accuracy in an FGO solution, due to the intricate interactions between the GNSS observations at each epochs and the propagation of the accuracy through the motion factors.

This paper aims at deriving a DOP-like factor for FGO solutions including a generic motion factor, that will provide a quality assessment of a FGO solution at each epoch, based on the constellation geometry and motion factor uncertainty only.

2. Materials and Methods

2.1. DOP in a snapshot least squares solution

The considered positioning algorithm is the SPP algorithm, consisting in the estimation of the 3D ENU position and the receiver clock bias from GNSS code observations at a single epoch k . j is used to index the scalar states within the state vector: $\mathbf{x}_k = [x_{j,k}]_{j \in [1,4]}$

The linearized observation model around an approximate position is:

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{n}_k, \quad \mathbf{n}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k) \quad (1)$$

where \mathbf{y}_k is observation vector comprising all GNSS code observations at epoch k , \mathbf{x}_k is the unknown state vector, \mathbf{H}_k is the Jacobian matrix of the code observation model and \mathbf{n}_k is the observation error vector, assumed normally-distributed, centered with a covariance \mathbf{R}_k .

The WLS estimate of \mathbf{x}_k [7], referred to as the *snapshot solution* is given by Eq. (2) and its covariance matrix is given by Eq. (3):

$$\hat{\mathbf{x}}_k^{(snap)} = (\mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k)^{-1} \mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{y}_k \quad (2)$$

$$\mathbf{P}_k^{(snap)} = (\mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k)^{-1} \quad (3)$$

Assuming that the observation error is independent and identically-distributed across all available observations, i.e. $\mathbf{R}_k = \sigma_{obs}^2 \mathbf{I}$, we obtain:

$$\mathbf{P}_k^{(snap)} = \sigma_{obs}^2 (\mathbf{H}_k^T \mathbf{H}_k)^{-1} \quad (4)$$

where σ_{obs} is the standard deviation of the GNSS code observations.

This expression separates the contribution of the measurement noise from the geometry of the constellation. We note $\mathbf{D}_k = (\mathbf{H}_k^T \mathbf{H}_k)^{-1}$ the DOP matrix. The different DOP factors are computed from \mathbf{D}_k . For example, assuming that the 3D position is placed on the first 3 elements of the state vector \mathbf{x}_k :

- Global DOP: $GDOP = \text{Trace}(\mathbf{D}_k)$
- Position DOP: $PDOP = (\mathbf{D}_k)_{1,1} + (\mathbf{D}_k)_{2,2} + (\mathbf{D}_k)_{3,3}$

where $(\mathbf{A})_{i,j}$ refers to the element at row i and column j of a matrix \mathbf{A} .

2.2. Factor Graph model

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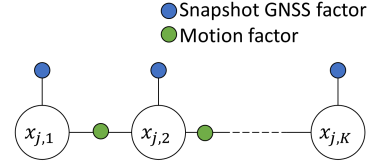


Figure 1. Factor graph considered in this study.

In this study, a simple factor graph (Figure 1) is considered for one *scalar* state x_j at different epochs indexed by $k \in \llbracket 1, K \rrbracket$. $x_{j,k}$ refers to one of the scalar state at epoch k , such as the position along one axis or the receiver clock bias.

We assume that the snapshot LS algorithm provides a solution for each of the K epochs of a trajectory. From each snapshot solution, the snapshot estimation accuracy $\sigma_{j,k}^2 = (\mathbf{P}_k^{(snap)})_{j,j}$ is used as a factor to constrain the scalar state x_j at epoch k and is referred to as *snapshot GNSS factor*.

$$y_{j,k} = x_{j,k} + n_{j,k}, \quad n_{j,k} \sim \mathcal{N}(0, \sigma_{j,k}^2) \quad (5)$$

An additional factor is introduced to model the probabilistic constraint linking the scalar states at two consecutive epochs. It assumes that the evolution of the scalar state between 2 epochs is known, with an uncertainty assumed to be normally-distributed with a standard deviation q . This constraint is referred to as a *motion factor* in this document. This generic motion factor uses the following affine model:

$$m_{k,k+1} = x_{j,k+1} - (x_{j,k} + b_{k+1}) + n_{m_{k,k+1}}, \quad n_{m_{k,k+1}} \sim \mathcal{N}(0, q^2) \quad (6)$$

where $n_{m_{k,k+1}}$ accounts for the uncertainty of the motion constraint, assumed to be a centered normal random variable with a constant standard deviation q and b_{k+1} is a known parameter, e.g. an integrated velocity coming from another sensor.

