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# Spectral sequences arising from the gauge equation

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**Abstract:** The gauge equation is a generalization of the conjugacy relation for Koszul connection to bundle morphisms that are not isomorphisms. The existence of non-trivial solution to this equation, especially when duality is imposed upon related connections, gives important information about the geometry of the manifolds considered. In this article, we use the gauge equation to introduce spectral sequences that are further specialized to Hessian structures.

**Keywords:** Gauge equation, spectral sequence, KV-cohomology, hessian manifold, statistical manifold.

**MSC:** 53A60, 53C07, 53C12

## 1. Introduction

The concept of a statistical manifold comes from the field of information geometry. Such an object is defined as a quadruple  $(M, g, \nabla, \nabla^+)$  where  $(M, g)$  is a smooth Riemannian manifold and  $\nabla, \nabla^+$  are torsion-free Koszul connections on  $TM$  that satisfy a metric relation [1]:

$$\forall X, Y, Z \in TM, Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z^+ Y)$$

One connection  $\nabla$  or  $\nabla^+$  entirely defines the other one, but the extra assumption of both being torsion-less is not automatically satisfied. Another important concept coming from the general theory of Koszul connections is the gauge equation. Two connections  $\nabla_1, \nabla_2$  on a vector bundle  $E \xrightarrow{\pi} M$  are said to satisfy a gauge equation if there exists a bundle morphism  $\theta: E \rightarrow E$  such that:  $\nabla_1 \theta = \theta \nabla_2$ . When  $\theta$  is an isomorphism, this is nothing but saying that  $\nabla_1, \nabla_2$  are in the same conjugacy class under the action of the gauge group, but in the general situation important information is gained on the geometry of  $M$ .

In the present work, we focus on the case where the gauge equation is satisfied by two connections  $\nabla, \nabla^+$  coming from a statistical manifold. In particular, two remarkable webs are defined, that give rise to spectral sequences of interest.

The paper is organized as follows: in section 2, basic facts about the gauge equation in the general settings are briefly recap, then some equivalent formulations are given and important parallel tensors are defined, in section 3 the cohomology of Koszul-Vinberg algebras is introduced and double complexes defined, in section 5 introductory material on spectral sequences is given and finally in 6 the special case of statistical manifolds is investigated.

### 1.1. Notations and writing conventions

Thorough this document, the next writing conventions are applied:  $M$  is a smooth connected manifold. For a vector bundle  $E \xrightarrow{\pi} M$ , the notation  $\Gamma(U; E)$  with  $U \subset M$  an open subset of the manifold  $M$  stands for the  $C^\infty(M)$ -module of smooth sections over

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$U$ . The functor  $U \mapsto \Gamma(U; E)$  defines a sheaf denoted by  $\Gamma_E$ . Finally,  $\Gamma(E)$  is a shorthand notation for  $\Gamma(M; E)$ . Lowercase letters are used for sections, uppercase ones for tangent vectors.

## 2. The gauge equation

Let  $E \xrightarrow{\pi} M$  be a vector bundle. A Koszul connection  $\nabla$  is a  $\mathbb{R}$ -linear mapping [2]:

$$\nabla: \Gamma(E) \rightarrow \Gamma(T^*M \otimes E) \quad (1)$$

such that for any  $f \in C^\infty(M)$ ,  $\nabla_X fs = df \otimes s + f \nabla_X s$ . Let  $E^* \xrightarrow{\pi^*} M$  be the bundle obtained by dualizing  $E$  fiberwise. A section  $\theta \in \Gamma(E^* \otimes E)$ , that is a  $(1,1)$ -tensor, defines two bundle morphisms:

$$\begin{array}{ccc} E & \xrightarrow{\theta} & E \\ & \searrow \pi & \downarrow \pi \\ & & M \end{array} \quad \begin{array}{ccc} E^* & \xrightarrow{\theta^t} & E^* \\ & \searrow \pi^* & \downarrow \pi^* \\ & & M \end{array} \quad (2)$$

where  $\theta^t$  is such that for any  $p \in M$ ,  $X \in T_p M$ ,  $\zeta \in T_p^* M$ :

$$\left( \theta_p^t \zeta \right) (X) = \zeta(\theta_p X) \quad (3)$$

**Definition 2.1.** Let  $(\nabla_1, \nabla_2)$  be a couple of Koszul connections. A  $(1,1)$ -tensor  $\theta$  is said to be a solution of the gauge equation if for any  $s \in \Gamma(E)$ :

$$\nabla_2 \theta s = \theta \nabla_1 s \quad (4)$$

or equivalently if the next diagram commutes:

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\nabla_1} & \Gamma(T^*M \otimes E) \\ \downarrow \theta & & \downarrow Id \otimes \theta \\ \Gamma(E) & \xrightarrow{\nabla_2} & \Gamma(T^*M \otimes E) \end{array} \quad (5)$$

Definition 2.1 can be made local, giving rise to diagrams:

$$\begin{array}{ccc} \Gamma(U; E) & \xrightarrow{\nabla_1} & \Gamma(U; T^*M \otimes E) \\ \downarrow \theta_U & & \downarrow Id \otimes \theta_U \\ \Gamma(U; E) & \xrightarrow{\nabla_2} & \Gamma(U; T^*M \otimes E) \end{array} \quad (6)$$

with  $U$  an open subset of  $M$  and  $\theta_U \in \Gamma(U; E^* \otimes E)$ . The above definitions can be generalized to arbitrary vector bundles over  $M$  giving rise to a category  $\mathcal{GC}$  whose objects are couples  $(E, \nabla)$ , with  $E$  a vector bundle on  $M$  and  $\nabla$  a Koszul connection on  $E$  and morphisms are bundle morphisms  $\theta: E \rightarrow F$  such that  $(E, \nabla_1) \rightarrow (F, \nabla_2)$  if the diagram:

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\nabla_1} & \Gamma(T^*M \otimes E) \\ \downarrow \theta & & \downarrow Id \otimes \theta \\ \Gamma(F) & \xrightarrow{\nabla_2} & \Gamma(T^*M \otimes F) \end{array} \quad (7)$$

commutes.

