Introduction to Robust Control Lecture notes -Draft

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Chapter 1

System norms

1.1 Introduction

In this chapter the main norms which will be used in this lecture are summarized.

1.2 Norms of vector

Let $\underline{x}^T$ be the transpose of vector $\underline{x}$. The norm $N(\underline{x})$ of a vector $\underline{x} \in \mathbb{R}^n$ is a function $N : \mathbb{R}^n \to \mathbb{R}^+$ satisfying the following properties:

- $N(\underline{x}) \geq 0$
- $N(\underline{x}) = 0 \Leftrightarrow \underline{x} = \underline{0}$
- $N(\underline{x} + \underline{y}) \leq N(\underline{x}) + N(\underline{y})$
- $N(\alpha \underline{x}) \leq |\alpha| N(\underline{x}) \quad \forall \alpha \in \mathbb{R}$

Let $\underline{x} = \begin{bmatrix} x_1 \\ \cdot \\ x_n \end{bmatrix}$ be a vector of $\mathbb{R}^n$. Denoting by $|x_i|$ the absolute value of x_i the l_2 norm of vector $\underline{x}$ is defined by:

$$\|\underline{x}\|_2 = \sqrt{\underline{x}^T \underline{x}} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \tag{1.1}$$

The l_2 norm is a special case of a more general family of norms, called l_p norm where $p \in \mathbb{N}^+$. The l_p norm of vector $\underline{x}$ is defined by:

$$\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \tag{1.2}$$

The l_∞ norm of vector $\underline{x}$ is defined by:

$$\|\underline{x}\|_\infty = \max_i |x_i| = \lim_{p \to \infty} \|\underline{x}\|_p \tag{1.3}$$

1.3 Norms of vector signal

Let $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ \cdot \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$ be a vector depending of time $t \geq 0$. Vector $\underline{x}(t)$ is called a vector signal. Assuming that the following integral is defined its represents the L_2 norm of vector signal $\underline{x}(t)$:

$$\|\underline{x}(t)\|_2 = \sqrt{\int_0^\infty \underline{x}^T(t)\underline{x}(t)dt} < \infty \tag{1.4}$$

The L_2 norm of a vector signal is also called its energy. Vector signal $\underline{x}(t)$ is said of finite energy, or equivalently $\underline{x}(t) \in L_2[0, \infty)$ where $L_2[0, \infty)$ is the Hilbert space of finite energy signal defined $\forall t \geq 0$, as soon as the integral in Equation (1.4) exists.

Let $\underline{X}(s)$ be the Laplace transform of vector signal $\underline{x}(t)$. As soon as the vector signal $\underline{x}(t) \in L_2$ its Laplace transform $\underline{X}(s)$ is strictly proper and all its poles have negative real part (or equivalently $\underline{x}(t) \in L_2[0, \infty) \Leftrightarrow \underline{X}(s) \in H_2$ where notation H is used after the mathematician G.H. Hardy). Let $\mathcal{C}^-$ be a contour following the imaginary axis of the complex plane and closed through a semicircle with infinite radius in the left half plane. By Parseval's theorem we have:

$$\begin{array}{rl} \|\underline{x}(t)\|_2 & = \sqrt{\frac{1}{2\pi j} \oint_{\mathcal{C}^-} \underline{X}^T(-s)\underline{X}(s)ds} \\ & \underbrace{=}_{s=j\omega} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{X}^T(-j\omega)\underline{X}(j\omega)d\omega} \end{array} \tag{1.5}$$

The use of the residue theorem finally leads to the following expression:

$$\|\underline{x}(t)\|_2 = \sqrt{\sum_{poles\ of\ X(s)} Res\left(\underline{X}^T(-s)\underline{X}(s)\right)} \tag{1.6}$$

Let λ_i be a pole (with negative real part) of $\underline{X}(s)$ with multiplicity n_i. Residue $Res\left(\underline{X}^T(-s)\underline{X}(s)\right)$ on pole λ_i is defined as follows:

$$Res_{s=\lambda_i}\left(\underline{X}^T(-s)\underline{X}(s)\right) = \frac{1}{(n_i-1)!}\frac{d^{n_i-1}}{ds^{n_i-1}}(s-\lambda_i)^{n_i}\underline{X}^T(-s)\underline{X}(s)\Big|_{s=\lambda_i} \tag{1.7}$$

The L_∞ norm of vector signal $\underline{x}(t)$ is defined by:

$$\|\underline{x}(t)\|_\infty = \sup_{t\in\mathbb{R}^+} \max_i |\underline{x}_i(t)| \tag{1.8}$$

Example 1.1. *Let $\underline{x}$) be defined as follows where τ is a positive time constant:*

$$\underline{x} = e^{-\frac{t}{\tau}} \quad \forall t \geq 0 \tag{1.9}$$

The square of the L_2 norm of signal $\underline{x}(t)$ is:

$$\begin{array}{rl} \|\underline{x}(t)\|_2^2 & = \int_0^\infty \underline{x}^T(t)\underline{x}(t)dt \\ & = \int_0^\infty e^{-\frac{2t}{\tau}}dt \\ & = \frac{\tau}{2}e^{-\frac{2t}{\tau}}\Big|_0^\infty \\ & = \frac{\tau}{2} \end{array} \tag{1.10}$$

And consequently:

$$\|\underline{x}(t)\|_2 = \sqrt{\frac{\tau}{2}} \tag{1.11}$$

The square of the L_2 norm of signal $\underline{x}(t)$ can equivalently be computed using the Laplace transform $\underline{X}(s)$ of $\underline{x}(t)$:

$$\begin{array}{l} \underline{X}(s) = \frac{1}{s+\frac{1}{\tau}} = \frac{\tau}{1+s\tau} \\ \Rightarrow \underline{X}^T(-s)\underline{X}(s) = \frac{\tau^2}{(1+s\tau)(1-s\tau)} \end{array} \tag{1.12}$$

Assuming $\tau > 0$ the unique pole $\lambda_1 = -\frac{1}{\tau}$ of $\underline{X}(s)$ has a negative real part and its multiplicity n_1 is equal to 1. Then residue of $\underline{X}^T(-s)\underline{X}(s)$ on pole $\lambda_1 = -\frac{1}{\tau}$ is computed as follows:

$$\begin{array}{ll} Res_{s=-\frac{1}{\tau}}\left(\underline{X}^T(-s)\underline{X}(s)\right) & = \frac{1}{(1-1)!}\frac{d^{1-1}}{ds^{1-1}}(s-\lambda_1)^1\underline{X}^T(-s)\underline{X}(s)\Big|_{s=-\frac{1}{\tau}} \\ & = (s+\frac{1}{\tau})\frac{\tau^2}{(1+s\tau)(1-s\tau)}\Big|_{s=-\frac{1}{\tau}} \\ & = \frac{\tau}{1-s\tau}\Big|_{s=-\frac{1}{\tau}} \\ & = \frac{\tau}{2} \end{array} \tag{1.13}$$

We finally get:

$$\|\underline{x}(t)\|_2 = \sqrt{\sum_{poles\ of\ X(s)} Res\left(\underline{X}^T(-s)\underline{X}(s)\right)} = \sqrt{\frac{\tau}{2}} \tag{1.14}$$

The L_∞ norm of signal $\underline{x}(t)$ is defined by:

$$\|\underline{x}(t)\|_\infty = \sup_{t\in\mathbb{R}^+} |e^{-\frac{t}{\tau}}| = 1 \tag{1.15}$$

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Example 1.2. *Let $\underline{x}(t)$ be defined as follows:*

$$\underline{x}(t) = \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} \quad \forall t \geq 0 \tag{1.16}$$

The square of the L_2 norm of signal $\underline{x}(t)$ is:

$$\begin{array}{ll} \|\underline{x}(t)\|_2^2 & = \int_0^\infty \underline{x}^T(t)\underline{x}(t)dt \\ & = \int_0^\infty e^{-2t} + e^{-4t}dt \\ & = \frac{e^{-2t}}{-2} + \frac{e^{-4t}}{-4}\Big|_0^\infty \\ & = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \end{array} \tag{1.17}$$

And consequently:

$$\|\underline{x}(t)\|_2 = \sqrt{\frac{3}{4}} \tag{1.18}$$

The square of the L_2 norm of signal $\underline{x}(t)$ can equivalently be computed using the Laplace transform $\underline{X}(s)$ of $\underline{x}(t)$:

$$\underline{X}(s) = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{bmatrix} \Rightarrow \underline{X}^T(-s)\underline{X}(s) = \frac{1}{(-s+1)(s+1)} + \frac{1}{(-s+2)(s+2)} \tag{1.19}$$

$\underline{X}(s)$ has two poles with negative real part, namely $\lambda_1 = -1$ and $\lambda_2 = -2$, each pole having a multiplicity equal to 1.

– *Residue of $\underline{X}^T(-s)\underline{X}(s)$ on pole $\lambda_1 = -1$ with multiplicity $n_1 = 1$ is computed as follows:*

$$\begin{array}{ll} Res_{s=-1}\left(\underline{X}^T(-s)\underline{X}(s)\right) & = \frac{1}{(1-1)!}\frac{d^{1-1}}{ds^{1-1}}(s-\lambda_1)^1 \underline{X}^T(-s)\underline{X}(s)\Big|_{s=-1} \\ & = (s+1)\left(\frac{1}{(-s+1)(s+1)} + \frac{1}{(-s+2)(s+2)}\right)\Big|_{s=-1} \\ & = \frac{1}{2} \end{array} \tag{1.20}$$

– *Residue of $\underline{X}^T(-s)\underline{X}(s)$ on pole $\lambda_1 = -2$ with multiplicity $n_2 = 1$ is computed as follows:*

$$\begin{array}{ll} Res_{s=-2}\left(\underline{X}^T(-s)\underline{X}(s)\right) & = \frac{1}{(1-1)!}\frac{d^{1-1}}{ds^{1-1}}(s-\lambda_2)^1 \underline{X}^T(-s)\underline{X}(s)\Big|_{s=-2} \\ & = (s+2)\left(\frac{1}{(-s+1)(s+1)} + \frac{1}{(-s+2)(s+2)}\right)\Big|_{s=-2} \\ & = \frac{1}{4} \end{array} \tag{1.21}$$

We finally get:

$$\begin{array}{ll} \|\underline{x}(t)\|_2 & = \sqrt{\sum_{poles\ of\ X(s)} Res\left(\underline{X}^T(-s)\underline{X}(s)\right)} \\ & = \sqrt{\frac{1}{2}+\frac{1}{4}} \\ & = \sqrt{\frac{3}{4}} \end{array} \tag{1.22}$$

The L_∞ norm of signal $\underline{x}(t)$ is defined by:

$$\begin{array}{ll} \|\underline{x}(t)\|_\infty & = \sup_{t\in\mathbb{R}^+} \max_i |\underline{x}_i(t)| \\ & = \sup_{t\in\mathbb{R}^+} \max\left\{e^{-t}, e^{-2t}\right\} \\ & = \sup_{t\in\mathbb{R}^+} e^{-t} \\ & = 1 \end{array} \tag{1.23}$$

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1.4 Frobenius norm

A Hermitian matrix $\mathbf{Q} = \mathbf{Q}^T$ is said to be positive semidefinite or nonnegative definite, written as $\mathbf{Q} \geq 0$, when the following relation holds:

$$\underline{x}^T \mathbf{Q} \underline{x} \geq 0 \ \forall \underline{x} \in \mathbb{R}^n \tag{1.24}$$

The trace of a square matrix $\mathbf{Q}$, which is denoted $\mathrm{tr}(\mathbf{Q})$, is the sum of its main diagonal entries, or, equivalently, the sum of its eigenvalues. Let $\mathbf{Q}$ be any $n \times m$ matrix. The Frobenius norm $\|\mathbf{Q}\|_F$ of matrix $\mathbf{Q}$ is defined by the square root of the sum of the absolute squares of its elements:

$$\mathbf{Q} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \Rightarrow \|\mathbf{Q}\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}|^2} \tag{1.25}$$

The Frobenius norm $\|\mathbf{Q}\|_F$ of matrix $\mathbf{Q}$ is equal to the square root of the trace of matrix $\mathbf{Q}\mathbf{Q}^T$:

$$\|\mathbf{Q}\|_F = \sqrt{\mathrm{tr}\left(\mathbf{Q}\mathbf{Q}^T\right)} \tag{1.26}$$

Finally, the Frobenius inner product is defined in $\mathbb{R}^{m\times m}$ by $\mathrm{tr}\left(\mathbf{Q}^T\mathbf{Q}\right)$.