Examples of motion constraints are the estimation of the receiver displacement by GNSS Doppler or Time Difference of Carrier Phase measurements [6], estimation of the receiver displacement by an external sensor such as an inertial measurement unit or a visual simultaneous localization and mapping solution [2], or simply a dynamic model, such as a random walk or constant velocity model with Gaussian uncertainty [3].

Finally, it is assumed that the errors of the snapshot GNSS factors and those of the motion factors are independent between each other and across epochs. Therefore, when stacking all the factors in a single vector \mathbf{y} , we obtain the following estimation problem:

$$\mathbf{y} = \begin{bmatrix} y_{j,1} \\ y_{j,2} \\ \vdots \\ y_{j,K} \\ m_{1,2} \\ m_{2,3} \\ \vdots \\ m_{K-1,K} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_{j,1} \\ x_{j,2} \\ \vdots \\ x_{j,K} \end{bmatrix} + \begin{bmatrix} n_{j,1} \\ n_{j,2} \\ \vdots \\ n_{j,K} \\ n_{m_{1,2}} \\ n_{m_{2,3}} \\ \vdots \\ n_{m_{K-1,K}} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \mathbf{I}_K \\ \mathbf{H}_m \end{bmatrix}}_{\mathbf{H}} \mathbf{x}_{j,1:K} + \underbrace{\begin{bmatrix} \mathbf{n}_{j,1:K} \\ \mathbf{n}_{m,1:K} \end{bmatrix}}_{\mathbf{n}_j}, \quad \mathbf{n}_j \sim \mathcal{N} \left(\mathbf{0}_{2K-1}, \begin{bmatrix} \text{diag}(\sigma_{j,1}^2, \dots, \sigma_{j,K}^2) & \mathbf{0}_{K \times K-1} \\ \mathbf{0}_{K-1 \times K} & q^2 \mathbf{I}_{K-1} \end{bmatrix} \right) \quad (7)$$

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2.3. Batch least squares solution of the considered Factor Graph

When ignoring the motion constraints in Eq. (7), we find the trivial solution which is called the **snapshot solution**. For this particular solution, the snapshot covariance matrix for state x_j across all epochs from 1 to K is:

$$\mathbf{P}_j^{(snap)} = \mathbf{diag}(\sigma_{j,1}^2, \dots, \sigma_{j,K}^2) \quad (8)$$

The solution of the factor graph mixing both snapshot GNSS factors and motion factors is referred to as the **batch solution**. The batch solution covariance matrix is obtained with the following equation.

$$\begin{aligned} \mathbf{P}_j^{(batch)} &= (\mathbf{H}^T \mathbf{Cov}(\mathbf{n}_j)^{-1} \mathbf{H})^{-1} = \left(\begin{bmatrix} \mathbf{I}_K & \mathbf{H}_m^T \end{bmatrix} \begin{bmatrix} \mathbf{P}_j^{(snap)} & \mathbf{0}_{K \times K-1} \\ \mathbf{0}_{K-1 \times K} & q^2 \mathbf{I}_{K-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I}_K \\ \mathbf{H}_m \end{bmatrix} \right)^{-1} \\ &= \left((\mathbf{P}_j^{(snap)})^{-1} + \frac{1}{q^2} \mathbf{H}_m^T \mathbf{H}_m \right)^{-1} \end{aligned} \quad (9)$$

The inverse of the batch solution covariance is a so-called a tri-diagonal matrix [8].

$$\mathbf{P}_j^{(batch)} = \begin{bmatrix} \frac{1}{q^2} + \frac{1}{\sigma_{j,1}^2} & -\frac{1}{q^2} & 0 & \dots & 0 \\ -\frac{1}{q^2} & \frac{2}{q^2} + \frac{1}{\sigma_{j,2}^2} & -\frac{1}{q^2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\frac{1}{q^2} & \frac{2}{q^2} + \frac{1}{\sigma_{j,K-1}^2} & -\frac{1}{q^2} \\ 0 & \dots & 0 & -\frac{1}{q^2} & \frac{1}{q^2} + \frac{1}{\sigma_{j,K}^2} \end{bmatrix}^{-1} \quad (10)$$