**Definition 2.2.** Let  $\nabla$  be an affine connection. Its dual is the affine connection:

$$\nabla^*: \Gamma(E^*) \rightarrow \Gamma(T^*M \otimes E^*) \quad (8)$$

defined by the relation:

$$(\nabla^* \zeta)(s) = d(\zeta(s)) - \zeta(\nabla s) \quad (9)$$

**Proposition 2.1.** If  $\theta$  is a solution of the gauge equation with connections  $(\nabla_1, \nabla_2)$ , then  $\theta^*$  is a solution of the gauge equation with connections  $(\nabla_2^*, \nabla_1^*)$

**Proof.** For  $s \in \Gamma(E)$ ,  $\zeta \in \Gamma(E^*)$ :

$$(\nabla_2^*(\theta^*\zeta))(s) = (\theta^*\zeta)(s) - (\theta^*\zeta)\nabla_2 s = \zeta(\theta s) - \zeta(\theta\nabla_2 s) \quad (10)$$

$$= \zeta(\theta s) - \zeta(\nabla_1 \theta s) = (\theta^*\nabla_1^* \zeta)(s) \quad (11)$$

□

Given a couple of connections  $(\nabla_1, \nabla_2)$ , the difference  $D_{1,2} = \nabla_1 - \nabla_2$  is a section of  $\Gamma(TM^* \otimes TM^* \otimes E)$ . Using it, the gauge equation 2.1 can be rewritten as:

$$\nabla_2 \theta - \theta \nabla_2 + D_{1,2} \theta = 0 \quad (12)$$

Considering  $\theta$  as a 0-form with values in  $E^* \otimes E$ , equation 12 may be rewritten as:

$$d^{\nabla_2} \theta + D_{1,2} \theta = 0 \quad (13)$$

where  $d^{\nabla_2}$  is the exterior covariant derivative associated with the connection  $\nabla_2$ . When  $\nabla_2$  is flat,  $d^{\nabla_2} d^{\nabla_2} = 0$  and thus  $d^{\nabla_2}(D_{1,2} \theta) = 0$ .

We recall that the Gauge group  $\mathcal{G}(E)$  is the set of bundle isomorphisms:

$$\begin{array}{ccc} & U & \\ E & \xrightarrow{\quad} & E \\ & \xleftarrow{U^{-1}} & \\ & \searrow & \downarrow \\ & & M \end{array} \quad (14)$$

Given a connection  $\nabla$ ,  $U\nabla U^{-1}$  is also a connection.

**Proposition 2.2.** Let the triple  $(\nabla_1, \nabla_2, \theta)$  be a solution of the gauge equation  $\nabla_1 \theta = \theta \nabla_2$ . For any couple  $(U, V)$  in  $\mathcal{G}(E)$ , the triple:

$$(U\nabla_1 U^{-1}, U\theta V^{-1}, V\nabla_2 V^{-1})$$

is a solution of a gauge equation.

**Proof.** Starting with  $\nabla_1 \theta = \theta \nabla_2$ , it comes:

$$U^{-1} U \nabla_1 U^{-1} U \theta V^{-1} V = U^{-1} U \theta V^{-1} V \nabla_2 V^{-1} V \quad (15)$$

Composing by  $U$  to the left and  $V^{-1}$  to the right yields the result. □

Proposition 2.2 indicates that the existence of a solution does not depend on a particular choice of frame-coframe for representing it. Furthermore, locally, it is always possible to assume a  $\theta$  of the form:

$$\theta = \left( \begin{array}{c|c} \text{Id} & 0 \\ \hline 0 & 0 \end{array} \right) \quad (16)$$

since a pair  $U, V$  such that  $U\theta V^{-1}$  has the reduced form of eq.16 exists by a standard linear algebra argument. Global reduction is, however, not possible as transition functions will

generally not preserve the diagonal structure. Let  $\tilde{E}$  be the bundle  $E \oplus E^*$ . The bilinear form:

$$B: (X + \alpha, Y + \beta) \in \tilde{E}^2 \rightarrow \beta(X) + \alpha(Y) \quad (17)$$

is non-degenerate, namely:

$$\forall Y + \beta \in \tilde{E} B(X + \alpha, Y + \beta) = 0 \Rightarrow X + \alpha = 0. \quad (18)$$

**Proposition 2.3.** *A  $(1, 1)$ -tensor  $\theta$  on  $E$  satisfies the gauge equation for a couple of connections  $(\nabla_1, \nabla_2)$  if and only the bilinear form:*

$$B_\theta: (X + \alpha, Y + \beta) \rightarrow B(X + \alpha, \theta Y + \theta^* Y) \quad (19)$$

is parallel with respect to the connection  $\tilde{\nabla} = \nabla_2 \oplus \nabla_1^*$ .

**Proof.** By definition:

$$B(X + \alpha, \theta Y + \theta^* Y) = \alpha(\theta Y) + \theta^* \beta(X) \quad (20)$$

Taking the differential yields:

$$d(\alpha(\theta Y)) = (\nabla_1^* \alpha)(\theta Y) + \alpha(\nabla_1 \theta Y) = (\theta^* \nabla_1 \star \alpha)(Y) + \alpha(\theta \nabla_2 Y) \quad (21)$$

$$= (\nabla_2^* \theta^* \alpha)(Y) + \alpha(\nabla_1 \theta Y) \quad (22)$$

and symmetrically:

$$d((\theta^* \beta)(X)) = d(\beta)(\theta X) \quad (23)$$

$$= (\nabla_2^* \theta^* \beta)(X) + \beta(\nabla_1 \theta X) \quad (24)$$

Now:

$$\tilde{B}_\theta(X + \alpha, Y + \beta) = \quad (25)$$

$$dB_\theta(X + \alpha, Y + \beta) - B_\theta(\tilde{\nabla}(X + \alpha), Y + \beta) - B_\theta(X + \alpha, \tilde{\nabla}(Y + \beta)) = \quad (26)$$

$$= dB_\theta(X + \alpha, Y + \beta) - \beta(\nabla_1 \theta X) + (\nabla_2^* \theta^* \alpha)(Y) - \alpha(\nabla_1 \theta X) + (\nabla_2^* \theta^*) \beta(X) = 0 \quad (27)$$

Conversely, if  $B_\theta$  is  $\tilde{\nabla}$ -parallel, then, for any couple  $(X + \alpha, Y + \beta)$ :

$$\alpha((\nabla_1 \theta - \theta \nabla_2) Y) + \beta((\nabla_1 \theta - \theta \nabla_2) X) = 0 \quad (28)$$

Taking, for example,  $\alpha = 0$ ,  $\beta$  arbitrary, it comes:

$$(\nabla_1 \theta - \theta \nabla_2) X = 0 \quad (29)$$

proving that the couple  $(\theta_1, \theta_2)$  satisfies the gauge equation.  $\square$

An immediate corollary is:

**Corollary 2.4.** *The kernel of  $\tilde{\theta} = \theta \oplus \theta^*$  is  $\tilde{\nabla}$ -invariant, hence the kernel of  $\theta$  (resp.  $\theta^*$ ) is  $\nabla_2$  (resp.  $\nabla_1^*$ ) invariant.*

**Proof.** The kernel of  $B$  is  $\{0\}$ , so if:

$$\forall Y + \beta \in \tilde{E}, B_\theta(X + \alpha, Y + \beta) = B(\theta X + \theta^* \alpha, Y + \beta) = 0 \quad (30)$$

Then  $\theta X + \theta^* \alpha = 0$  and  $X + \alpha \in \ker \tilde{\theta}$ . Given a basis of  $\ker \tilde{\theta}$  at a point  $p \in M$ , parallel transporting it by  $\tilde{\nabla}$  yields another basis of  $\ker \tilde{\theta}$  at an arbitrary point  $q \in M$ , hence the claim.  $\square$

*Remark 2.1.* Corollary 2.4 implies by parallel transport that the dimension of the kernel of  $\theta$  (resp.  $\theta^*$ ) is a constant, hence the rank of  $\theta$  (resp.  $\theta^*$ ) is also a constant.