More generally it can be shown that $\mathrm{tr}\left(f\left(\mathbf{A}\mathbf{B}^2\mathbf{A}\right)\right) = \mathrm{tr}\left(f\left(\mathbf{B}\mathbf{A}^2\mathbf{B}\right)\right)$ for any function $f : \mathbb{R} \to \mathbb{R}$.

1.5 Singular Value Decomposition

Let $\lambda(\mathbf{A})$ be the set of eigenvalues of square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{\Lambda}$ the diagonal matrix formed by the eigenvalues of $\mathbf{A}$ and $\mathbf{X}$ the matrix formed by the right eigenvectors of $\mathbf{A}$ and. We have by definition of eigenvalues and right eigenvectors:

$$\begin{array}{c} \mathbf{A}\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_n \end{bmatrix} = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \\ \Leftrightarrow \mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{\Lambda} \\ \Leftrightarrow \mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^H \end{array} \tag{1.27}$$

In the following $\mathbf{U}^H$ stands for complex conjugate transpose of matrix $\mathbf{U}$. Matrix $\mathbf{U}$ is said unitary when the following relation holds:

$$\mathbf{U} = \begin{bmatrix} \underline{u}_1 & \cdots & \underline{u}_m \end{bmatrix} \in \mathbb{C}^{m\times m}, \quad \mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbb{I} \tag{1.28}$$

Now let $\mathbf{A} \in \mathbb{R}^{m\times p}$. Then there exists unitary matrices $\mathbf{U}$ and $\mathbf{V}$ such that the following decomposition of matrix $\mathbf{A}$ holds:

$$\left\{ \begin{array}{l} \mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H \\ \mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Lambda}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \mathbf{\Lambda}_q = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & 0 & \sigma_q \end{bmatrix} \end{array} \right. \tag{1.29}$$

where:

$$\left\{\begin{array}{l} \sigma_i \in \mathbb{R}^+ \quad \forall i = 1, 2, \cdots, q \\ \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q > 0 \\ q = \min(m, p) \text{ assuming that } \mathbf{A} \text{ has no eigenvalue equal to } 0 \end{array}\right. \tag{1.30}$$

The preceding decomposition of matrix $\mathbf{A}$ is called the Singular Value Decomposition (SVD) of matrix $\mathbf{A}$. From the preceding definition we get the following properties:

$$\left\{\begin{array}{l} \mathbf{A}\mathbf{A}^H = \left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H\right)\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H\right)^H = \mathbf{U}\left[\begin{array}{cc} \boldsymbol{\Lambda}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{array}\right]\mathbf{U}^H \\ \mathbf{A}^H\mathbf{A} = \left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H\right)^H\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H\right) = \mathbf{V}\left[\begin{array}{cc} \boldsymbol{\Lambda}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{array}\right]\mathbf{V}^H \end{array}\right. \tag{1.31}$$

The preceding equations indicate that $\mathbf{U}$ is the matrix formed by the right eigenvectors of $\mathbf{A}\mathbf{A}^H$ whereas $\mathbf{V}$ is the matrix formed by the right eigenvectors of $\mathbf{A}^H\mathbf{A}$. Moreover $\boldsymbol{\Lambda}^2$ is composed of the non-zero eigenvalues of either $\mathbf{A}\mathbf{A}^H$ or $\mathbf{A}^H\mathbf{A}$.

Singular-values of matrix $\mathbf{A}$ are defined as the root square of the non-zero eigenvalues of either $\mathbf{A}^H\mathbf{A}$ or $\mathbf{A}\mathbf{A}^H$. The set $\sigma_i(\mathbf{A})$ is the set of singular-values of matrix $\mathbf{A}$ and we equivalently have:

$$\sigma_i(\mathbf{A}) = \sqrt{\lambda_i(\mathbf{A}^H\mathbf{A})|_{\lambda_i \neq 0}} = \sqrt{\lambda_i(\mathbf{A}\mathbf{A}^H)|_{\lambda_i \neq 0}} \tag{1.32}$$

Geometrically, the singular-values of a matrix $\mathbf{A}$ are the lengths of the semi-axes of the hyper-ellipsoid E defined by:

$$E = \{\underline{y} : \underline{y} = \mathbf{A}\underline{x}, \underline{x} \in \mathbb{R}^n, \|\underline{x}\|_2 = 1\} \tag{1.33}$$

Consider a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and its singular-value decomposition $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H$ where $\sigma_1 \geq \cdots \geq \sigma_n > 0$. Then the inverse of $\mathbf{A}$ only requires calculating the inverse of n real numbers since:

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H \Rightarrow \mathbf{A}^{-1} = \mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{U}^H = \mathbf{V}\left[\begin{array}{cccc} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_n} \end{array}\right]\mathbf{U}^H \tag{1.34}$$

Example 1.3. *Let* $\mathbf{A}$ *be defined as follows:*

$$\mathbf{A} = \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -9 \end{array}\right] \tag{1.35}$$

It is clear that the root square of the non-zero eigenvalues of either $\mathbf{A}^H\mathbf{A}$ *or* $\mathbf{A}\mathbf{A}^H$ *are* $\{1, 2, 9\}$*: those are the singular-values* $\sigma_i(\mathbf{A})$ *of* $\mathbf{A}$.

Furthermore it can be shown that the singular-value decomposition of $\mathbf{A}$ *is the following:*

$$\left\{ \begin{array}{l} \mathbf{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ \mathbf{\Sigma} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \mathbf{V} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{array} \right. \tag{1.36}$$

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1.6 Induced norms

As far as matrices can be viewed as a linear operators, the norm of a matrix is defined by extending the l_p-norm of a vector to matrices. The norm is said to be an *induced norm* since it depends on the choice of the vector norm.

In addition to the four norms properties of section 1.2 and in the case of square matrices induced matrix norms satisfy the following multiplicative property (also called the consistency condition):

$$\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\| \tag{1.37}$$

The p-norm (induced by vector l_p-norm) of a matrix $\mathbf{A}$ is defined by:

$$\|\mathbf{A}\|_p = \max_{\underline{x} \neq \underline{0}} \frac{\|\mathbf{A}\underline{x}\|_p}{\|\underline{x}\|_p} \tag{1.38}$$

It can be shown that the 2-norm of matrix $\mathbf{A}$ is the largest singular-value of $\mathbf{A}$, which is denoted $\overline{\sigma}(\mathbf{A})$:

$$\|\mathbf{A}\|_2 = \max_i \sigma_i(\mathbf{A}) = \overline{\sigma}(\mathbf{A}) = \max_{\|\underline{x}\|_2=1} \|\mathbf{A}\underline{x}\|_2 = \max_{\underline{x} \neq \underline{0}} \frac{\|\mathbf{A}\underline{x}\|_2}{\|\underline{x}\|_2} \tag{1.39}$$

For the smallest singular-value $\underline{\sigma}(\mathbf{A})$ of matrix $\mathbf{A}$ we have the following property:

$$\underline{\sigma}(\mathbf{A}) = \min_{\|\underline{x}\|_2=1} \|\mathbf{A}\underline{x}\|_2 = \min_{\underline{x} \neq \underline{0}} \frac{\|\mathbf{A}\underline{x}\|_2}{\|\underline{x}\|_2} \tag{1.40}$$

The inverse of the smallest singular-value of $\mathbf{A}$ is the 2-norm of matrix $\mathbf{A}^{-1}$:

$$\|\mathbf{A}^{-1}\|_2 = \frac{1}{\min_i \sigma_i(\mathbf{A})} = \frac{1}{\underline{\sigma}(\mathbf{A})} \tag{1.41}$$

The condition number $\kappa(\mathbf{A})$ of matrix $\mathbf{A}$ is defined as the ratio between the maximum singular-value and the minimum singular-value of $\mathbf{A}$. We get from the preceding properties:

$$\kappa(\mathbf{A}) = \frac{\sigma_1}{\sigma_n} = \frac{\overline{\sigma}(\mathbf{A})}{\underline{\sigma}(\mathbf{A})} = \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2 \tag{1.42}$$

If $\kappa(\mathbf{A})$ is large the matrix is said to be ill-conditioned and it may be difficult to numerically compute its inverse or pseudo inverse.

Let $\mathbf{A}$ be a matrix (non necessarily square) written as follows:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \tag{1.43}$$

It can be shown that:

– The 1-norm of matrix $\mathbf{A}$ is the so called maximum *column* sum and can be computed as follows:

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \tag{1.44}$$

– The ∞-norm of matrix $\mathbf{A}$ is the so called maximum *row* sum and can be computed as follows:

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{m} |a_{ij}| \tag{1.45}$$

Notice that not all matrix norms are induced norms. An example is the Frobenius norm. Indeed $\|\mathbb{I}_n\|_p = 1$ for any induced p-norm whereas $\|\mathbb{I}_n\|_F = \sqrt{n}$.