2.4. Efficient computation of the batch estimation covariance

To compute efficiently the batch solution covariance matrix, one can use the particular shape of (10). Additionally, we are interested only in the diagonal elements of the matrix, to characterize the uncertainty of the considered scalar state estimate at each epoch k . The symbolic expression of the diagonal elements for the first few values of K is shown in Annex A. It is observed that the k -th diagonal element is a ratio of 2 expressions:

$$(\mathbf{P}_j^{(batch)})_{k,k} = \frac{N_{j,k}(q, \sigma_{j,1}, \dots, \sigma_{j,K})}{D_j(q, \sigma_{j,1}, \dots, \sigma_{j,K})} \quad (11)$$

where $N_{j,k}$ and D_j are expressions depending on $q, \sigma_{j,1}, \dots, \sigma_{j,K}$

The denominator D_j is common to all epochs of the considered trajectory. It can be computed using a second-order linear recurrence relation:

$$\begin{aligned} p_0 &= \frac{1}{q^2}, \quad p_1 = 1 + \frac{\sigma_{j,1}^2}{q^2} \\ p_k &= (q^2 + 2\sigma_{j,k}^2)p_{k-1} - \sigma_{j,k}^2 \sigma_{j,k-1}^2 p_{k-2}, \quad \text{for } 2 \leq k < K \\ D_j &= p_K = (q^2 + \sigma_{j,K}^2)p_{K-1} - \sigma_{j,K}^2 \sigma_{j,K-1}^2 p_{K-2} \end{aligned} \quad (12)$$

The numerator $N_{j,k}$ depends on the considered epochs within the trajectory and can be computed using the following expression:

$$N_{j,k} = D_j(\sigma_{j,k} = 0) \sigma_{j,k}^2 \quad (13)$$

where $D_j(\sigma_{j,k} = 0)$ is the evaluation of expression D_j at $\sigma_{j,k} = 0$ and leaving other parameters untouched.

The expressions of $N_{j,k}$ and D_j have been validated by comparing the diagonal elements of Eq. (10) to the values computed using Eq. (11), using a symbolic mathematical software up to $K = 10$, and numerical simulations up to $K = 100$. Elements of the mathematical demonstration to obtain those expressions are available in Annex B.

2.5. DOP computation for a Factor Graph Optimization solution

By definition, the DOP factor is defined as the ratio between the uncertainty of the combined estimated states and the uncertainty of the GNSS observations, assuming that all GNSS observations are independent and affected by an error with the same variance σ_{obs}^2 :

$$DOP_k = \frac{\sqrt{\sum_{j \in \mathcal{S}} \text{var}(\hat{x}_{j,k})}}{\sigma_{obs}} \quad (14)$$

where $\text{var}(\hat{x}_{j,k})$ is the estimation variance of state $x_{j,k}$, and \mathcal{S} is the set of indexes of the scalar states of interest. For example, if the state vector at a particular epoch is composed of $[east, north, up, clock_bias]$, to compute the horizontal DOP, one has to combine the east and north position uncertainty and $\mathcal{S} = [1, 2]$.

The previous section provided the estimation uncertainty computation for both snapshot and batch solutions, for a scalar element of the state vector. The computation can then be performed for the scalar states of interest before combining them into a DOP factor.

We define the DOP factors coming from the snapshot solution, noted $DOP_k^{(snap)}$, and the one coming from the batch solution including a motion factor, noted $DOP_k^{(batch)}$

$$DOP_k^{(snap)} = \frac{\sqrt{\sum_{j \in \mathcal{S}} (\mathbf{P}_k^{(snap)})_{j,j}}}{\sigma_{obs}} = \sqrt{\sum_{j \in \mathcal{S}} (\mathbf{H}_k^T \mathbf{H}_k)_{j,j}} \quad (15)$$

$$DOP_k^{(batch)} = \frac{\sqrt{\sum_{j \in \mathcal{S}} (\mathbf{P}_k^{(batch)})_{j,j}}}{\sigma_{obs}} \approx \frac{\sqrt{\sum_{j \in \mathcal{S}} \frac{N_{j,k}(q, \sigma_1, \dots, \sigma_K)}{D_j(q, \sigma_1, \dots, \sigma_K)}}}{\sigma_{obs}} \quad (16)$$

Note that while the snapshot DOP does not depend on σ_{obs} , the batch DOP depends on the ratio between the motion uncertainty and the observation uncertainty q/σ_{obs} .