*Remark 2.2.* The kernel of  $\theta^*$  is the set of differential forms vanishing on the image of  $\theta$ . The knowledge of the kernel of  $B_\theta$  thus completely characterize  $\ker \theta$  and  $\text{im } \theta$ . In particular,  $\theta$  has constant rank. 98  
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When there exists a Riemannian metric on the manifold  $M$ , the gauge equation can be specialized to pairs of connections on  $TM$  related by duality. 101  
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**Definition 2.3.** Let  $\nabla$  be an affine connection. Its conjugate with respect to  $g$  (often referred as the dual connection) is the connection  $\nabla^+$  defined by the relation: 103  
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$$\forall Z \in TM, \forall r, s \in \Gamma(TM), Z(g(r, s)) = g(\nabla_Z r, s) + g(r, \nabla_Z^+ s) \quad (31)$$

*Remark 2.3.* The most common notation for the conjugate connection is  $\nabla^*$ . In the present text, we adopt  $\nabla^+$  to distinguish from the connection on  $E^*$ . 105  
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**Definition 2.4.** Let  $\theta$  be a bundle morphism on  $TM$ . Its conjugate, denoted  $\theta^+$ , is the bundle morphism defined by: 107  
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$$\forall X, Y \in TM, g(\theta X, Y) = g(X, \theta^+ Y) \quad (32)$$

**Proposition 2.5.** If  $U: TM \rightarrow TM$  is a unitary bundle isomorphism, that is: 109

$$\forall X, Y \in TM, g(UX, UY) = g(X, Y)$$

Then  $U^{-1} = U^+$ . 110

**Proposition 2.6.** Let  $\nabla$  be a connection and  $U$  be a unitary bundle isomorphism, then: 111

$$(U\nabla U^+)^+ = U\nabla^+ U^+ \quad (33)$$

**Proof.** If  $U$  is unitary, so is  $U^+$ . Let  $Z \in TM, r, s \in \Gamma(TM)$ . Then: 112

$$Z(g(r, s)) = Z(g(U^+ r, U^+ s)) \quad (34)$$

$$= g(\nabla_Z U^+ r, U^+ s) + g(U^+ r, \nabla_Z^+ s) \quad (35)$$

$$= g(U\nabla_Z U^+ r, s) + g(r, U\nabla_Z^+ U^+ s) \quad (36)$$

and the claim follows.  $\square$  113

A is proposition 2.2: 114

**Proposition 2.7.** If the triple  $(\nabla, \nabla^+, \theta)$  satisfies the gauge equation  $\nabla\theta = \theta\nabla^+$ , so does  $(U\nabla, U^+, U\nabla^+ U^+, U^+\theta U)$  for any unitary isomorphism  $U$ . 115  
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*Remark 2.4.* If  $\theta$  is normal, that is  $[\theta, \theta^+] = 0$ , and the triple  $(\nabla, \nabla^+, \theta)$  satisfies the gauge equation  $\nabla\theta = \theta\nabla^+$ , then, locally, there exists a unitary isomorphism  $U$  such that  $U^+\theta U$  is diagonal and  $(U\nabla, U^+, U\nabla^+ U^+, U^+\theta U)$  satisfies a gauge equation. This is again a well-known fact from linear algebra since  $\theta$  is locally diagonalizable in an orthonormal frame. As in the case of eq. 16, this is generally not true globally. 117  
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**Proposition 2.8.** Using the musical isomorphisms  $TM \xrightleftharpoons[\sharp]{\flat} T^*M$ , it comes: 122

$$\forall X \in TM, \alpha \in T^*M \left( \nabla_X^+ \alpha^\sharp \right) = (\nabla_X \alpha)^\sharp$$

**Proof.** For any  $X, Y, Z \in TM$ : 123

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z^+ Y)$$

Passing to forms, for any  $Z \in TM$  and  $X \in T^*M$ :

$$Z(g(X, \alpha^\sharp)) = g(\nabla_Z X, \alpha^\sharp) + g(X, \nabla_Z^+ \alpha^\sharp)$$

Now:

$$\begin{aligned} Z(g(X, \alpha^\sharp)) &= Z(\alpha(X)) = (\nabla^* \alpha)(X) + \alpha(\nabla_Z X) \\ &= g(X, (\nabla^* \alpha)^\sharp) + g(\nabla_Z X, \alpha^\sharp) \end{aligned}$$

and the claim follows by identification.  $\square$

*Remark 2.5.*

**Proposition 2.9.** *Let the triple  $(\nabla, \nabla^+, \theta)$  satisfies the gauge equation  $\nabla\theta = \theta\nabla^+$ . The tensor:*

$$g_\theta: (X, Y) \mapsto g(\theta X, Y) \quad (37)$$

is  $\nabla$ -parallel.

**Proof.** The tensor  $B_\theta$  of proposition 2.3 can be written, using the metric, as:

$$B_\theta(X + \alpha, Y + \beta) = \beta(\theta X) + \alpha(\theta Y) \quad (38)$$

$$= g(\theta X, \beta^\sharp) + g(\alpha^\sharp, \theta Y) \quad (39)$$

Since  $B_\theta$  is  $\tilde{\nabla}$ -parallel, the proposition follows.  $\square$

*Remark 2.6.* Defining a metric  $g^*$  on  $T^*M$  by:

$$\forall \alpha, \beta \in T^*M, g^*(\alpha, \beta) = g(\alpha^\sharp, \beta^\sharp) \quad (40)$$

The proof of proposition 2.9 also shows that the tensor:

$$g_\theta^*: (\alpha, \beta) \mapsto g^*(\theta^* \alpha, \beta)$$

is  $\nabla^*$ -parallel.

Proposition 2.9 has the important consequence that  $TM$  can be split in two ways:

$$TM = \ker \theta \oplus \text{im } \theta, \quad TM = \ker \theta^+ \oplus \text{im } \theta^+. \quad (41)$$

It is clear from proposition 2.9 that if  $\theta$  is symmetric, that is  $\theta = \theta^+$ , the tensor:

$$(X, Y) \mapsto \frac{1}{2}g(\theta X, Y) + g(\theta X, Y) \quad (42)$$

is  $\nabla$ -parallel. When  $\theta$  is skew symmetric  $\theta = -\theta^+$ , the same is true for:

$$(X, Y) \mapsto \frac{1}{2}g(\theta X, Y) - g(\theta X, Y) \quad (43)$$

As in 7, a category so that morphisms represent gauge equation solutions. The situation is nevertheless a little bit more complicated as the dimension of the vector bundle may not agree. We recall the well-known definition:

**Definition 2.5.** Let  $E \rightarrow M$  be a vector bundle. A pseudo-riemannian metric on  $E$  is a smooth bilinear  $C^\infty(M)$ -mapping  $g_E: \Gamma(E) \times \Gamma(E) \rightarrow C^\infty(M)$  such that:

- $\forall s, s' \in \Gamma(E), g_E(s, s') = g_E(s', s).$

- There exists an isomorphisme  ${}^b: \Gamma(E) \rightarrow \Gamma(E^*)$  such that, for any  $s, s' \in \Gamma(E)$ :

$$s^b(s') = g_E(s, s').$$

A pseudo-riemannian metric is Riemannian if  $g_E(s, s) > 0$  for any  $s \neq 0$  in  $\Gamma(E)$ .