Example 1.4. *Let* $\mathbf{A}$ *be defined as follows:*

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -9 \end{bmatrix} \tag{1.46}$$

We have seen that the singular-values $\sigma_i(\mathbf{A})$ *of* $\mathbf{A}$ *are* $\{1, 2, 9\}$. *The* 2*-norm of matrix* $\mathbf{A}$ *is the largest singular-value of* $\mathbf{A}$*:*

$$\|\mathbf{A}\|_2 = \max_i \sigma_i(\mathbf{A}) = \overline{\sigma}(\mathbf{A}) = 9 \tag{1.47}$$

The ∞*-norm of matrix* $\mathbf{A}$ *is the maximum row sum:*

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| = \max(1, 2, 9) = 9 \tag{1.48}$$

■

Example 1.5. *Let* $\mathbf{A}$ *be defined as follows:*

$$\mathbf{A} = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix} \tag{1.49}$$

Matrix $\mathbf{\Sigma}$ *of the singular-value decomposition of* $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$ *is:*

$$\mathbf{\Sigma} = \begin{bmatrix} 7.1476828 & 0 & 0 \\ 0 & 3.5440075 & 0 \\ 0 & 0 & 0.5921495 \end{bmatrix} \tag{1.50}$$

The 2*-norm of matrix* $\mathbf{A}$ *is the largest singular-value of* $\mathbf{A}$*:*

$$\|\mathbf{A}\|_2 = \max_i \sigma_i(\mathbf{A}) = \overline{\sigma}(\mathbf{A}) = 7.1476828 \tag{1.51}$$

The ∞*-norm of matrix* $\mathbf{A}$ *is the maximum row sum:*

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| = \max(11, 6, 3) = 11 \tag{1.52}$$

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1.7 H_2-norm of stable plants

1.7.1 H_2-norm definition

Let $\mathbf{F}(s)$ be a strictly proper transfer matrix of a stable (i.e. all the eigenvalues of the matrix lies in $\mathbb{C}^-$) linear time invariant system. The set $\mathcal{RH}_\infty$ is the set of all rational stable and proper (but not necessarily strictly proper) transfer matrices. We will denote by $\mathbf{g}(t)$ the impulse response of the system and by $\mathcal{L}[\mathbf{g}(t)]$ the Laplace transform of $\mathbf{g}(t)$:

$$\mathbf{F}(s) = \mathcal{L}[\mathbf{g}(t)] \Leftrightarrow \mathbf{g}(t) = \mathcal{L}^{-1}[\mathbf{F}(s)] \tag{1.53}$$

The H_2-norm of transfer matrix $\mathbf{F}(s)$ is defined through the impulse response $\mathbf{g}(t)$ of the system:

$$\begin{array}{rl} \|\mathbf{F}(s)\|_2 & = \sqrt{\int_0^\infty \operatorname{tr}\left(\mathbf{g}^T(t)\mathbf{g}(t)\right) dt} \\ & = \sqrt{\int_0^\infty \operatorname{tr}\left(\mathbf{g}(t)\mathbf{g}^T(t)\right) dt} \end{array} \tag{1.54}$$

Let's write matrix $\mathbf{g}(t)$ by column as:

$$\mathbf{g}(t) = \begin{bmatrix} \underline{g}_1(t) & \underline{g}_2(t) & \cdots & \underline{g}_m(t) \end{bmatrix} \tag{1.55}$$

Then we get:

$$\|\mathbf{F}(s)\|_2 = \sqrt{\sum_{i=1}^{m} \int_0^\infty \underline{g}_i^T(t)\underline{g}_i(t)dt} \tag{1.56}$$

From the preceding definition it is clear that the H_2-norm of transfer matrix $\mathbf{F}(s)$ is the L_2-norm of the sum of output signal $\underline{g}_i^T(t)$ when impulses (or Delta Dirac functions) are applied to each input channels.

The H_2-norm of transfer matrix $\mathbf{F}(s)$ can be equivalently defined by using the Parseval's theorem. Let $\mathcal{C}^-$ be a contour following the imaginary axis of the complex plane and closed through a semicircle with infinite radius in the left half plane. By Parseval's theorem we get:

$$\begin{aligned}
\|\mathbf{F}(s)\|_2 &= \sqrt{\tfrac{1}{2\pi j}\oint_{\mathcal{C}^-} \operatorname{tr}\left(\mathbf{F}^T(-s)\mathbf{F}(s)\right) ds} \\
&= \sqrt{\tfrac{1}{2\pi j}\oint_{\mathcal{C}^-} \operatorname{tr}\left(\mathbf{F}(s)\mathbf{F}^T(-s)\right) ds} \\
&\underbrace{=}_{s=j\omega} \sqrt{\tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \operatorname{tr}\left(\mathbf{F}(j\omega)\mathbf{F}^T(-j\omega)\right) d\omega} \\
&= \sqrt{\tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \operatorname{tr}\left(\mathbf{F}^T(-j\omega)\mathbf{F}(j\omega)\right) d\omega}
\end{aligned} \tag{1.57}$$

Using the definition of singular-values the H_2-norm of transfer matrix $\mathbf{F}(s)$ can also be computed using its singular-values:

$$\begin{aligned}
\|\mathbf{F}(s)\|_2 &= \sqrt{\tfrac{1}{2\pi j}\sum_{i=1}^{q}\oint_{\mathcal{C}^-} \sigma_i\left(\mathbf{F}\left(s\right)\right) ds} \\
&= \sqrt{\tfrac{1}{2\pi}\sum_{i=1}^{q}\int_{-\infty}^{\infty} \sigma_i\left(\mathbf{F}\left(j\omega\right)\right) d\omega}
\end{aligned} \tag{1.58}$$

As soon as $\mathbf{F}(s)$ is strictly proper and all its poles have negative real part its H_2-norm exists. The use of the residue theorem leads to the following expression:

$$\begin{aligned}
\|\mathbf{F}(s)\|_2 &= \sqrt{\sum_{poles\ of\ F(s)} Res\left(\operatorname{tr}\left(\mathbf{F}^T(-s)\mathbf{F}(s)\right)\right)} \\
&= \sqrt{\sum_{poles\ of\ F(s)} Res\left(\operatorname{tr}\left(\mathbf{F}(s)\mathbf{F}^T(-s)\right)\right)}
\end{aligned} \tag{1.59}$$

Another interpretation of the H_2-norm is the following: assume that $\underline{y}(s) = \mathbf{F}(s)\,\underline{u}(s)$, where $\underline{y}(s) := \mathcal{L}\left[\underline{y}(t)\right]$ is the Laplace transform of the output signal $\underline{y}(t)$ of the linear system and $\underline{u}(s) := \mathcal{L}\left[\underline{u}(t)\right]$ the Laplace transform of its input signal $\underline{u}(t)$. Then $\|\mathbf{F}(s)\|_2^2$ represents the steady-state variance of the output signal $\underline{y}(t)$ when each component of $\underline{u}(t)$ is a stochastic white noise with unit covariance, i.e. $E\left[\underline{u}(t)\underline{u}^T(\tau)\right] = \delta(t-\tau)\mathbb{I}$.

It is worth noticing that the H_2-norm is not an induced norm because it does not satisfy the multiplicative property (1.37). This implies that the H_2-norm does not provide any information on how the series (cascade) interconnection will behave[1].

Example 1.6. *To calculate the H_2-norm of $F(s) = 1/(s+a)$ where $a > 0$ (indeed $F(s)$ shall be stable to compute its H_2-norm) we use the fact that:*

$$g(t) = \mathcal{L}^{-1}\left[\mathbf{F}(s)\right] = \mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = \exp\left(-at\right) \tag{1.60}$$

and hence:

$$\|\mathbf{F}(s)\|_2^2 = \int_0^{\infty} \exp\left(-2at\right) dt = \frac{1}{2a} \Rightarrow \|\mathbf{F}(s)\|_2 = \sqrt{\frac{1}{2a}} \text{ where } a > 0 \tag{1.61}$$

[1] Multivariable Feedback Control: Analysis and Design, Sigurd Skogestad & Ian Postlethwaite, Wiley-Interscience 2005

For this example, it is clear that the H_2-norm does not satisfy the multiplicative property (1.37)*. Indeed we have:*

$$\begin{array}{ll} \|\mathbf{F}(s)\mathbf{F}(s)\|_2^2 & = \|\mathbf{F}^2(s)\|_2 \\ & = \int_0^\infty \left(\mathcal{L}^{-1}\left[\mathbf{F}^2(s)\right]\right)^2 dt \\ & = \int_0^\infty \left(\mathcal{L}^{-1}\left[\frac{1}{(s+a)^2}\right]\right)^2 dt \\ & = \int_0^\infty \left(t\,e^{-at}\right)^2 dt \\ & = \int_0^\infty t^2\,e^{-2at} dt \end{array} \tag{1.62}$$

Once the integration by parts is accomplished, we finally get:

$$\|\mathbf{F}(s)\mathbf{F}(s)\|_2^2 = \frac{1}{4a^3} \Rightarrow \|\mathbf{F}(s)\mathbf{F}(s)\|_2 = \frac{1}{2a\sqrt{a}} \tag{1.63}$$

Thus, we conclude that for $0 < a < 1$ *the multiplicative property* (1.37) *does not hold. Indeed:*

$$0 < a < 1 \Rightarrow \|\mathbf{F}(s)\mathbf{F}(s)\|_2 = \frac{1}{2a\sqrt{a}} > \frac{1}{2a} = \|\mathbf{F}(s)\|_2\|\mathbf{F}(s)\|_2 \tag{1.64}$$

■

Example 1.7. *We wish to calculate the H_2-norm of a second order system whose input-output relation reads:*

$$\frac{1}{\omega_n^2}\ddot{y}(t) + \frac{2\zeta}{\omega_n}\dot{y}(t) + y(t) = Ku(t) \tag{1.65}$$

where ζ is the damping ratio ($\zeta \geq 0$), ω_n the natural frequency ($\omega_n \geq 0$) and K the static gain.

Taking the Laplace transform of the preceding input-output relation (without initial conditions) leads to the transfer function $F(s)$ of the system:

$$F(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{1.66}$$

We will assume in the following that the damping ratio ζ is lower than 1 *such that the* 2 *poles λ_1 and λ_2 of the transfer function $F(s)$ are complex conjugates:*

$$\left\{ \begin{array}{l} \lambda_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \\ \lambda_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2} \end{array} \right. \tag{1.67}$$

Consequently transfer function $F(s)$ reads:

$$F(s) = \frac{K\omega_n^2}{(s-\lambda_1)(s-\lambda_2)} \tag{1.68}$$

Rather than computing the H_2-norm thanks to the impulse response $g(t) = \mathcal{L}^{-1}[\mathbf{F}(s)]$ we will use the residue theorem. Indeed th transfer function $F(s)$ is *strictly proper and all its poles have negative real part thus its H_2-norm exists and the residue theorem can be applied. This leads to the following expression:*

$$\|F(s)\|_2 = \sqrt{\sum_{poles\ of\ F(s)} Res\left(F^T(-s)F(s)\right)} \tag{1.69}$$

where:

$$Res_{s=\lambda_i}\left(F^T(-s)F(s)\right) = \frac{1}{(n_i-1)!}\frac{d^{n_i-1}}{ds^{n_i-1}}(s-\lambda_i)^{n_i}F^T(-s)F(s)\Big|_{s=\lambda_i} \tag{1.70}$$

The multiplicity of pole λ_1 is $n_1 = 1$. We get:

$$\begin{array}{ll} Res_{s=\lambda_1}\left(F^T(-s)F(s)\right) & = (s-\lambda_1)F^T(-s)F(s)\big|_{s=\lambda_1} \\ & = \cancel{(s-\lambda_1)}\frac{K\omega_n^2}{(-s-\lambda_1)(-s-\lambda_2)}\frac{K\omega_n^2}{\cancel{(s-\lambda_1)}(s-\lambda_2)}\Big|_{s=\lambda_1} \\ & = \frac{(K\omega_n^2)^2}{(-\lambda_1-\lambda_1)(-\lambda_1-\lambda_2)(\lambda_1-\lambda_2)} \\ & = \frac{(K\omega_n^2)^2}{2\lambda_1(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)} \end{array} \tag{1.71}$$

Similarly the multiplicity of pole λ_2 is $n_2 = 1$. We get:

$$\begin{array}{ll} Res_{s=\lambda_2}\left(F^T(-s)F(s)\right) & = (s-\lambda_2)F^T(-s)F(s)\big|_{s=\lambda_2} \\ & = \cancel{(s-\lambda_2)}\frac{K\omega_n^2}{(-s-\lambda_1)(-s-\lambda_2)}\frac{K\omega_n^2}{(s-\lambda_1)\cancel{(s-\lambda_2)}}\Big|_{s=\lambda_2} \\ & = \frac{(K\omega_n^2)^2}{(-\lambda_2-\lambda_1)(-\lambda_2-\lambda_2)(\lambda_2-\lambda_1)} \\ & = \frac{(K\omega_n^2)^2}{2\lambda_2(\lambda_2+\lambda_1)(\lambda_2-\lambda_1)} \end{array} \tag{1.72}$$