3. Results

3.1. Validation method of the proposed batch DOP formula

The formula, Eq. (11), to obtain uncertainty for a batch solution from the snapshot scalar uncertainty has been validated for a single scalar state.

However, when combining the uncertainty of several scalar states to compute a DOP factor, the correlated impact of an observation error on the different states should be accounted. This is not the case for the batch DOP computed using Eq. (16), since the diagonal elements of the covariance matrix of the batch solution are approximated by considering each scalar state uncertainty independently from the other states.

The following subsections aim at quantifying the impact of this approximation. To do so, the batch DOP obtained from Eq. (16) and the exact batch DOP, obtained by computing the covariance of the full state vectors with Eq. (10), are compared based on the difference of HDOP and the ratio of HDOP. The results are summarized in Table 1.

Table 1. HDOP difference and ratio between proposed formula and exact computation.

Scenario	Factor graph length K	Motion to observation uncertainty ratio q/σ	Max HDOP difference	Max HDOP ratio
fixed geometry	[10, 50]	[0.01, 0.1, 0.5, 1]	0.008	1 %
varying geometry	50	[0.01, 0.1, 0.5, 1]	0.02	3 %
urban scenario	1009	[0.01, 0.1, 0.5]	0.15	22 %

3.2. Validation the FGO DOP formula in a fixed constellation case

When considering several consecutive epochs at a high rate (typically higher than 0.1 Hz), the constellation geometry can be considered as fixed over a window of a few tens of epochs. In the case of an open sky receiver, the snapshot DOP will typically remain constant, except when one satellite appear or disappear in the antenna's field of view.

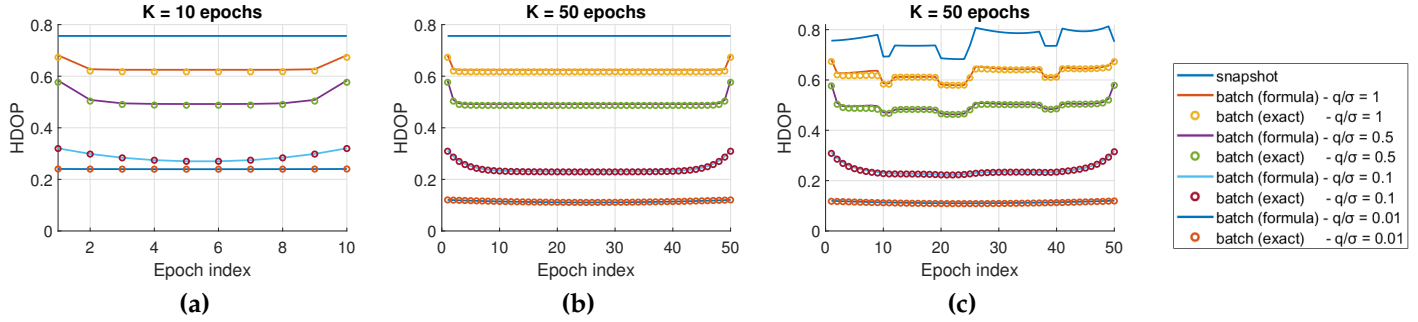


Figure 2. HDOP comparison for fixed (a),(b) and time-varying (c) constellation geometry

Figure 2 (a),(b) shows the batch HDOP for all epochs for a mix of factor graph length K and motion to observation uncertainty ratio q/σ . The HDOP value computed with the formula is close to the exact HDOP value, especially for low values of q/σ .