**Definition 2.6.** Let  $E, F$  be two vector bundles on  $M$  equipped with respective Riemannian metrics  $g_E, g_F$ . A partial isometry from  $E$  to  $F$  is a bundle morphism  $U$  such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{{}^b} & E^* \\ \downarrow U & & \uparrow U^* \\ F & \xrightarrow{{}^b} & F^* \end{array} \quad (44)$$

*Remark 2.7.* Definition 2.6 is equivalent to the fact that for any  $s, s' \in \Gamma(E)$ :

$$g_F(Us, Us') = g_E(s, s').$$

**Definition 2.7.** Let  $U: E \rightarrow F$  be a partial isometry and  $\nabla_2$  be a Koszul connection on  $F$ . Its dual  $\nabla_2^+$  is the connection on  $E$  defined by the relation

$$U\nabla_2^+ = \nabla_2 U. \quad (45)$$

**Definition 2.8.** The category  $\mathcal{GU}$  has objects  $(E, \nabla)$  where  $\nabla$  is a Koszul connection on  $E$  and morphisms  $(U, \theta): (E, \nabla_1) \rightarrow (F, \nabla_2)$  where  $U: E \rightarrow F$  is a partial isometry,  $\theta: E \rightarrow F$  is a bundle morphism and  $\nabla_1 = \nabla_2^+, \nabla_2\theta = \theta\nabla_1$ .

### 3. KV cohomology

The cochain complex of Koszul-Vinberg algebras may be introduced by the following three ways:[3,4]

Way1: From the point of view of the tensorial calculus: The brut formula,

Way2: From the pointof view of the theory of categories: Simplicial objects.

Way3: The point of view of the Anomalies, viz the calculation rules.

In this work, we take into account forthcoming our applications interests in relationships between the information Geometry and the Differential topology.

#### 3.1. Koszul-Vinberg algebras.

We go to recall useful basic definitions.

**Definition 3.1.** A real Koszul-Vinberg algebra is a real vector space  $A$  endowed with a product

$$A \times A \ni (a, b) \rightarrow ab \in A$$

subject the following identity,

$$(a, b, c) = (b, a, c)$$

where

$$(a, b, c) = (ab)c - a(bc).$$

Examples.

(a) Associative algebras are Koszul-Vinberg algebras.

(b) The vector of vector fields on a smooth manifold  $M$  endowed with symmetric flat Koszul connection  $\nabla$ .



### 3.2. KV modules of Koszul-Vinberg algebras

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**Definition 3.2.** A real left module of a real Koszul-Vinberg algebra  $A$  is a real vector space  $V$  endowed with a bilinear mapping

$$A \times V \ni (a, v) \rightarrow a.v \in V$$

which satisfies the following identity,

$$(a, a', v) = (a', a, v),$$

where

$$(a, a', v) = (aa').v - a.(a'.v).$$

In this paper we will be dealing with Koszul-Vinberg algebra of vector fields  $\mathcal{X}(M)$  on a differentiable manifold  $M$  endowed with a Koszul connection  $\nabla$  whose both the curvature tensor  $R^\nabla$  and the torsion tensor  $T^\nabla$  vanish identically.

We put

$$A := (\mathcal{X}(M), \nabla).$$

The product on  $A$  is defined as it follows,

$$a.a' = \nabla_a a'.$$

Of course  $\mathcal{X}(M)$  is a left Koszul-Vinberg module of  $A$

The space of smooth functions  $C^\infty(M)$  is a left Koszul-Vinberg module of  $A$  under the left action

$$A \times C^\infty(M) \ni (a, f) \rightarrow df(a) \in C^\infty(M)$$

### 3.3. Vectorial cochain complexes

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Given a Koszul-Vinberg algebra  $A$ , to any left module  $V$  are associated the following two cochain complexes of  $A$  with coefficients in  $V$ . One is denoted by  $C_{KV}(A, V)$  and is named KV complex. The other is denoted  $C_\tau(A, V)$  and is named total KV complex. We go to remind the definition of these complexes and point some domains of their efficiency.

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#### 3.3.1. The complex $C_{KV}(A, V)$

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We set

$$J(V) = \{v \in V \text{ s.t. } (a, a', v) = 0 \quad \forall (a, a') \subset A\}.$$

Given

$$\xi = a_1 \otimes \dots \otimes a_{q+1} \in A^{\otimes q+1}$$

we put

$$\partial_i \xi = \dots \otimes \hat{a}_i \otimes \dots$$

and

$$a.\xi = \sum_i \dots \otimes a.a_i \otimes \dots$$

The vector space  $C_{KV}(A, V)$  is  $\mathbb{Z}$ -graded by the homogeneous subspaces  $C_{KV}^q$  which are defined as it follows:

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$$\begin{aligned} C_{KV}^q &= 0 \quad \text{if } q < 0, C_{KV}^0 = J(V), \\ C_{KV}^q &= \text{Hom}(A^{\otimes q}, V), \quad q > 0. \end{aligned}$$

The operator

$$C_{KV}^q \ni f \rightarrow \delta.f C_{KV}^{q+1}$$

is defined as it follows,

$$\delta.v(a) = a.v \quad (a, v) \in A \times J(V),$$

Let

$$f \in C_{KV}^q$$

and

$$\xi = a_1 \otimes \dots \otimes a_{q+1}$$

then

$$\delta.f(\xi) = \sum_1^q (-1)^{i-1} [a_i.f(\partial_i \xi) - f(X_i.\partial_i \xi)].$$

The q-th cohomology space is denoted by

$$H_{KV}^q(A, V) = \frac{\ker(\delta : C_{KV}^q \rightarrow C_{KV}^{q+1})}{\delta(C_{KV}^{q-1})}.$$

This cohomology is the solution of the conjecture of Gerstenhaber for the deformations of Hyperbolic structure in the sense of Koszul. The conjecture of Gerstenhaber claim that *Every restrict theory of deformation generates it proper theory of cohomology*, [5].