Using the fact that λ_2 is the complex conjugate of λ_1, $\lambda_2 = \lambda_1^*$, we get:

$$Res_{s=\lambda_2}\left(F^T(-s)F(s)\right) = \left(Res_{s=\lambda_1}\left(F^T(-s)F(s)\right)\right)^* \tag{1.73}$$

Denoting by Re the real part of a complex number we get:

$$\begin{array}{ll} \|F(s)\|_2^2 & = \sum_{poles\ of\ F(s)} Res\left(F^T(-s)F(s)\right) \\ & = 2Re\left(Res_{s=\lambda_1}\left(F^T(-s)F(s)\right)\right) \\ & = \cancel{2}Re\left(\frac{(K\omega_n^2)^2}{\cancel{2}\lambda_1(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)}\right) \end{array} \tag{1.74}$$

Let's multiply the numerator and the denominator by $\lambda_2 = \lambda_1^*$ such that $\lambda_1\lambda_2 = |\lambda_1|^2 = \omega_n^2$:

$$\begin{array}{ll} \|F(s)\|_2^2 & = Re\left(\frac{\left(K\omega_n^2\right)^2\lambda_2}{\lambda_1\lambda_2(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)}\right) \\ & = Re\left(\frac{\left(K\omega_n^2\right)^2\lambda_2}{\omega_n^2(-2\zeta\omega_n)\left(2j\omega_n\sqrt{1-\zeta^2}\right)}\right) \\ & = Re\left(\frac{K^2\left(-\zeta\omega_n-j\omega_n\sqrt{1-\zeta^2}\right)}{(-2\zeta)\left(2j\omega_n\sqrt{1-\zeta^2}\right)}\right) \\ & = Re\left(\frac{K^2\left(-j\zeta\omega_n+\omega_n\sqrt{1-\zeta^2}\right)}{2\zeta\left(2\omega_n\sqrt{1-\zeta^2}\right)}\right) \\ & = \frac{K^2}{4}\frac{\omega_n}{\zeta} \end{array} \tag{1.75}$$

and hence:

$$\|\mathbf{F}(s)\|_2 = \frac{|K|}{2}\sqrt{\frac{\omega_n}{\zeta}} \tag{1.76}$$

■

1.7.2 H_2-norm and grammians

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ be a minimal realization of a stable and strictly proper transfer matrix $\mathbf{F}(s)$:

$$\mathbf{F}(s) = \mathbf{C}\left(s\mathbb{I} - \mathbf{A}\right)^{-1}\mathbf{B} \Leftrightarrow \mathbf{g}(t) = \mathbf{C}\exp\left(\mathbf{A}t\right)\mathbf{B} \tag{1.77}$$

Then it can be shown that the square of the H_2-norm of transfer matrix $\mathbf{F}(s)$ can be computed as follows:

$$\begin{array}{ll}\|\mathbf{F}(s)\|_2^2 & = \int_0^\infty \operatorname{tr}\left(\mathbf{g}(t)\mathbf{g}^T(t)\right)dt \\ & = \int_0^\infty \operatorname{tr}\left(\mathbf{g}^T(t)\mathbf{g}(t)\right)dt \\ & = \int_0^\infty \operatorname{tr}\left(\mathbf{B}^T\exp\left(\mathbf{A}^Tt\right)\mathbf{C}^T\mathbf{C}\exp\left(\mathbf{A}t\right)\mathbf{B}\right)dt \\ & = \operatorname{tr}\left(\mathbf{B}^T\mathbf{W}_o\mathbf{B}\right) \\ & = \operatorname{tr}\left(\mathbf{C}\mathbf{W}_c\mathbf{C}^T\right)\end{array} \tag{1.78}$$

Where $\mathbf{W}_o$ and $\mathbf{W}_c$ are the observability grammian and the controllability grammian respectively. They are defined as follows:

$$\mathbf{W}_o = \mathbf{W}_o^T = \int_0^\infty \exp(\mathbf{A}^Tt)\mathbf{C}^T\mathbf{C}\exp(\mathbf{A}t)dt > 0 \tag{1.79}$$

And:

$$\mathbf{W}_c = \mathbf{W}_c^T = \int_0^\infty \exp(\mathbf{A}t)\mathbf{B}\mathbf{B}^T\exp(\mathbf{A}^Tt)dt > 0 \tag{1.80}$$

Matrices $\mathbf{W}_o$ and $\mathbf{W}_c$ are the solutions of the following Lyapunov equations:

$$\mathbf{A}^T\mathbf{W}_o + \mathbf{W}_o\mathbf{A} + \mathbf{C}^T\mathbf{C} = \mathbf{0} \tag{1.81}$$

And:

$$\mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{0} \tag{1.82}$$

Indeed when $\mathbf{A}$ is an Hurwitz matrix we have:

$$\mathbf{g}(t) = \mathbf{C}\exp(\mathbf{A}t)\mathbf{B} \tag{1.83}$$

Thus:

$$\begin{array}{ll}\|\mathbf{F}(s)\|_2^2 & = \int_0^\infty \operatorname{tr}\left(\mathbf{g}^T(t)\mathbf{g}(t)\right)dt \\ & = \operatorname{tr}\left(\int_0^\infty \mathbf{B}^T\exp(\mathbf{A}^Tt)\mathbf{C}^T\mathbf{C}\exp(\mathbf{A}t)\mathbf{B}dt\right) \\ & = \operatorname{tr}\left(\mathbf{B}^T\left(\int_0^\infty \exp(\mathbf{A}^Tt)\mathbf{C}^T\mathbf{C}\exp(\mathbf{A}t)dt\right)\mathbf{B}\right) \\ & = \operatorname{tr}\left(\mathbf{B}^T\mathbf{W}_o\mathbf{B}\right)\end{array} \tag{1.84}$$

where $\mathbf{W}_o$ is the observability grammian:

$$\mathbf{W}_o = \int_0^\infty \exp(\mathbf{A}^Tt)\mathbf{C}^T\mathbf{C}\exp(\mathbf{A}t)dt \tag{1.85}$$

When multiplying $\mathbf{W}_o$ by $\exp(\mathbf{A}^T t_0)$ on the left and by $\exp(\mathbf{A} t_0)$ on the right, we get $\forall\ t_0 \in \mathbb{R}$:

$$\begin{array}{ll} \exp(\mathbf{A}^T t_0)\mathbf{W}_o \exp(\mathbf{A} t_0) & = \int_0^\infty \exp(\mathbf{A}^T (t+t_0))\mathbf{C}^T\mathbf{C}\exp(\mathbf{A}(t+t_0))dt \\ & = \int_{t_0}^\infty \exp(\mathbf{A}^T t)\mathbf{C}^T\mathbf{C}\exp(\mathbf{A}t)dt \end{array} \tag{1.86}$$

Differentiation with respect to t_0 and using the facts that matrix $\mathbf{A}$ is assumed to be stable (that is $\lim_{t\to\infty}\exp(\mathbf{A}t) = \lim_{t\to\infty}\exp(\mathbf{A}^T t) = \mathbf{0}$) and that $g(x) = \int_{a(x)}^{b(x)} f(\tau)\, d\tau \Rightarrow g'(x) = f(b(x))\, b'(x) - f(a(x))\, a'(x)$ yields:

$$\begin{array}{r} \mathbf{A}^T \exp(\mathbf{A}^T t_0)\mathbf{W}_o \exp(\mathbf{A}t_0) + \exp(\mathbf{A}^T t_0)\mathbf{W}_o \exp(\mathbf{A}t_0)\mathbf{A} \\ = -\exp(\mathbf{A}^T t_0)\mathbf{C}^T\mathbf{C}\exp(\mathbf{A}t_0) \end{array} \tag{1.87}$$

Finally setting $t_0 = 0$ leads to the following Lyapunov equation:

$$\mathbf{A}^T\mathbf{W}_o + \mathbf{W}_o\mathbf{A} = -\mathbf{C}^T\mathbf{C} \tag{1.88}$$

We use the relation $\|\mathbf{F}(s)\|_2^2 = \int_0^\infty \mathrm{tr}\left(\mathbf{g}(t)\mathbf{g}^T(t)\right) dt$ to get the result involving the controllability grammian.

1.7.3 Balanced realization

We have seen in section 1.7.2 that controllability grammian $\mathbf{W}_c$ and observability grammian $\mathbf{W}_o$ can be obtained as the solution to the following Lyapunov equations:

$$\mathbf{A}^T\mathbf{W}_o + \mathbf{W}_o\mathbf{A} + \mathbf{C}^T\mathbf{C} = \mathbf{0} \tag{1.89}$$

And:

$$\mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{0} \tag{1.90}$$

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ be a minimal realization of a stable and strictly proper transfer matrix $\mathbf{F}(s)$:

$$\mathbf{F}(s) = \mathbf{C}\left(s\mathbb{I} - \mathbf{A}\right)^{-1}\mathbf{B} := \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right) \tag{1.91}$$

The minimal realization of $\mathbf{F}(s)$ is said to be balanced as soon as its controllability grammian $\mathbf{W}_c$ and observability grammian $\mathbf{W}_o$ are equal:

$$\mathbf{W}_c = \mathbf{W}_o \tag{1.92}$$

In the balanced realization basis, the realization of $\mathbf{F}(s)$ reads:

$$\mathbf{F}(s) := \left(\begin{array}{c|c} \mathbf{P}_b^{-1}\mathbf{A}\mathbf{P}_b & \mathbf{P}_b^{-1}\mathbf{B} \\ \hline \mathbf{C}\mathbf{P}_b & \mathbf{0} \end{array}\right) \tag{1.93}$$

Furthermore, by writing the Lyapunov equations that controllability grammian $\mathbf{W}_{cb}$ and observability grammian $\mathbf{W}_{ob}$ solve in the balanced realization basis, we obtain:

$$\left\{ \begin{array}{l} \mathbf{W}_{cb} = \mathbf{P}_b^{-1}\mathbf{W}_c\left(\mathbf{P}_b^{-1}\right)^T \\ \mathbf{W}_{ob} = \mathbf{P}_b^T\mathbf{W}_o\mathbf{P}_b \end{array} \right. \tag{1.94}$$

We define the Hankel singular-values σ_i as the square roots of the eigenvalues of the product $\mathbf{W}_c \mathbf{W}_o$:

$$\sigma_i := \sqrt{\lambda_i \left(\mathbf{W}_c \mathbf{W}_o\right)} \tag{1.95}$$

The realization is balanced as soon as:

$$\mathbf{W}_{cb} = \mathbf{W}_{ob} = \mathbf{\Sigma} := \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \tag{1.96}$$

To get the balanced realization we can proceed as follows: since $\mathbf{W}_o = \mathbf{W}_o^T$ is positive definite, we can factor it as $\mathbf{W}_o = \mathbf{L}_o^T \mathbf{L}_o$ by Cholesky factorization, where $\mathbf{L}_o$ is an invertible matrix. Similarly since $\mathbf{W}_c = \mathbf{W}_c^T$ is positive definite, we can factor it as $\mathbf{W}_c = \mathbf{L}_c^T \mathbf{L}_c$ by Cholesky factorization, where $\mathbf{L}_c$ is an invertible matrix. Then the singular value decomposition of product $\mathbf{L}_o \mathbf{L}_c^T$ is used, where $\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbb{I}$ and $\mathbf{V}\mathbf{V}^T = \mathbf{V}^T\mathbf{V} = \mathbb{I}$:

$$\mathbf{L}_o \mathbf{L}_c^T = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \tag{1.97}$$