On the general shape of the batch DOP, we can observe that the DOP is lower in the middle of the graph. This is due to the fact that all surrounding epochs contribute to the accuracy at a particular epoch. Secondly, the overall accuracy depends on the value of q/σ : the lower q/σ is, the better the accuracy is. Finally, for larger q/σ , the DOP usually reaches a floor value for the central part of the trajectory that is common for different values of K . This points towards the fact that increasing the factor graph length may not result in better accuracy once the floor has been reached.

3.3. Validation of the FGO DOP formula in a varying constellation case

In this section, the nominal GPS constellation from IS-GPS-200M is considered. The scenario considers a fixed receiver in Toulouse on the day 2023-01-01 with a 5-min interval over the first hours of the day. This scenario allows to have varying and realistic constellation geometries, leading to temporal variations of the snapshot DOP.

Figure 2 (c) shows the exact HDOP and the HDOP computed using equation (16) for a fixed value of $K = 50$ and various values of q/σ . Again, the HDOP values obtained with the formula are close to the exact values.

An interesting observation can be made. For high accuracy motion constraints (e.g. $q/\sigma \leq 0.1$), variations of HDOP are attenuated and the batch HDOP reaches a floor value for central epochs. In this particular cases, the accurate motion constraints allow to have a good relative accuracy between consecutive position estimates, and the observations from every epoch to contribute to the absolute accuracy of the whole trajectory.

3.4. Validation of the FGO DOP formula in constrained urban case

In this section, we study the case where the constellation geometry is changing in time due to the masking by building as a GNSS receiver is moving across a city. The considered trajectory comes from the Google Smartphone Decimeter Challenge 2021 [9], and has been chosen as one that passes through an urban center. Figure 3 shows the 2D trajectory. Frequent satellite masking occur leading to frequent snapshot HDOP degradation.

Figure 4 shows the exact HDOP and the HDOP computed using Eq. (16) for $K = 1009$, i.e. all available epochs at 0.5 Hz, and various values of q/σ . Again, the HDOP values obtained with the formula are close to the exact values. The largest difference corresponds to epochs where fewer satellites are available. At those epochs, sparse GNSS observations results in larger correlation between the east and north position estimation errors, and therefore, larger approximation error of the proposed formula.

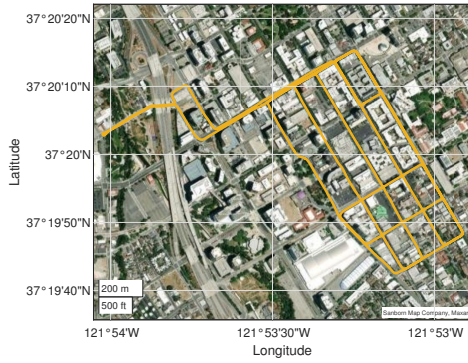


Figure 3. Trajectory "2021-04-28-US-SJC-1/Pixel4" from the GSDC2021 dataset.

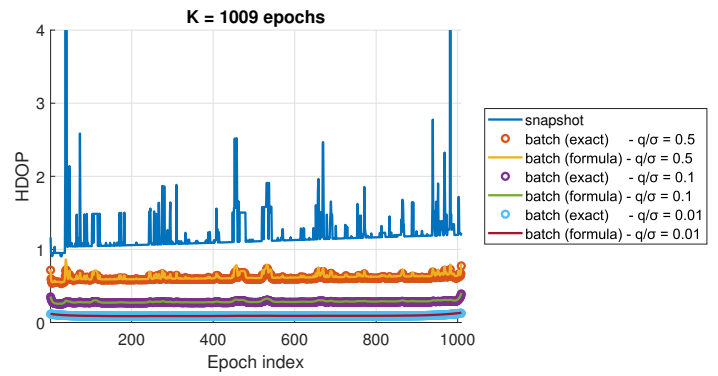


Figure 4. HDOP comparison for urban masking scenario

The FGO solution improves the accuracy of the solution, even in the epochs without enough satellites to compute a solution. For those epochs, the snapshot uncertainty has been clipped to a value of 100 m for each position state. Again, the temporal DOP variations are attenuated in the case of highly accurate motion uncertainties.