This cohomology also manages equivalence relation between extensions of Koszul-Vinberg algebras by Koszul-Vinberg modules. More explicitly let  $Ext(A, V)$  the set of equivalence classes of extensions of A by V, then one can write

$$Ext(A, V) = H_{KV}^2(A, V).$$

In the category of modules of associative algebras and as well as in the category of modules of Lie algebras respectively, the 2nd Hochschild space  $HH^2(-, -)$  and the 2nd Chevalley-Eilenberg space  $H_{CE}^2(-, -)$  play the similar role. 183  
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### 3.3.2. Total KV complex $C_\tau(A, V)$ . 186

The cochain complex  $C_\tau(A, V)$  is  $\mathbb{Z}$ -graded by the homogeneous subspaces  $C_\tau^q(A, V)$  which as it follows,

$$C_\tau^q = 0 \text{ if } q < 0,$$

$$C_\tau^0 = V,$$

$$C_\tau^q = Hom(A^{\otimes q}, V) \text{ if } q > 0.$$

We keep using the notation as in the subsection above. The operation  $\delta_\tau$  is defined as it follows:

$$\delta_\tau v(a) = a.v \text{ if } v \in C^0,$$

$$\text{If } f \in C_\tau^q,$$

$$\delta_\tau f(\xi) = \sum_1^{q+1} (-1)^{i+1} [a_i.f(\partial_i \xi) - f(a_i.\partial_i \xi)].$$

The q-th cohomology space of the total complex is denoted by

$$H_\tau^q(A, V) = \frac{\ker(\delta_\tau : C_\tau^q \rightarrow C_\tau^{q+1})}{\delta_\tau(C_\tau^{q-1})}.$$

### 3.4. Scalar Complexes $C_{KV}(A, R)$ and $C_\tau(A, R)$ . 187

Henceforth:

$$A = (\mathcal{X}(M), \nabla)$$

and:

$$V = C^\infty(M).$$

The scalar cochain complexes  $C_{KV}(A, R)$  is  $\mathbb{Z}$ -graded as it follows, 188

$$C_{KV}^q(A, R) = 0 \text{ for } q < 0, \quad \text{189}$$

$$C_{KV}^0(A, R) = J(C^\infty(M)) := \text{the space of affine functions}, \quad \text{190}$$

$$C_{KV}^q(A, R) = \text{Hom}(A^{\otimes q}, C^\infty(R)) \text{ if } q > 0. \quad 191$$

Regarding the total KV cohomology the vector space  $C_\tau(A, R)$  is graded as it follows, 192

$$C_\tau^q(A, R) = 0 \text{ if } q < 0, \quad 193$$

$$C_\tau^0(A, R) = C^\infty(M), \text{ if } q > 0. \quad 194$$

$$C_\tau^q(A, R) = \text{Hom}(A^{\otimes q}, C^\infty(M)) \text{ if } q > 0. \quad 195$$

### 3.5. Links with the De Rham complex. 196

We consider the real De Rham complex which is graded as it follows, 197

$$\Omega(M) = \bigoplus \Omega^q(M, R),$$

Where

$$\Omega^q(A, R) = \text{Hom}(\Lambda^q A, C^\infty(M)).$$

The operator

$$d : \Omega^q(M) \rightarrow \Omega^{q+1}(M)$$

is defined as it follows,

$$d\omega(a_0 \wedge \dots \wedge a_q) = \sum_0^q (-1)^i a_i \cdot \omega(\dots \wedge \hat{a}_i \wedge \dots) + \sum_{i < j} (-1)^{i+j} \omega([a_i, a_j] \wedge \dots \wedge \hat{a}_i \wedge \dots \wedge \hat{a}_j \wedge \dots).$$

The inclusion map

$$\Omega^q(M) \subset C_\tau^q(A, R)$$

yields the following cochain complex injective morphism

$$(\Omega(M), d) \rightarrow (C_\tau(A, R), \delta_\tau).$$

The quotient complex is denoted by

$$(Q, d) = \frac{(C_\tau(A, R), \delta)}{(\Omega(M), d)},$$

Links of de Rham cohomology with the total KV cohomology emerge from the following short exact sequence of cochain complexes,

$$0 \rightarrow (\Omega(M), d) \rightarrow (C_\tau(A, R), \delta_\tau) \rightarrow (Q, d) \rightarrow 0.$$

The short exact sequence above yields the following long cohomology exact sequence

$$\rightarrow H_{dR}^q(M, R) \rightarrow H_\tau^q(A, R) \rightarrow H^q(Q) \rightarrow H_{dR}^{q+1}(M, R) \rightarrow$$

### 3.6. Tensor product of two KV complexes 199

In the category of statistical geometry we will be interested in spectral sequences which arise from particular double complexes. We consider two Koszul-Vinberg algebras  $A$  and  $A^*$ . Let  $W$  be of left KV module of the both  $A$  and  $A^*$ . From this situation arise the following four cochain complexes:

$$(I) : C_{KV}(A, W),$$

$$(II) : C_{KV}(A^*, W),$$

$$(III) : C_\tau(A, W),$$

$$(IV) : C_\tau(A^*, W).$$

Let us consider the vector space  $C(W)$  which is by-graded by the vector space

$$C^{q,p} = C_\tau^q(A, W) \otimes C_\tau^p(A^*, W).$$

We set

$$C^m = \sum_{q+p=m} C^{q,p}.$$

Given

$$\alpha \otimes \beta \in C^{q,p}$$

we put

$$\delta(\alpha \otimes \beta) = [\delta_\tau \alpha \otimes \beta + (-1)^q \alpha \otimes \delta_\tau \beta] \in C^{q+1,p} \oplus C^{q,p+1}$$

It is obvious that

$$\delta \circ \delta = 0.$$

The cohomology space  $H^{q,p}$  is defined as it follows,

$$H^{q,p} = \frac{\ker(\delta : C^{q,p} \rightarrow C^{q+1,p} \oplus C^{q,p+1})}{\text{im}(\delta) \cap C^{q,p}}.$$

In Hessian statistical manifolds solutions of gauge equations give rise to statistical 2-webs. These 2-webs are canonically associated with tensor products of cochain complexes the cohomology of which can be calculated with spectral sequences. Situations as in (I), (II), (III), (IV) arise in any Hessian manifold  $(M, g, \nabla)$ .

#### 4. Statistical structures

Henceforth the framework is the category  $\mathcal{GU}$  restricted to objects  $(M, g, \nabla)$  where:

- (i)  $g$  is a Riemannian metric on  $TM$ .
- (ii)  $(\nabla, \nabla^+)$  are both torsion free Koszul connections on  $TM$ .

and morphisms Version March 20, 2024 submitted toof the form  $(\text{Id}, \theta)$ .

We keep the notation in Section 2 and in Section 3.