Then the similarity transformation $\mathbf{P}_b$ is defined as follows:

$$\mathbf{P}_b = \mathbf{L}_c^T \mathbf{V} \mathbf{\Sigma}^{-0.5} \tag{1.98}$$

Indeed, we can check that the controllability grammian in the balanced basis is the following:

$$\begin{aligned} \mathbf{W}_{cb} &= \mathbf{P}_b^{-1} \mathbf{W}_c \left(\mathbf{P}_b^{-1}\right)^T \\ &= \left(\mathbf{\Sigma}^{0.5} \mathbf{V}^T \left(\mathbf{L}_c^{-1}\right)^T\right) \mathbf{L}_c^T \mathbf{L}_c \left(\mathbf{\Sigma}^{0.5} \mathbf{V}^T \left(\mathbf{L}_c^{-1}\right)^T\right)^T \\ &= \left(\mathbf{\Sigma}^{0.5} \mathbf{V}^T \left(\mathbf{L}_c^{-1}\right)^T\right) \mathbf{L}_c^T \mathbf{L}_c \left(\mathbf{L}_c^{-1} \mathbf{V} \mathbf{\Sigma}^{0.5}\right) \\ &= \mathbf{\Sigma} \end{aligned} \tag{1.99}$$

Similarly using the fact that $\mathbf{L}_o \mathbf{L}_c^T = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \Leftrightarrow \mathbf{L}_c \mathbf{L}_o^T = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^T$, we can check that the observability grammian in the balanced basis is the following:

$$\begin{aligned} \mathbf{W}_{ob} &= \mathbf{P}_b^T \mathbf{W}_o \mathbf{P}_b \\ &= \left(\mathbf{L}_c^T \mathbf{V} \mathbf{\Sigma}^{-0.5}\right)^T \mathbf{L}_o^T \mathbf{L}_o \left(\mathbf{L}_c^T \mathbf{V} \mathbf{\Sigma}^{-0.5}\right) \\ &= \left(\mathbf{\Sigma}^{-0.5} \mathbf{V}^T \mathbf{L}_c\right) \mathbf{L}_o^T \mathbf{L}_o \left(\mathbf{L}_c^T \mathbf{V} \mathbf{\Sigma}^{-0.5}\right) \\ &= \mathbf{\Sigma}^{-0.5} \mathbf{V}^T \left(\mathbf{L}_c \mathbf{L}_o^T\right) \left(\mathbf{L}_o \mathbf{L}_c^T\right) \mathbf{V} \mathbf{\Sigma}^{-0.5} \\ &= \mathbf{\Sigma} \end{aligned} \tag{1.100}$$

1.8 H_∞-norm of a stable plant

1.8.1 Frequency response of a stable system

Let $\underline{u}(t)$ be the input vector signal of the system, $\underline{u}(s)$ the Laplace transform of the input vector signal $\underline{u}(t)$, $\underline{y}(t)$ the output vector signal of the system and $\underline{y}(s)$ the Laplace transform of the output vector signal $\underline{y}(t)$.

The output vector signal $\underline{y}(t)$ can be computed thanks to the input vector signal $\underline{u}(t)$ and the impulse response $\mathbf{g}(t)$. Indeed denoting by $*$ the convolution product we have:

$$\underline{y}(s) = \mathbf{F}(s)\underline{u}(s) \Leftrightarrow \underline{y}(t) = \mathbf{g}(t) * \underline{u}(t) = \int_0^\infty \mathbf{g}(\tau)\underline{u}(t-\tau)d\tau \tag{1.101}$$

We will specialized this relation assuming that the input vector signal $\underline{u}(t)$ is an harmonic vector signal with frequency ω and constant amplitude $\underline{a}$:

$$\underline{u}(t) = \underline{a}\exp(j\omega t) \tag{1.102}$$

This leads to the following expression of the output signal vector $\underline{y}(t)$:

$$\begin{array}{ll} \underline{y}(t) & = \int_0^\infty \mathbf{g}(\tau)\underline{u}(t-\tau)d\tau \\ & = \int_0^\infty \mathbf{g}(\tau)\underline{a}\exp(j\omega(t-\tau))\,d\tau \\ & = \left(\int_0^\infty \mathbf{g}(\tau)\exp(-j\omega\tau)\,d\tau\right)\underline{a}\exp(j\omega t) \\ & = \left(\left.\int_0^\infty \mathbf{g}(\tau)\exp(-s\tau)\,d\tau\right|_{s=j\omega}\right)\underline{a}\exp(j\omega t) \\ & = \left.\mathbf{F}(s)\right|_{s=j\omega}\underline{a}\exp(j\omega t) \\ & = \mathbf{F}(j\omega)\underline{a}\exp(j\omega t) \end{array} \tag{1.103}$$

The output signal $\underline{y}(t)$ has the same frequency than the input signal $\underline{u}(t) = \underline{a}\exp(j\omega t)$. $\mathbf{F}(j\omega)$ is the factor by which magnitude and phase of the harmonic vector input signal $\underline{u}(t) = \underline{a}\exp(j\omega t)$ are modified. Notice that the output signal $\underline{y}(t)$ does not have an L_2 norm because the integral $\int_0^\infty \underline{y}^H(t)\underline{y}(t)dt$ is unbounded. Nevertheless we can compute the square of the amplitude of the output signal as $\underline{y}^H(t)\underline{y}(t)$. using the fact that $\exp(j\omega t)$ is scalar we get:

$$\begin{array}{ll} \underline{y}^H(t)\underline{y}(t) & = \exp(-j\omega t)\,\underline{a}^T\mathbf{F}^H(j\omega)\mathbf{F}(j\omega)\underline{a}\exp(j\omega t) \\ & = \underline{a}^T\mathbf{F}^H(j\omega)\mathbf{F}(j\omega)\underline{a} \end{array} \tag{1.104}$$

From(1.31) the singular-value decomposition of $\mathbf{F}(j\omega)$ is written as follows:

$$\mathbf{F}^H(j\omega)\mathbf{F}(j\omega) = \\ \mathbf{V}(j\omega)\begin{bmatrix} \sigma_1^2(j\omega) & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2(j\omega) & 0 & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 0 & & & \\ 0 & \dots & 0 & \sigma_q^2(j\omega) & & & \\ \vdots & \dots & & & 0 & & \\ & \dots & & & & \ddots & \\ 0 & & & & & & 0 \end{bmatrix}\mathbf{V}^H(j\omega) \tag{1.105}$$

Let $\overline{\sigma}(\mathbf{F}(j\omega))$ be the largest singular-value of matrix $\mathbf{F}(j\omega)$, that is the ω valued upper bound of the set of singular-values $\{\sigma_i(\mathbf{F}(j\omega))\}$ and $\underline{\sigma}(\mathbf{F}(j\omega))$ be the smallest singular-value of matrix $\mathbf{F}(j\omega)$, that is the ω valued lower bound of the set of singular-values $\{\sigma_i(\mathbf{F}(j\omega))\}$:

$$0 < \underline{\sigma}(\mathbf{F}(j\omega)) \le \sigma_i(\mathbf{F}(j\omega)) \le \overline{\sigma}(\mathbf{F}(j\omega)) \quad \forall\, i,\, \omega \tag{1.106}$$

Then using the fact that $\mathbf{V}(j\omega)$ is a unitary matrix, i.e. $\mathbf{V}(j\omega)\mathbf{V}^H(j\omega) = \mathbb{I}$, we get :

$$\underline{\sigma}^2 \left(\mathbf{F}\left(j\omega\right)\right) \underline{a}^T \underline{a} \leq \underline{y}^H(t)\underline{y}(t) \leq \overline{\sigma}^2 \left(\mathbf{F}\left(j\omega\right)\right) \underline{a}^T \underline{a} \tag{1.107}$$

Thus the ratio between the amplitude $\sqrt{\underline{y}^H(t)\underline{y}(t)} = |y(t)|$ of the output signal $\underline{y}(t)$ and the amplitude $\sqrt{\underline{a}^T \underline{a}} = |\underline{a}|$ of the input harmonic signal $\underline{u}(t) = \underline{a} \exp\left(j\omega t\right)$ is comprised between $\underline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right)$ and $\overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right)$:

$$\underline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) \leq \frac{|y(t)|}{|\underline{a}|} \leq \overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) \tag{1.108}$$

For SISO systems the unique singular-value of the transfer function $\mathbf{F}\left(j\omega\right)$ represents its magnitude:

$$\textit{SISO system} \Rightarrow \underline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) = \overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) = \left|\mathbf{F}\left(j\omega\right)\right| \tag{1.109}$$

For MIMO systems the singular-values $\sigma_i(j\omega)$ of the transfer matrix $\mathbf{F}\left(j\omega\right)$ are called the principal gains.

1.8.2 H_∞-norm definition

A system with transfer matrix $\mathbf{F}(s)$ can be viewed as a mapping between an input signal $\underline{u}(t)$ and an output signal $\underline{y}(t)$. If $\underline{u}(t)$ has finite energy, then $\max_{\underline{u}(t)\neq\underline{0}} \frac{\|\underline{y}(t)\|_2}{\|\underline{u}(t)\|_2}$ is the maximum size of the output response of the system in terms of its energy.

The energy gain (or L_2 gain) γ of a system with transfer matrix $\mathbf{F}(s)$ is defined as:

$$\gamma = \max_{0<\|\underline{u}(t)\|_2<\infty} \frac{\|\underline{y}(t)\|_2}{\|\underline{u}(t)\|_2} = \max_{0<\|\underline{u}(t)\|_2<\infty} \frac{\left\|\mathcal{L}^{-1}\left[\mathbf{F}(s)\underline{u}(s)\right]\right\|_2}{\|\underline{u}(t)\|_2} \tag{1.110}$$

Alternatively one can characterize the energy gain (or L_2 gain) of a stable plant as the minimal value of γ such that:

$$\|\underline{y}(t)\|_2 = \left\|\mathcal{L}^{-1}\left[\mathbf{F}(s)\underline{u}(s)\right]\right\|_2 \leq \gamma \|\underline{u}(t)\|_2 \text{ for all finite energy inputs } \underline{u}(t) \tag{1.111}$$

If system with transfer matrix $\mathbf{F}(s)$ is not stable then no such γ exists and $\gamma = \infty$. On the other hand if $\mathbf{F}(s)$ is stable then γ is finite. Consequently a transfer matrix $\mathbf{F}(s)$ is said to be stable if and only if all finite energy signals $\underline{u}(t) \in L_2$ are mapped by $\mathbf{F}(s)$ into finite energy signals $\underline{y}(t) \in L_2$. Alternatively the transfer matrix $\mathbf{F}(s)$ is stable if and only if $\gamma < \infty$. Furthermore if $\mathbf{F}(s)$ is a stable transfer matrix, its H_∞-norm is equal to the energy gain of the system:

$$\|\mathbf{F}(s)\|_\infty = \gamma = \max_\omega \overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) \tag{1.112}$$

where $\overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right)$ is such that:

$$0 < \underline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) \leq \sigma_i\left(\mathbf{F}(j\omega)\right) \leq \overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) \quad \forall\, i,\, \omega \tag{1.113}$$

Note that $\sigma_i\left(\mathbf{F}(j\omega)\right)$ is the *root square of any non-zero eigenvalue* of either $\mathbf{F}^H(j\omega)\,\mathbf{F}(j\omega)$ or $\mathbf{F}(j\omega)\,\mathbf{F}^H(j\omega)$, that is any singular value of $\mathbf{F}(j\omega)$.