4. Discussion

This paper proposes a definition of the notion of DOP for batch solution including a generic motion factor, applicable to FGO solutions. The obtained DOP factor mainly depends on the snapshot DOP, the FG length K and the motion to observation uncertainty ratio q/σ . Depending on q/σ , the batch DOP reaches a floor value for large FG length, meaning that increasing the FG length provides low marginal gains in terms of accuracy.

A formula is derived for fast computation of batch DOP factors. The approximation error comes from ignoring the correlated impact of observation errors on multiple states when computing the scalar uncertainty for each state. The error on HDOP is of the order of a few percents in open sky conditions, and rises to 22% in constrained conditions, when a lower number of satellites is available.

Based on this formula, a future work could try to derive an optimal factor graph length depending on the satellite visibility conditions of consecutive epochs, in order to limit the computational complexity of large factor graph solution.

The formula could also be improved by considering partial observability of the state vector, where the snapshot solution cannot be computed (e.g. when less than 4 satellites are available). Solutions to consider the covariance between the scalar state of the snapshot solution could also be investigated in order to reduce the approximation error.

Finally, the proposed formula could be applied to other types of states, such as the float carrier ambiguities. This could provide insight to optimize factor graph length of a float solution, before trying to fix the carrier ambiguities.

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Appendix A. Symbolic expression of the first few diagonal elements of $\mathbf{P}_j^{(batch)}$

In the annexes, a simplified notation will be used by dropping the index j referring to a particular scalar state in the state vector: $\mathbf{P}^{(batch)} \triangleq \mathbf{P}_j^{(batch)}$ and $\sigma_k \triangleq \sigma_{j,k}$.

It can be observed that the k -th diagonal element of $P^{(batch)}$ is a ratio of 2 expressions: 227

$$(\mathbf{P}^{(batch)})_{k,k} = \frac{N_k(q, \sigma_1, \dots, \sigma_K)}{D(q, \sigma_1, \dots, \sigma_K)}$$

where N_k and D are expressions depending on $q, \sigma_1, \dots, \sigma_K$ 228

The first few diagonal elements of $P^{(batch)}$ are given in Table A1. 229

Table A1. Symbolic expression of the diagonal elements of $\mathbf{P}_j^{(batch)}$ for the first values of K .