We are concerned with the gauge equation of the couple  $(\nabla, \nabla^*)$ . Let  $\theta$  be a solution of that gauge equation, then we define another pair of solutions  $(\Theta, \Theta^*)$  by the following identities:

$$2g(\Theta(X), Y) = g(\theta(X), Y) + g(X, \theta(Y)),$$

$$2g(\Theta^*(X), Y) = g(\theta(X), Y) - g(X, \theta(Y)).$$

All of the following four distributions are regular and are in involution,  $\{Ker(\Theta), Im(\Theta), Ker(\Theta^*), Im(\Theta^*)\}$ . Furthermore one has the following 2-webs,

$$TM = K \oplus I,$$

$$TM = K^* \oplus I^*$$

where

$$K = \ker(\Theta),$$

$$I = \text{im}(\Theta),$$

$$K^* = \ker(\Theta^*),$$

$$I^* = \text{im}(\Theta^*).$$

Other remarkable properties of these distributions are the following identities:

$$\nabla_X K = K,$$

$$\nabla_X K^* = K^*,$$

$$\nabla_X^* I = I,$$

$$\nabla_X^* I^* = I^*.$$

FACT.1: If either  $(M, g, \nabla)$  or  $(M, g, \nabla^*)$  are Hessian structure then  $K, K^*, I$  and  $I^*$  are Hessian foliations. Thus any of the pairs  $(K, I)$  and  $(K^*, I^*)$  gives rise to double cochain complex. 212

FACT.2: It would be remembered that a foliated manifold carries in addition to its total de Rham complex two other remarkable complexes which are the complex of foliated forms and the complex of basic forms. 213

FACT.3: Regrading the picture  $\theta \rightarrow (\Theta, \Theta^*)$  any of the three distributions  $\ker(\theta), \ker(\Theta)$  and  $\ker(\Theta^*)$  has a constant rank, nevertheless these three rank may be pairwise different each from other. 214

#### 4.1. Tensor products 216

For every non negative integer  $q$  the dual vector spaces  $\Gamma(K^{\otimes q})$  and of  $\Gamma(\Lambda^q K)$  are denoted by  $C^q(K)$  and  $\Omega_K^q(M)$  respectively. Let us put 217

$$C(K) = \bigoplus_q C^q(K),$$

$$\Omega_K(M) = \bigoplus_q \Omega_K^q(M).$$

It makes sense to restrict the de Rham operation to  $\Omega_K(M)$  in order to define the following cochain complex, 218

$$\rightarrow \Omega_K^{q-1}(M) \rightarrow \Omega_K^q(M) \rightarrow \Omega_K^{q+1}(M) \rightarrow$$

In any Hessian manifold we involve FACT.1 and we use the operators  $\delta_{KV}$  and  $\delta_\tau$  to write down the KV cochain complexes  $[C(K), \delta_\tau]$  and  $[C(K), \delta_{KV}]$ , 219

$$[C(K), \delta_{KV}] : 0 \rightarrow J(C^\infty(M)) \rightarrow \dots \rightarrow C^{q-1}(K) \rightarrow C^q(K) \rightarrow C^{q+1}(K) \rightarrow$$

$$[C(K), \delta_\tau] : 0 \rightarrow C^\infty(M) \rightarrow \dots \rightarrow C^{q-1}(K) \rightarrow C^q(K) \rightarrow C^{q+1}(K) \rightarrow$$

The cochain complex above is nothing else than

$$\rightarrow C_{KV}^{q-1}(A_K, R) \rightarrow C_{KV}^q(A_K, R) \rightarrow C_{KV}^{q+1}(A_K, R) \rightarrow,$$

where

$$A_K = \Gamma(K = \ker(\Theta)).$$

Similar complexes are attached to three other distributions  $I, K^*$  and  $I^*$ . 220

#### 4.2. Double complexes in a statistical manifold 221

In a statistical manifold  $(M, g, \nabla, \nabla^*)$  let  $\theta$  be a solution of the jauge equation of  $(\nabla^*, \nabla)$ . The situation just described to the following two 2-webs  $(K, I)$  and  $(K^*, I^*)$ . We are concerned with the two de Rham double complexes: 222

$$\Omega(K, I) = \bigoplus_{q,p} \Omega_K^q(M) \otimes \Omega_I^p(M),$$

$$\Omega(K^*, I^*) = \bigoplus_{q,p} \Omega_{K^*}^q(M) \otimes \Omega_{I^*}^p(M)$$

#### 4.3. Double complexes in a Hessian manifold 223

In a Hessian manifold  $(M, g, \nabla, \nabla^*)$  we are concerned with de Rham double complexes which derive from the following 2-webs

$$\Omega(K, I)$$

$$\Omega(K^*, I^*)$$

To investigate properties of the statistical structure  $(g, \nabla, \nabla^*)$  we involve the two Koszul-Vinberg algebras:

$$A = \Gamma(TM, \nabla),$$

$$A^* = \Gamma(TM), \nabla^*).$$

Thus we face four complexes:

$$\begin{aligned} &C_{KV}(A, R), \\ &C_{KV}(A^*, R), \\ &C_\tau(A, R), \\ &C_\tau(A^*, R). \end{aligned}$$

We are interested two double KV complexes

$$\begin{aligned} C_{KV}(A, A^*) &= \oplus_{q,p} C_{KV}^q(A, R) \otimes C_{KV}^p(A^*, R) \\ C_\tau(A, A^*) &= \oplus C_\tau^q(A, R) \otimes C_\tau^p(A^*, R). \end{aligned}$$

These double complexes give rise to the total complexes

$$(C_{KV}(M), d_{KV})$$

and

$$(C_\tau(M), d_\tau).$$

Here

$$C_\tau(M) = \oplus_n C_\tau^n(M)$$

where

$$C_\tau^n(M) = \oplus_{[q+p=n]} C_\tau^q(A, R) \otimes C_\tau^p(A^*, R).$$

The following operator

$$d_\tau : C_\tau^n(M) \rightarrow C_\tau^{n+1}(M)$$

is defined as it follows, given

$$u \otimes v \in C_\tau^q(A, R) \otimes C_\tau^p(A^*, R)$$

$$d_\tau(u \otimes v) = \delta_\tau(u) \otimes v + (-1)^q u \otimes \delta_\tau(v).$$

Mutatis mutandis  $(C_{KV}, d_{KV})$  is defined similarly.

Let  $G$  be the group of symmetries of  $(M, g, \nabla, \nabla^*)$ ;  $G$  is the following finite dimensional Lie group

$$G = \text{Isom}(M, g) \cap \text{Aff}(M, \nabla).$$

Of course the cohomology spaces of the complexes which are introduced above are geometric invariant of  $G$ . 224

A forthcoming section will be devoted to calculate the cohomology of these complexes. Before dealing with these calculations we go introduction other useful materials which will be involved. These materials derive from persistent simplicial homologies which are related with the gauge equation as in the subsection just below. 225  
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#### 4.4. Gauge equation and homology persistence. 232

We fix a statistical structure  $(M, g, \nabla, \nabla^*)$ . Let  $\theta$  be a solution of the gauge equation of  $(\nabla^*, \nabla)$ . According to the notation used in the precedent sections  $\theta$  gives rise to two 2-webs

$$(K, I)$$

and

$$(K^*, I^*).$$

The foliation defined by  $I^*$  is denoted by  $\mathcal{F}_\theta$ .

Let  $r^*(\theta)$  be the rank of the distribution  $I^*$ . We put

$$r^*(M) = \max_{\theta} \{r^*(\theta)\}.$$

STEP.1:

Now we choose a  $\theta_1$  such that

$$r^*(\theta_1) = r^*(M).$$

Henceforth we fix  $x \in M$ .

Let  $F_1(x)$  be the leaf of  $\mathcal{F}_{\theta_1}$  which contains  $x$ .