The eigenvalues of $\mathbf{F}^H(j\omega)\,\mathbf{F}(j\omega)$ are the values of λ such that $\det\left(\lambda\,\mathbb{I}-\mathbf{F}^H(j\omega)\,\mathbf{F}(j\omega)\right)=0$. Similarly the eigenvalues of $\mathbf{F}(j\omega)\,\mathbf{F}^H(j\omega)$ are the values of λ such that $\det\left(\lambda\,\mathbb{I}-\mathbf{F}(j\omega)\,\mathbf{F}^H(j\omega)\right)=0$.

In other words the H_∞-norm of transfer matrix $\mathbf{F}(s)$ is the induced norm obtained by the L_2 norms of input and output vector signals:

$$\|\mathbf{F}(s)\|_\infty=\max_{\underline{u}(t)\neq\underline{0}}\frac{\|\underline{y}(t)\|_2}{\|\underline{u}(t)\|_2}=\max_{\omega}\overline{\sigma}\left(\mathbf{F}(j\omega)\right)\tag{1.114}$$

This equation indicates that H_∞-norm is a measure of the maximum gain of a stable system over all frequencies ω; it is the worst case frequency response of the system.

For a stable SISO (Single Input Single Output) linear system with transfer function $F(s)$ the H_∞-norm represents the maximal peak value on the Bode magnitude plot of $\mathbf{F}(j\omega)$:

$$\|F(s)\|_\infty=\max_{\omega}|F(j\omega)|\tag{1.115}$$

For MIMO (Multi Input Multi Output) systems, it represents the maximal peak value on the singular-values magnitude plot of $\mathbf{F}(j\omega)$; in other words, it is the largest gain when the system is fed by an harmonic input vector.

It is worth noticing that the H_∞-norm is an induced norm and thus satisfies the multiplicative property (1.37).

1.8.3 H_∞-norm and Hamiltonian matrix

Unlike H_2-norm , the H_∞-norm cannot be computed analytically. Only numerical solutions can be obtained. It can be shown that if $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ is a minimal realization of transfer matrix $\mathbf{F}(s)$ then its H_∞-norm can be computed as follows:

$$\|\mathbf{F}(s)\|_\infty=sup_{\gamma\geq 0}\{\gamma:\mathbf{H}\text{ has eigenvalue with null real part}\}\tag{1.116}$$

The Hamiltonian matrix $\mathbf{H}$ is defined as follows[2]:

$$\begin{aligned}\mathbf{H}&=\begin{bmatrix}\mathbf{A}+\mathbf{B}\mathbf{R}_\gamma^{-1}\mathbf{D}^T\mathbf{C} & \mathbf{B}\mathbf{R}_\gamma^{-1}\mathbf{B}^T\\ -\mathbf{C}^T\left(\mathbb{I}+\mathbf{D}\mathbf{R}_\gamma^{-1}\mathbf{D}^T\right)\mathbf{C} & -\left(\mathbf{A}+\mathbf{B}\mathbf{R}_\gamma^{-1}\mathbf{D}^T\mathbf{C}\right)^T\end{bmatrix}\\ &=\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ -\mathbf{C}^T\mathbf{C} & -\mathbf{A}^T\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ -\mathbf{C}^T\mathbf{D}\end{bmatrix}\mathbf{R}_\gamma^{-1}\begin{bmatrix}\mathbf{D}^T\mathbf{C} & \mathbf{B}^T\end{bmatrix}\end{aligned}\tag{1.117}$$

Where $\mathbf{R}_\gamma=\mathbf{R}_\gamma^T>0$ reads:

$$\mathbf{R}_\gamma=\gamma^2\mathbb{I}-\mathbf{D}^T\mathbf{D}\tag{1.118}$$

For the usual case where $\mathbf{D}=\mathbf{0}$ the Hamiltonian matrix $\mathbf{H}$ reduces as follows:

$$\mathbf{H}=\begin{bmatrix}\mathbf{A} & \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T\\ -\mathbf{C}^T\mathbf{C} & -\mathbf{A}^T\end{bmatrix}\tag{1.119}$$

[2]Robust Control Design: An Optimal Control Approach, Feng Lin, Wiley–Blackwell, 2007

The way to get such Hamiltonian matrix is based on the bounded-real lemma which is presented in Section 8.2.3.

We may use the Routh–Hurwitz stability criterion on the characteristic polynomial $\chi_H(s) := \det(s\mathbb{I} - \mathbf{H})$ to get H_∞-norm of $\mathbf{F}(s)$. Because $\chi_H(s)$ is a even polynomial involving powers of s^2, a row of zeros will appear in the Routh table of $\chi_H(s)$. In order to build the Routh table, the row of zeros shall be replaced by the coefficients of the auxiliary polynomial obtained by taking the derivative with respect to s of the polynomial corresponding to the previous row. Let $\{\gamma_i\}$ be the set of values of $\gamma \geq 0$ such that the γ-dependent coefficients in the first column of the Routh table become zero. Then, following Aghdam[3] the H_∞-norm of $\mathbf{F}(s)$ is obtained as:

$$\|\mathbf{F}(s)\|_\infty = \max_i \{\gamma_i\} \tag{1.120}$$

As shown by Doyle & al.[4] the algorithm for the calculation of the H_∞-norm can be obtained by observing that $\|\mathbf{F}(s)\|_\infty < \gamma$ is equivalent to the fact that $\mathbb{I} - \frac{1}{\gamma^2}\mathbf{F}^T(-j\omega)\mathbf{F}(j\omega)$ is invertible for all $\omega \in \mathbb{R}$, meaning that $\left(\mathbb{I} - \frac{1}{\gamma^2}\mathbf{F}^T(-s)\mathbf{F}(s)\right)^{-1}$ has no pole on the imaginary axis. Then construct a realization of $\left(\mathbb{I} - \frac{1}{\gamma^2}\mathbf{F}^T(-s)\mathbf{F}(s)\right)^{-1}$ and use the fact that the poles coincide with the eigenvalues of the corresponding state matrix $\mathbf{H}$. The key point is the achievement of the realization, which is obtained through the following following lemma[4]: let $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ be the realization of a strictly proper transfer matrix $\mathbf{F}(s)$ and $\mathbf{H}$ the Hamiltonian matrix defined in (1.119). Then:

$$\begin{array}{rl} \left(\mathbb{I} - \frac{1}{\gamma^2}\mathbf{F}^T(-s)\mathbf{F}(s)\right)^{-1} & = \left(\begin{array}{cc|c} \mathbf{A} & \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T & \frac{\mathbf{B}}{\gamma} \\ -\mathbf{C}^T\mathbf{C} & -\mathbf{A}^T & \mathbf{0} \\ \hline \mathbf{0} & \frac{\mathbf{B}^T}{\gamma} & \mathbb{I} \end{array}\right) \\ & := \left[\begin{array}{cc} \mathbf{0} & \frac{\mathbf{B}^T}{\gamma} \end{array}\right] (s\mathbb{I} - \mathbf{H})^{-1} \left[\begin{array}{c} \frac{\mathbf{B}}{\gamma} \\ \mathbf{0} \end{array}\right] + \mathbb{I} \end{array} \tag{1.121}$$

To get this result, consider Figure 1.1. The relation between $\underline{e}(s)$ and $\underline{r}(s)$ is obtained by reading Figure 1.1 *against* the arrows:

$$\begin{array}{rl} \underline{e}(s) & = \underline{r}(s) + \frac{1}{\gamma^2}\mathbf{F}^T(-s)\mathbf{F}(s)\underline{e}(s) \\ \Rightarrow \underline{e}(s) & = \left(\mathbb{I} - \frac{1}{\gamma^2}\mathbf{F}^T(-s)\mathbf{F}(s)\right)^{-1} \underline{r}(s) \end{array} \tag{1.122}$$

On the other hand, the realization of $\mathbf{F}^T(-s)$ is obtained from the realization

[3] Amir G. Aghdam, A Method to Obtain the Infinity-Norm of Systems using the Routh Table, World Automation Congress (WAC) 2006, July 24-26, Budapest, Hungary

[4] Doyle J.C. ,Glover K., Khargonekar P.P. and Francis B.A., State-space solutions to standard H_2 and H_∞ control problems, IEEE Transaction on Automatic Control, Vol 34, n 8, 1989, pp 831-847

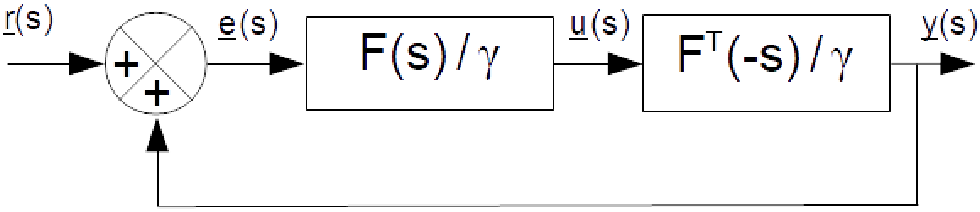

Figure 1.1: Closed-loop Hamiltonian transfer function

of $\mathbf{F}(s)$ as follows:

$$\begin{array}{rl} \mathbf{F}(s) & = \left(\begin{array}{c|c} \mathbf{A} & \frac{\mathbf{B}}{\gamma} \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right) := \mathbf{C}\left(s\mathbb{I}-\mathbf{A}\right)^{-1}\frac{\mathbf{B}}{\gamma} \\ \Rightarrow \mathbf{F}^T(-s) & = \left(\mathbf{C}\left(-s\mathbb{I}-\mathbf{A}\right)^{-1}\frac{\mathbf{B}}{\gamma}\right)^T = -\frac{\mathbf{B}^T}{\gamma}\left(s\mathbb{I}-\left(-\mathbf{A}^T\right)\right)^{-1}\mathbf{C}^T \\ & = \left(\begin{array}{c|c} -\mathbf{A}^T & -\mathbf{C}^T \\ \hline \frac{\mathbf{B}^T}{\gamma} & \mathbf{0} \end{array}\right) \end{array} \tag{1.123}$$

Thus, in the time domain we have:

$$\left\{\begin{array}{l} \mathbf{F}(s) = \left(\begin{array}{c|c} \mathbf{A} & \frac{\mathbf{B}}{\gamma} \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right) \Rightarrow \left\{\begin{array}{l} \underline{\dot{x}}_1 = \mathbf{A}\underline{x}_1 + \frac{\mathbf{B}}{\gamma}\underline{e} \\ \underline{u} = \mathbf{C}\underline{x}_1 \end{array}\right. \\ \mathbf{F}^T(-s) = \left(\begin{array}{c|c} -\mathbf{A}^T & -\mathbf{C}^T \\ \hline \frac{\mathbf{B}^T}{\gamma} & \mathbf{0} \end{array}\right) \Rightarrow \left\{\begin{array}{l} \underline{\dot{x}}_2 = -\mathbf{A}^T\underline{x}_2 - \mathbf{C}^T\underline{u} \\ \underline{y} = \frac{\mathbf{B}^T}{\gamma}\underline{x}_2 \end{array}\right. \end{array}\right. \tag{1.124}$$

From Figure 1.1 we see that $\underline{e} = \underline{r} + \underline{y}$. Thus the realization of Figure 1.1 reads as follows:

$$\left\{\begin{array}{l} \underline{e} = \underline{r} + \underline{y} \\ \underline{u} = \mathbf{C}\underline{x}_1 \\ \underline{y} = \frac{\mathbf{B}^T}{\gamma}\underline{x}_2 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \begin{bmatrix} \underline{\dot{x}}_1 \\ \underline{\dot{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T \\ -\mathbf{C}^T\mathbf{C} & -\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{\mathbf{B}}{\gamma} \\ \mathbf{0} \end{bmatrix} \underline{r} \\ \underline{y} = \begin{bmatrix} \mathbf{0} & \frac{\mathbf{B}^T}{\gamma} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \mathbb{I}\,\underline{r} \end{array}\right. \tag{1.125}$$

In the frequency domain we get:

$$\underline{e}(s) = \left(\begin{bmatrix} \mathbf{0} & \frac{\mathbf{B}^T}{\gamma} \end{bmatrix} \left(s\mathbb{I}-\mathbf{H}\right)^{-1} \begin{bmatrix} \frac{\mathbf{B}}{\gamma} \\ \mathbf{0} \end{bmatrix} + \mathbb{I}\right)\underline{r}(s) \tag{1.126}$$

When identifying (1.122) with (1.126) we get relation (1.121). An alternate relation can also be obtained by replacing $\frac{\mathbf{F}(s)\mathbf{F}^T(-s)}{\gamma^2}$ in Figure 1.1 by $\frac{\mathbf{F}^T(-s)\mathbf{F}(s)}{\gamma^2}$.