Factor graph length	Epoch index	Numerator N_k	Denominator D
$K = 3$	$k = 1$	$\sigma_1^2(q^4 + q^2(2\sigma_2^2 + \sigma_3^2) + \sigma_2^2\sigma_3^2)$	$q^4 + q^2(\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2) + \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2$
$K = 3$	$k = 2$	$\sigma_2^2(q^4 + q^2(\sigma_1^2 + \sigma_3^2) + \sigma_1^2\sigma_3^2)$	
$K = 4$	$k = 1$	$\sigma_1^2(q^6 + q^4(2\sigma_2^2 + 2\sigma_3^2 + \sigma_4^2) + q^2(3\sigma_2^2\sigma_3^2 + 2\sigma_2^2\sigma_4^2 + \sigma_3^2\sigma_4^2) + \sigma_2^2\sigma_3^2\sigma_4^2)$	$q^6 + q^4(\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 + \sigma_4^2) + q^2(\sigma_1^2\sigma_2^2 + 2\sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_4^2 + 3\sigma_2^2\sigma_3^2 + 2\sigma_2^2\sigma_4^2 + \sigma_3^2\sigma_4^2) + \sigma_1^2\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_2^2\sigma_4^2 + \sigma_1^2\sigma_3^2\sigma_4^2 + \sigma_2^2\sigma_3^2\sigma_4^2$
$K = 4$	$k = 2$	$\sigma_2^2(q^6 + q^4(\sigma_1^2 + 2\sigma_3^2 + \sigma_4^2) + q^2(2\sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_4^2 + \sigma_3^2\sigma_4^2) + \sigma_1^2\sigma_3^2\sigma_4^2)$	
$K = 5$	$k = 1$	$\sigma_1^2(q^8 + q^6(2\sigma_2^2 + 2\sigma_3^2 + 2\sigma_4^2 + \sigma_5^2) + q^4(3\sigma_2^2\sigma_3^2 + 4\sigma_2^2\sigma_4^2 + 2\sigma_2^2\sigma_5^2 + 3\sigma_3^2\sigma_4^2 + 2\sigma_3^2\sigma_5^2 + \sigma_4^2\sigma_5^2) + q^2(4\sigma_2^2\sigma_3^2\sigma_4^2 + 3\sigma_2^2\sigma_3^2\sigma_5^2 + 2\sigma_2^2\sigma_4^2\sigma_5^2 + \sigma_3^2\sigma_4^2\sigma_5^2) + \sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2)$	$q^8 + q^6(\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 + 2\sigma_4^2 + \sigma_5^2) + q^4(\sigma_1^2\sigma_2^2 + 2\sigma_1^2\sigma_3^2 + 2\sigma_1^2\sigma_4^2 + \sigma_1^2\sigma_5^2 + 3\sigma_2^2\sigma_3^2 + 4\sigma_2^2\sigma_4^2 + 2\sigma_2^2\sigma_5^2 + 3\sigma_3^2\sigma_4^2 + 2\sigma_3^2\sigma_5^2 + \sigma_4^2\sigma_5^2) + q^2(\sigma_1^2\sigma_2^2\sigma_3^2 + 2\sigma_1^2\sigma_2^2\sigma_4^2 + \sigma_1^2\sigma_2^2\sigma_5^2 + 3\sigma_1^2\sigma_3^2\sigma_4^2 + 2\sigma_1^2\sigma_3^2\sigma_5^2 + \sigma_1^2\sigma_4^2\sigma_5^2 + 4\sigma_2^2\sigma_3^2\sigma_4^2 + 3\sigma_2^2\sigma_3^2\sigma_5^2 + 2\sigma_2^2\sigma_4^2\sigma_5^2 + \sigma_3^2\sigma_4^2\sigma_5^2) + \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 + \sigma_1^2\sigma_2^2\sigma_3^2\sigma_5^2 + \sigma_1^2\sigma_2^2\sigma_4^2\sigma_5^2 + \sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2$
$K = 5$	$k = 2$	$\sigma_2^2(q^8 + q^6(\sigma_1^2 + 2\sigma_3^2 + 2\sigma_4^2 + \sigma_5^2) + q^4(2\sigma_1^2\sigma_3^2 + 2\sigma_1^2\sigma_4^2 + \sigma_1^2\sigma_5^2 + 3\sigma_3^2\sigma_4^2 + 2\sigma_3^2\sigma_5^2 + \sigma_4^2\sigma_5^2) + q^2(3\sigma_1^2\sigma_3^2\sigma_4^2 + 2\sigma_1^2\sigma_3^2\sigma_5^2 + \sigma_1^2\sigma_4^2\sigma_5^2 + \sigma_3^2\sigma_4^2\sigma_5^2) + \sigma_1^2\sigma_3^2\sigma_4^2\sigma_5^2)$	
$K = 5$	$k = 3$	$\sigma_3^2(q^8 + q^6(\sigma_1^2 + 2\sigma_2^2 + 2\sigma_4^2 + \sigma_5^2) + q^4(\sigma_1^2\sigma_2^2 + 2\sigma_1^2\sigma_4^2 + \sigma_1^2\sigma_5^2 + 4\sigma_2^2\sigma_4^2 + 2\sigma_2^2\sigma_5^2 + \sigma_4^2\sigma_5^2) + q^2(2\sigma_1^2\sigma_2^2\sigma_4^2 + \sigma_1^2\sigma_2^2\sigma_5^2 + \sigma_1^2\sigma_4^2\sigma_5^2 + 2\sigma_2^2\sigma_4^2\sigma_5^2) + \sigma_1^2\sigma_2^2\sigma_4^2\sigma_5^2)$	

Appendix B. Computation of the denominator of the elements of $\mathbf{P}_j^{(batch)}$ 230

The denominator D of the elements of $\mathbf{P}^{(batch)}$ is equal to the determinant of $(\mathbf{P}^{(batch)})^{-1}$. 231
This comes directly from the Laplace formula to compute the inverse of a matrix. 232

This section reveals how to compute the determinant of matrix $(\mathbf{P}^{(batch)})^{-1}$. The particular shape of $(\mathbf{P}^{(batch)})^{-1}$ allows to have interesting expressions to compute its determinant. 233
Indeed, $(\mathbf{P}^{(batch)})^{-1}$ is a tri-diagonal matrix, i.e. a banded matrix with 3 non-zero diagonals 234
elements. For such matrix, a recurrence formula exist to compute its determinant. 235
236