So we introduce the  $r^*(M)$ -dimensional statistical submanifold

$$(F_1(x), g, \nabla^*) \subset (M, g, \nabla^*).$$

STEP.2:

We use the gauge equation of  $F_x(g, \nabla^*)$  to define  $r^*(F_1(x))$ . Then we impliment the manipulation as in STEP.1 to introduce the statistical submanifold

$$(F_2(x), g, \nabla^*) \subset (F_1(x), g, \nabla^*).$$

We set

$$(F_0, g, \nabla^*) = (M, g, \nabla^*).$$

Then inductively we construct the following statistical filtration:

$$SP(M) : (F_q(x), g, \nabla^*) \subset (F_{q+1}(x), g, \nabla^*) \subset \dots \subset (M, g, \nabla^*). \quad (46)$$

We consider the real singular chain complex of  $M$ :

$$Sing(M) : \dots \rightarrow C_{q+1}(M) \rightarrow C_q(M) \rightarrow C_{q-1}(M) \rightarrow \dots \quad (47)$$

The topology persistence ( $SP(M)$ ) yields the following homology persistence:

$$HP(M) : \dots \rightarrow Sing(F_{q+1}(x)) \rightarrow Sing(F_q(x)) \rightarrow Sing(F_{q-1}(x)) \rightarrow \dots \quad (48)$$

## 5. Spectral sequences

In this section, we briefly recall the definition of spectral sequences of cochain complexes. A good recent reference on the subject is [6].

**Definition 5.1.** A graded differential sheaf  $(\mathcal{S}, d)$  is a graded sheaf  $(\mathcal{S}^p)_{p \in \mathbb{Z}}$  together with a graded morphism  $d: \mathcal{S}^p \rightarrow \mathcal{S}^{p+1}$  satisfying  $d^2 = 0$ .

**Definition 5.2.** The derived cohomology sheaf is the graded sheaf  $H(\cdot)$ :

$$H^p(\mathcal{S}) = \frac{\ker\{d^p: \mathcal{S}^p \rightarrow \mathcal{S}^{p+1}\}}{\text{im}\{d^{p-1}: \mathcal{S}^{p-1} \rightarrow \mathcal{S}^p\}} \quad (49)$$

*Remark 5.1.* The derived cohomology sheaf is the sheafification of the local cohomology presheaf:

$$U \mapsto H^p(\mathcal{S}(U))$$

In the sequel, a ring  $R$  is fixed.

**Definition 5.3.** A bigraded module  $E$  over  $R$  is a double indexed collection of  $R$ -modules  $E^{p,q}$ ,  $p, q \in \mathbb{Z}$ .

**Definition 5.4.** Let  $E$  be a bigraded module over  $R$  and let  $r \in \mathbb{N}$ . A differential over  $E$  of bidegree  $(r, 1 - r)$  is double indexed collections of  $R$ -morphisms  $d: E^{p,q} \rightarrow E^{p+r, q+1-r}$  such that  $d^2 = 0$ .

**Definition 5.5.** A differential bigraded  $R$ -module is a couple  $(E, d)$  with  $E$  a bigraded module and  $d$  a differential of bidegree  $(r, 1 - r)$ ,  $r$  a fixed integer.

**Definition 5.6.** A cohomology spectral sequence is a sequence of bigraded differential modules  $(E_r, d_r), r = 1, 2, \dots$  where  $d_r$  has bidegree  $(r, 1 - r)$  and for all  $p, q, r, E_{r+1}^{p,q} \sim H^{p,q}(E_r, d_r)$ .

*Remark 5.2.* A spectral sequence can be viewed as a successive approximation process and in most cases,  $(E_2, d_2)$  is known and is the starting point of the sequence. Now, looking at stage  $n$ , that is  $(E_n, d_n)$ , the defining property of the spectral sequence indicates that if  $Z_n = \ker d_n, B_n = \text{im } d_{n-1}$ , then, as a bigraded module,  $E_{n+1} \sim Z_n/B_n$ . Now, if  $\bar{Z}_{n+1} = \ker d_{n+1}, \bar{B}_{n+1} = \text{im } d_n$ , there exist modules  $Z_{n+1}, B_{n+1}$  such that  $\bar{Z}_{n+1} = Z_{n+1}/B_n, \bar{B}_{n+1} = B_{n+1}/B_n$  and, by Noether's isomorphism,  $Z_{n+1}/B_{n+1} = \bar{Z}_{n+1}/\bar{B}_{n+1}$ . Furthermore, since  $d_{n+1}$  is a differential,  $B_{n+1} \supset B_n, Z_{n+1} \subset Z_n$ , hence  $B_n \subset B_{n+1} \subset Z_{n+1} \subset Z_n$ . Proceeding by recurrence, there exist limiting modules:

$$B_\infty = \cup_n B_n, Z_\infty = \cap_n Z_n$$

and the purpose of the spectral sequence is to obtain  $Z_\infty/B_\infty$ .

**Definition 5.7.** A spectral sequence is said to converge if there exists, for each couple of integers  $(p, q)$  an integer  $r(p, q)$  such that all differentials  $d_r: E_r^{p,q} \rightarrow E_{p+r, q+1-r}$  are 0 for  $r \geq r(p, q)$ .

**Proposition 5.1.** If a spectral sequence converges, then, for any couple of integers  $p, q$ , the module  $E_\infty^{p,q}$  is isomorphic to the direct limit of the diagram:

$$\begin{array}{ccccc} E_{r(p,q)}^{p,q} & \longrightarrow & E_{r(p,q)+1}^{p,q} & \longrightarrow & \dots \\ & \searrow & \downarrow & \swarrow & \\ & & E_\infty^{p,q} & & \end{array} \quad (50)$$

**Definition 5.8.** An exact couple is a pair of modules  $M, E$  and morphisms  $i, j, k$  fitting in the exact diagram:

$$\begin{array}{ccc} M & \xrightarrow{i} & M \\ & \swarrow k & \searrow j \\ & & E \end{array} \quad (51)$$

**Proposition 5.2.** Given an exact couple as in definition 5.8,  $E$  is differential module with differential  $d = j \circ k$

The next proposition can be found in [6].

**Proposition 5.3.** Let  $(M, E, i, j, k)$  be an exact couple. The derived couple:  $M_1 = \text{im}(i), E_1 = H(E)$  is exact with morphisms:

$$i_1 = i|_{M_1}, j_1 = j \circ i + dE, k_1(e + dE) = k(e)$$

Passing to bigraded modules and iterating the process defines a spectral sequence  $(E_r, d_r)$ , where  $E_r$  is the  $r$ -th derived module of  $E$  and  $d_r = j_r \circ k_r$ .

Finally, still using [6], a filtered complex  $F^p C \subset F^{p+1} C \subset \dots$  defines an exact couple by passing to cohomology. Namely, starting with the short exact sequence:

$$0 \longrightarrow F^p C \longrightarrow F^{p+1} C \longrightarrow F^{p+1} C / F^p C \longrightarrow 0 \quad (52)$$



one obtain a long homology sequence:

$$\dots \rightarrow H^{p+q}(F^{p+1}\mathcal{C}) \xrightarrow{i} H^{p+q}(F^p\mathcal{C}) \xrightarrow{j} H^{p+q}(F^{p+1}\mathcal{C}/F^p\mathcal{C}) \xrightarrow{k} H^{p+q+1}(F^{p+1}\mathcal{C}) \quad (53)$$

Putting:

$$E^{p,q} = H^{p+q}(F^{p+1}\mathcal{C}/F^p\mathcal{C}), \quad D^{p,q} = H^{p+q}(F^p\mathcal{C})$$

one obtain an exact couple, hence a spectral sequence. This construction will be part of the next section, where our aim will be to point out that from the methods of the information geometry emerge relevant spectral sequences.