Having in mind that for any square invertible matrix $\mathbf{Y}$ we have $\mathbf{X}\mathbf{Y}^{-1}\mathbf{Z} = \frac{\mathbf{X}\,\mathrm{adj}(\mathbf{Y})\mathbf{Z}}{\det(\mathbf{Y})}$ (here $\mathbf{X} = \begin{bmatrix} \mathbf{0} & \frac{\mathbf{B}^T}{\gamma} \end{bmatrix}$, $\mathbf{Y} = (s\mathbb{I} - \mathbf{H})$ and $\mathbf{Z} = \begin{bmatrix} \frac{\mathbf{B}}{\gamma} \\ \mathbf{0} \end{bmatrix}$), we conclude that relation (1.121) indicates that the eigenvalues of Hamiltonian matrix $\mathbf{H}$ are the roots of $\det\left(\mathbb{I} - \frac{1}{\gamma^2}\mathbf{F}^T(-s)\mathbf{F}(s)\right)$.

Example 1.8. *Let $F(s)$ be defined as follows:*

$$F(s) = \frac{1}{s+2} \tag{1.127}$$

To calculate the H_∞-norm of $F(s)$ we use the fact that:

$$F(j\omega) = \frac{1}{2+j\omega} \Rightarrow |F(j\omega)| = \frac{1}{\sqrt{4+\omega^2}} \tag{1.128}$$

And hence :

$$\|F(s)\|_\infty = \max_\omega |F(j\omega)| = |F(0)| = \frac{1}{\sqrt{4}} = \frac{1}{2} \tag{1.129}$$

A realization of $F(s)$ is the following:

$$\begin{cases} \mathbf{A} = -2 \\ \mathbf{B} = 1 \\ \mathbf{C} = 1 \end{cases} \tag{1.130}$$

Then the Hamiltonian matrix $\mathbf{H}$ is defined as follows:

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T \\ -\mathbf{C}^T\mathbf{C} & -\mathbf{A}^T \end{bmatrix} = \begin{bmatrix} -2 & \frac{1}{\gamma^2} \\ -1 & 2 \end{bmatrix} \tag{1.131}$$

The characteristic polynomial $\chi_H(s)$ of $\mathbf{H}$ reads:

$$\chi_H(s) := det(s\mathbb{I} - \mathbf{H}) = det\left(\begin{bmatrix} s+2 & -\frac{1}{\gamma^2} \\ 1 & s-2 \end{bmatrix}\right) = s^2 - 4 + \frac{1}{\gamma^2} \tag{1.132}$$

The Routh table corresponding to $\chi_H(s)$ reads:

s^2	1	$-4+\frac{1}{\gamma^2}$
s^1	$0 \to 2$	
s^0	$-4+\frac{1}{\gamma^2}$	

(1.133)

Note that coefficient 2 in row s^1 has been obtained by taking the derivative with respect to s of the polynomial corresponding to the previous row:

$$\frac{d}{ds}\left(s^2 - 4 + \frac{1}{\gamma^2}\right) = 2 \tag{1.134}$$

The γ-dependent coefficient in the first column of the Routh table become zero for $\gamma \geq 0$ such that $\gamma = \frac{1}{2}$: this is actually H_∞-norm of $F(s)$.

■

Example 1.9. *Let $\mathbf{F}(s)$ be defined as follows:*

$$\mathbf{F}(s) = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{bmatrix} \tag{1.135}$$

To calculate the H_∞-norm of $\mathbf{F}(s)$ we use the fact that:

$$
\begin{aligned}
&\mathbf{F}(j\omega) = \begin{bmatrix} \frac{1}{j\omega+1} \\ \frac{1}{j\omega+2} \end{bmatrix} \\
&\Rightarrow \mathbf{F}^H(j\omega)\,\mathbf{F}(j\omega) = \begin{bmatrix} \frac{1}{-j\omega+1} & \frac{1}{-j\omega+2} \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega+1} \\ \frac{1}{j\omega+2} \end{bmatrix} = \frac{1}{\omega^2+1} + \frac{1}{\omega^2+4}
\end{aligned} \tag{1.136}
$$

Thus:

$$
\overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) = \sqrt{\frac{1}{\omega^2+1} + \frac{1}{\omega^2+4}} \tag{1.137}
$$

and:

$$
\|\mathbf{F}(s)\|_\infty = \gamma = \max_\omega \overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) = \sqrt{1+\frac{1}{4}} = \sqrt{\frac{5}{4}} \tag{1.138}
$$

Alternatively we can write:

$$
\begin{aligned}
\mathbf{F}(j\omega) &= \begin{bmatrix} \frac{1}{j\omega+1} \\ \frac{1}{j\omega+2} \end{bmatrix} \\
\Rightarrow \mathbf{F}(j\omega)\,\mathbf{F}^H(j\omega) &= \begin{bmatrix} \frac{1}{j\omega+1} \\ \frac{1}{j\omega+2} \end{bmatrix} \begin{bmatrix} \frac{1}{-j\omega+1} & \frac{1}{-j\omega+2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{(1+j\omega)\,(1-j\omega)} & \frac{1}{(1+j\omega)\,(2-j\omega)} \\ \frac{1}{(2+j\omega)\,(1-j\omega)} & \frac{1}{(2+j\omega)\,(2-j\omega)} \end{bmatrix}
\end{aligned} \tag{1.139}
$$

In order to compute the singular values of $\mathbf{F}(j\omega)$, lets compute the characteristic polynomial of matrix $\mathbf{F}(j\omega)\,\mathbf{F}^H(j\omega)$:

$$
\begin{aligned}
det\left(\lambda\mathbb{I} - \mathbf{F}(j\omega)\,\mathbf{F}^H(j\omega)\right) &= \frac{\lambda^2\,\omega^4+5\,\lambda^2\,\omega^2+4\,\lambda^2-2\,\lambda\,\omega^2-5\,\lambda}{(1+j\omega)\,(1-j\omega)\,(2+j\omega)\,(2-j\omega)} \\
&= \frac{\lambda\left(\lambda\left(\omega^4+5\,\omega^2+4\right)-2\,\omega^2-5\right)}{(1+j\omega)\,(1-j\omega)\,(2+j\omega)\,(2-j\omega)}
\end{aligned} \tag{1.140}
$$

Thus the eigenvalues of $\mathbf{F}(j\omega)\,\mathbf{F}^H(j\omega)$ are $\lambda = 0$ and $\lambda = \frac{2\,\omega^2+5}{\omega^4+5\,\omega^2+4}$. The root square of the non-zero eigenvalue *of $\mathbf{F}\left(j\omega\right)\mathbf{F}^H\left(j\omega\right)$ reads:*

$$
\sigma\left(\mathbf{F}(j\omega)\right) = \sqrt{\frac{2\,\omega^2+5}{\omega^4+5\,\omega^2+4}} = \sqrt{\frac{1}{\omega^2+1} + \frac{1}{\omega^2+4}} \tag{1.141}
$$

Thus for this example:

$$
\overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) = \sigma\left(\mathbf{F}(j\omega)\right) = \sqrt{\frac{1}{\omega^2+1} + \frac{1}{\omega^2+4}} \tag{1.142}
$$

We then retrieve the preceding result:

$$
\|\mathbf{F}(s)\|_\infty = \gamma = \max_\omega \overline{\sigma}\left(\mathbf{F}\left(j\omega\right)\right) = \sqrt{1+\frac{1}{4}} = \sqrt{\frac{5}{4}} \tag{1.143}
$$

Alternatively, a realization of $\mathbf{F}(s)$ *reads:*

$$\mathbf{F}(s) = \frac{1}{s^2+3s+2}\begin{bmatrix} s+2 \\ s+1 \end{bmatrix} := \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline 2 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \tag{1.144}$$

Then the Hamiltonian matrix $\mathbf{H}$ *is defined as follows:*

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T \\ -\mathbf{C}^T\mathbf{C} & -\mathbf{A}^T \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & \frac{1}{\gamma^2} \\ \hline -5 & -3 & 0 & 2 \\ -3 & -2 & -1 & 3 \end{array}\right] \tag{1.145}$$

The characteristic polynomial $\chi_H(s)$ *of* $\mathbf{H}$ *reads:*

$$\begin{aligned} \chi_H(s) &:= det(s\mathbb{I} - \mathbf{H}) \\ &= det\left(\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 2 & \lambda+3 & 0 & -\frac{1}{\gamma^2} \\ 5 & 3 & \lambda & -2 \\ 3 & 2 & 1 & \lambda-3 \end{bmatrix}\right) \\ &= s^4 + \left(\frac{2}{\gamma^2} - 5\right) s^2 + 4 - \frac{5}{\gamma^2} \end{aligned} \tag{1.146}$$

The Routh table corresponding to $\chi_H(s)$ *reads:*

s^4	1	$\frac{2}{\gamma^2} - 5$	$4 - \frac{5}{\gamma^2}$
s^3	$0 \to 4$	$0 \to 2\left(\frac{2}{\gamma^2} - 5\right)$	
s^2	$2\left(\frac{2}{\gamma^2} - 5\right)$	$4\left(4 - \frac{5}{\gamma^2}\right)$	
s^1	$\frac{88}{\gamma^2} - 84$		
s^0	$4\left(4 - \frac{5}{\gamma^2}\right)$		

(1.147)

Note that coefficients in row s^3 *have been obtained by taking the derivative with respect to* s *of the polynomial corresponding to the previous row:*

$$\frac{d}{ds}\left(s^4 + \left(\frac{2}{\gamma^2} - 5\right) s^2 + 4 - \frac{5}{\gamma^2}\right) = 4\,s^3 + 2\left(\frac{2}{\gamma^2} - 5\right) s \tag{1.148}$$

Let $\{\gamma_i\} = \{\sqrt{\frac{2}{5}} \approx 0.6325\,, \sqrt{\frac{88}{84}} \approx 1.0235\,, \sqrt{\frac{5}{4}} \approx 1.118\}$ *be the set of values of* $\gamma \geq 0$ *such that the* γ*-dependent coefficients in the first column of the Routh table become zero. Then the* H_∞*-norm of* $\mathbf{F}(s)$ *is obtained as:*

$$\|\mathbf{F}(s)\|_\infty = \max_i\{\gamma_i\} = \sqrt{\frac{5}{4}} \tag{1.149}$$

■

Chapter 2

Sensitivity function

2.1 Open-loop versus closed-loop

Usually a plant alone do not fit with the industrial constraints within which it will be used. Typically a plant without controller will not be neither enough precise nor fast. Thus a controller shall be added to the plant (equipped with sensors and actuators) to satisfy industrial specifications. We will denote:

- $F(s)$ the transfer function of the plant;
- $C(s)$ the transfer function of the controller;
- $u(t)$ the plant input (actuator signal) and $y(t)$ the plant output (sensor signal) whose Laplace transform are respectively $\mathcal{L}(y(t)) = Y(s)$ and $\mathcal{L}(u(t)) = U(s)$;
- $r(t)$ the reference input whose Laplace transform is $\mathcal{L}(r(t)) = R(s)$. Reference input $r(t)$ represents what we would like $y(t)$ to be;
- $\epsilon(t) = r(t) - y(t)$ the tracking error as depicted in Figure 2.2. Its Laplace transform is $\mathcal{L}(\epsilon(t)) = \epsilon(s)$.