Theorem A1 (Determinant of a tri-diagonal matrix). (Muir, 1960) [8] 237

Let us consider the following $K \times K$ tri-diagonal matrix 238

$$\begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 & 0 \\ c_1 & a_2 & b_2 & \dots & 0 & 0 \\ 0 & c_2 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{K-1} & b_{K-1} \\ 0 & 0 & 0 & \dots & c_{K-1} & a_K \end{bmatrix}$$

The determinant of such matrix, noted d_K , can be computed with the following recurrence relation: 239

$$d_0 = 1, \quad d_1 = a_1$$

$$d_k = a_k d_{k-1} - b_{k-1} c_{k-1} d_{k-2}, \quad \text{for } 2 \leq k \leq K$$

When applying this theorem to $(\mathbf{P}^{(batch)})^{-1}$, we obtain a ratio of 2 expressions where 240
the denominator is always $q^{2(K-1)} \prod_{n=1}^K \sigma_n^2$. Therefore, noting the numerator p_K , we have: 241

$$d_K = \frac{p_K}{q^{2(K-1)} \prod_{n=1}^K \sigma_n^2}$$

As we may have to evaluate this expression for $\sigma_n = 0$, it is more convenient to work with $p_K = d_K q^{2(K-1)} \prod_{n=1}^K \sigma_n^2$.

Lemma A1 (recurrence relation considering the sequence $s_k = d_k \prod_{j=1}^k (q^2 \sigma_j^2)$).

$$\begin{aligned} s_0 &= 1, & s_1 &= q^2 + \sigma_1^2 \\ s_k &= (q^2 + 2\sigma_k^2)s_{k-1} - \sigma_k^2 \sigma_{k-1}^2 s_{k-2}, & \text{for } 2 \leq k < K \\ s_K &= (q^2 + \sigma_K^2)s_{K-1} - \sigma_K^2 \sigma_{K-1}^2 s_{K-2} \end{aligned}$$

Proof.

$$\begin{aligned} s_k &= d_k \prod_{j=1}^k (q^2 \sigma_j^2) \\ &= (a_k d_{k-1} - b_{k-1} c_{k-1} d_{k-2}) \prod_{j=1}^k (q^2 \sigma_j^2) \quad (\text{using Theorem A1}) \\ &= a_k q^2 \sigma_k^2 \underbrace{d_{k-1} \prod_{j=1}^{k-1} (q^2 \sigma_j^2)}_{s_{k-1}} - b_{k-1} c_{k-1} q^4 \sigma_k^2 \sigma_{k-1}^2 \underbrace{d_{k-2} \prod_{j=1}^{k-2} (q^2 \sigma_j^2)}_{s_{k-2}} \\ &= a_k q^2 \sigma_k^2 s_{k-1} - \sigma_k^2 \sigma_{k-1}^2 s_{k-2} \quad (\text{replacing } b_k \text{ and } c_k \text{ by their values}) \end{aligned}$$

Theorem A1 provides the initial values of the sequence d_k , which are used to define the initial values of s_k .

The term p_K is actually equal to $p_K = \frac{1}{q^2} d_K \prod_{j=1}^K (q^2 \sigma_j^2) = \frac{1}{q^2} s_K$. The common factor to all terms $\frac{1}{q^2}$ is applied to the initial values of a new sequence p_k . Finally, we can also replace by the values of a_k . Then, we have:

$$\begin{aligned} p_0 &= \frac{1}{q^2} d_0 = \frac{1}{q^2}, & p_1 &= \frac{1}{q^2} d_1 q^2 \sigma_1^2 = 1 + \frac{\sigma_1^2}{q^2} \\ p_k &= (q^2 + 2\sigma_k^2)p_{k-1} - \sigma_k^2 \sigma_{k-1}^2 p_{k-2}, & \text{for } 2 \leq k < K \\ p_K &= (q^2 + \sigma_K^2)p_{K-1} - \sigma_K^2 \sigma_{K-1}^2 p_{K-2} \end{aligned}$$

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