## 6. Application to statistical manifolds

In an introductory manner the notion a spectral sequence has been reminded in Section 5. As machineries, spectral sequences are powerful tools for homological calculations.

In a statistical structure  $(M, g, \nabla, \nabla^*)$  we have identified chain complexes and cochain complexes which are attached to solutions of the gauge equation of  $((\nabla^*, \nabla))$ .

The homology of these complexes may be approximated by some well known spectral sequences. We aim to point out some spectral sequences which are linked with the complexes which have been introduced in the precedent sections.

### 6.1. The spectral sequences of a double complex

We focus the complexes which emerge from the statistical geometry.

In a Hessian structure  $(M, g, \nabla, \nabla^*)$  we fix a solution  $\theta$  of the gauge equation of  $((\nabla^*, \nabla))$ . We focus on the total KV complex

$$C_\tau(M)^n = \bigoplus_{[j+i=n]} C_\tau^j(A, R) \otimes C_\tau^i(A^*, R).$$

#### 6.1.1. The filtration $F_A^p(C_\tau(M))$ .

Before pursuing we define  $(d'_\tau, d''_\tau)$  as it follows:given

$$u \otimes v \in C_\tau^j(A, R) \otimes C_\tau^i(A^*, R),$$

$$d'_\tau(u \otimes v) = \delta_\tau(u) \otimes v,$$

$$d''_\tau(u \otimes v) = (-1)^j u \otimes \delta_\tau(v).$$

$$d''_\tau = 1 \otimes \delta_\tau$$

so that without any confusion one has

$$d_\tau = d'_\tau + d''_\tau.$$

To any couple  $(p \leq n)$  of positive integer we set

$$F_{A,n}^p(C_\tau(M)) = \bigoplus_{[j \leq p]} C_\tau^j(A, R) \otimes C_\tau^{n-j}(A^*, R),$$

$$F_{A^*,n}^p(C_\tau(M)) = \bigoplus_{[j \leq p]} C_\tau^{n-j}(A, R) \otimes C_\tau^j(A^*, R).$$

It is easy to verify the following claims,

$$F_{A,n}^p(C_\tau(M)) \subset F_{A,n}^{p+1}(C_\tau(M))$$

$$d''_\tau(F_{A,n}^p(C_\tau(M))) \subset F_{A,n+1}^p(C_\tau(M));$$

$$F_{A^*,n}^p(C_\tau(M)) \subset F_{A^*,n}^{p+1}(C_\tau(M)),$$

$$d'_\tau(F_{A^*,n}^p)(C_\tau(M)) \subset F_{A^*,n+1}^p(C_\tau(M)).$$

Each filtration yields a spectral sequence that we denote by

$$(E_r, d_r(A)),$$

$$E_r(A^*, d_r).$$

Consider the tensor

$$\Omega_\tau = \Omega(M) \otimes \Omega(M), d_R$$

where  $\Omega(M)$  is the de Rham complex of  $M$ .

*Remark,[3]*

The inclusion mapping

$$\Omega_\tau(M) \rightarrow C_\tau(M)$$

is a complex morphism.

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## 6.2. Singular persistence of a compact statistical manifold

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Let  $(M, g, \nabla, \nabla^*)$  be a compact statistical manifolds. We have involved solutions of gauge equation of  $(\nabla^*, \nabla)$  to obtain an homological persistence on  $M$ :

$$\subset \text{Sing}(F_{p+1}(x)) \subset \text{Sing}(F_p(x)) \subset \text{Sing}(F_{p-1}(x)) \subset$$

By canonical construction we deduce a short exact sequences

$$0 \rightarrow \text{Sing}(F^{p+1}(x)) \rightarrow \text{Sing}(F^p(x)) \rightarrow E^p(x) \rightarrow 0,$$

with

$$E^p(x) = \frac{\text{Sing}(F^p(x))}{\text{Sing}(F^{p+1}(x))}.$$

This sort exact sequence yields the following long exact sequence of singular homology spaces

$$\rightarrow H_{q+1}(F^{p+1}(x)) \rightarrow H_{q+1}(F^p(x)) \rightarrow H_{q+1}(E^p(x)) \rightarrow H_q(F^{p+1}(x)) \rightarrow$$

One uses the topology persistence to construct an homological exact couple whose spectral sequence converge to the singular homology  $H(M)$ .

By theorem of de Rham the approach leads to de Rham algebra of  $M$ .

Indeed by setting

$$M = \oplus_p H(F^p(x)),$$

$$E = \oplus_p H(E^p),$$

the long exact homology sequence above yields the exact couple

$$i : M \rightarrow M,$$

$$j : M \rightarrow E,$$

$$k : E \rightarrow M.$$

In section 5 it is explained how to construct inductively the derived exact couples. Here let us sketch another construction to be applied to the total complex of a Hessian structure  $(M, g, \nabla, \nabla^*)$ , namely

$$C_\tau(M)$$

which is bi-graded by the sub-spaces

$$C_\tau^{q,p}(M) = C_\tau^q(A, R) \otimes C_\tau^p(A^*)$$

We remind the filtration

$$F_{A,n}^p(C_\tau(M)) = \bigoplus_{[j \leq p]} C_\tau^j(A) \otimes C_\tau^{n-j}(A^*).$$

**At one side**, according to general process this filtration gives rise to an exact couple which yields a spectral sequence

$$E(A) = \{E_r^{j,i}\}$$

**At another side**, the operators  $d'_\tau$  and  $d''_\tau$  are defined as in 6.1.

Set

$$H''^{j,i}(C_\tau(M)) = \frac{\ker(d''_\tau : C_\tau^{j,i}(M) \rightarrow C_\tau^{j,i+1}(M))}{d''_\tau(C_\tau^{j,i-1}(M))},$$

and

$$H^j H''^i(C_\tau(M)) = \frac{\ker(d'_\tau : H''^{j,i}(C_\tau(M)) \rightarrow H''^{j+1,i}(C_\tau(M)))}{d'_\tau(H''^{j-1,i}(C_\tau(M)))}.$$

Now we are in position to implement some powerful well known machineries. 298

**Theorem 6.1.** *The term  $E_2^{j,i}$  of the spectral sequence  $E(A)$  is isomorphic to  $H^j H''^i(C_\tau(M))$ . ♣* 299

**Theorem 6.2.** *The spectral sequence  $E(A)$  converges to the total cohomology of the total complex*

$$(C_\tau(M), d_\tau) \clubsuit$$

For technical details readers are referred to [6], [7], [8]. 300

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