In that section we will compare open-loop control versus closed-loop control. In open-loop control the output signal $y(t)$ of the plant to be controlled has no effect upon the input of the plant to be controlled as depicted in Figure 2.1. It is not the case for closed-loop control.

Denoting by $C_o(s)$ the open-loop controller, simple algebra shows that the input output relation $G_o(s)$ of the open-loop control depicted in Figure 2.1

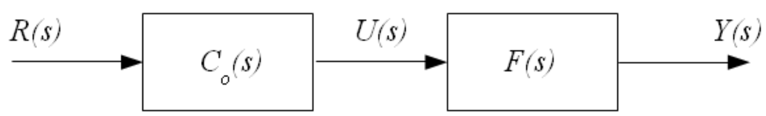

Figure 2.1: Open-loop control

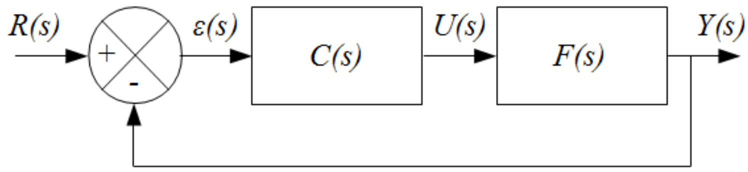

Figure 2.2: Closed-loop control with controller in the direct path

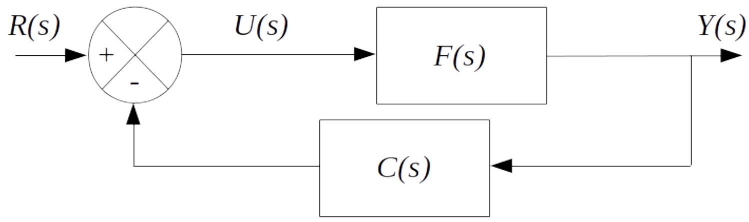

Figure 2.3: Closed-loop control with controller in the feedback path

reads:

$$Y(s) = C_o(s)F(s)R(s) \Rightarrow G_o(s) = \frac{Y(s)}{R(s)} = C_o(s)F(s) \tag{2.1}$$

On the other hand closed-loop control, or feedback control loop, is a control pattern within which the output signal $y(t)$ of the plant to be controlled is returned back and compared to the reference input to form the system control as depicted in Figure 2.2.

Denoting by $C(s)$ the closed-loop controller, simple algebra shows that the input output relation $G(s)$ of the closed-loop control depicted in Figure 2.2 reads:

$$Y(s) = F(s)C(s)\left(R(s) - Y(s)\right) \Rightarrow G(s) = \frac{Y(s)}{R(s)} = \frac{F(s)C(s)}{1 + F(s)C(s)} \tag{2.2}$$

It is worth noticing that other feedback loop configurations exist. For example the closed-loop controller $C(s)$ may be put is the feedback path as depicted in Figure 2.3.

The input output relation of Figure 2.3 reads:

$$Y(s) = F(s)\left(R(s) - C(s)Y(s)\right) \Rightarrow \frac{Y(s)}{R(s)} = \frac{F(s)}{1 + F(s)C(s)} \tag{2.3}$$

That is:

$$Y(s) = \frac{F(s)C(s)}{1 + F(s)C(s)} \frac{R(s)}{C(s)} \tag{2.4}$$

The preceding relation indicates that Figure 2.3 is equivalent to Figure 2.2 when the reference input $R(s)$ is replaced by $\frac{R(s)}{C(s)}$.

In the following we will focus on the feedback loop where the controller is situated in the direct path as depicted in Figure 2.2.

When comparing (2.1) and (2.2) it is clear that the open-loop controller $C_o(s)$ can be obtained from the closed-loop controller $C(s)$ by choosing:

$$C_o(s)F(s) = \frac{F(s)C(s)}{1+F(s)C(s)} \Rightarrow C_o(s) = \frac{C(s)}{1+F(s)C(s)} \tag{2.5}$$

Consequently it seems that open-loop control and closed-loop control are equivalent control. Nevertheless we shall have in mind that the plant model $F(s)$ often comes from linearization and simplification, and thus is uncertain. Thus we will study the sensitivity of both open-loop scheme and closed-loop scheme with respect to the plant model uncertainty thanks to the sensitivity function S_α^H defined by:

$$S_\alpha^H = \frac{\partial H/H}{\partial\alpha/\alpha} = \frac{\alpha}{H}\frac{\partial H}{\partial\alpha} \tag{2.6}$$

Basically S_α^H relates the relative change of quantity H with respects to the relative change of quantity α.

Specializing the sensitivity function definition to the case where H is the open-loop transfer function $G_o(s)$ and α the uncertain transfer function $F(s)$ of the plant we get:

$$S_F^{G_o} = \frac{F}{G_o}\frac{\partial G_o}{\partial F} = \frac{F}{C_oF}\frac{\partial}{\partial F}(C_oF) = \frac{F}{C_oF}C_o = 1 \tag{2.7}$$

The preceding relation indicates that any change in the plant transfer function $F(s)$ is totally transferred into the open-loop control scheme whatever controller $C_0(s)$ is.

On the other hand we will now specialize the sensitivity function definition to the case where H is the closed-loop transfer function $G(s)$ and α the uncertain transfer function $F(s)$ of the plant:

$$S_F^G = \frac{F}{G}\frac{\partial G}{\partial F} = \frac{F}{\frac{CF}{1+CF}}\frac{\partial}{\partial F}\left(\frac{CF}{1+CF}\right) = \frac{1+CF}{C}\frac{\partial}{\partial F}\left(\frac{CF}{1+CF}\right) \tag{2.8}$$

Let's compute the following expression:

$$\frac{\partial}{\partial F}\left(\frac{CF}{1+CF}\right) = \frac{C\,(1+CF) - CFC}{(1+CF)^2} = \frac{C}{(1+CF)^2} \tag{2.9}$$

We finally get:

$$S_F^G = \frac{1}{1+CF} \tag{2.10}$$

Thus the sensitivity function $S(s)$ of the closed-loop system reads:

$$S_F^G := S(s) = \frac{1}{1+C(s)F(s)} \tag{2.11}$$

The preceding relation clearly indicates that as soon as the product $C(s)F(s)$ is *high* within the frequency range of the uncertain plant $F(s)$ then the closed-loop control scheme allows a great reduction of the sensitivity of the controlled system with respects to uncertainties. Same result can be achieved when comparing the sensitivity of open-loop control scheme and closed-loop control scheme with respect to external disturbances.

As a consequence the central idea to control a plant is the feedback loop where the output signal $y(t)$ of the plant to be controlled is returned back and compared to the reference input to form the system control as depicted in Figure 2.2.

2.2 SISO systems

For SISO systems $S(s) = \frac{e(s)}{r(s)} = \frac{1}{1+L(s)}$ represents the sensitivity function. It can be shown that sensitivity function quantifies how sensitive is the closed loop system to variations of the plant.

Indeed let $L(s) = F(s)\,C(s)$, where $F(s)$ is the transfer function of the plant and $C(s)$ is the transfer function of the corrector. Then according to Figure 4.3 the closed loop transfer function $G(s)$ reads:

$$L(s) = F(s)\,C(s) \Rightarrow G(s) = \frac{y(s)}{r(s)} = \frac{L(s)}{1+L(s)} = \frac{F(s)\,C(s)}{1+F(s)\,C(s)} \tag{2.12}$$

Then the *relative* sensitivity of the closed loop transfer function $G(s)$ with respect to variations of the plant transfer function $F(s)$ reads:

$$\begin{aligned} \frac{\partial G(s)/\partial F(s)}{G(s)/F(s)} &= \frac{\partial G(s)}{\partial F(s)}\frac{F(s)}{G(s)} \\ &= \frac{C(s)\,(1+F(s)\,C(s)) - F(s)\,C(s)\,C(s)}{(1+F(s)\,C(s))^2}\frac{1+F(s)\,C(s)}{C(s)} \\ &= \frac{1}{1+F(s)\,C(s)} \\ &= \frac{1}{1+L(s)} \\ &= S(s) \end{aligned} \tag{2.13}$$

The same result is obtained when the controller $C(s)$ is laid in the feedback loop:

$$\begin{aligned} G(s) &= \frac{F(s)}{1+F(s)\,C(s)} \\ \Rightarrow \frac{\partial G(s)/\partial F(s)}{G(s)/F(s)} &= \frac{\partial G(s)}{\partial F(s)}\frac{F(s)}{G(s)} \\ &= \frac{1+F(s)\,C(s) - F(s)\,C(s)}{(1+F(s)\,C(s))^2}\,(1+F(s)\,C(s)) \\ &= \frac{1}{1+F(s)\,C(s)} \\ &= \frac{1}{1+L(s)} \\ &= S(s) \end{aligned} \tag{2.14}$$

Therefore one of the objective of robust control is to minimize the H_∞-norm of the sensitivity function $S(s)$.

Nevertheless the minimization of the H_∞-norm of the sensitivity function $S(s)$ alone turns out to be an ill-posed problem because the gain of the resulting controller would be infinite.

The complementary sensitivity function $T(s) = 1 - S(s) = \frac{L(s)}{1+L(s)}$ represents the closed loop transfer function. The inverse of the infinity-norm of the complementary sensitivity function $\|T(s)\|_\infty^{-1}$ is thus the reciprocal of the resonance peak of the closed loop transfer function and constitutes an important performance indicator for the closed loop system.

2.3 Bode's integral theorem

Let $S(s) = \frac{1}{1+L(s)}$ be the sensitivity function of a SISO system. We will assume that $S(s)$ goes to zero faster than $1/s$ for large s. Let λ_k be the poles of the open loop transfer function $L(s)$ in the right-half plane. Then denoting by λ_k the unstable poles of $L(s)$ (that are the poles in the right-half plane), and assuming the the closed loop is stable, it can be shown that the sensitivity function $S(s)$ satisfies the following relation[1], which is called the Bode's integral theorem:

$$S(s) = \frac{1}{1+L(s)} \Rightarrow \int_0^\infty ln\left(|S(j\omega)|\right) d\omega = \pi \sum_k \lambda_k \tag{2.15}$$

For an open loop transfer function without pole in the right-half plane the Bode's integral theorem reads:

$$\int_0^\infty ln\left(|S(j\omega)|\right) d\omega = 0 \tag{2.16}$$

This result can be interpreted as follows: the integrated value of the log of the magnitude of the sensitivity function remains the same whatever the action of the feedback control. At low frequencies, in order to have good tracking performance, the magnitude of the sensitivity function $S(s)$ shall be much lower than 1, thus $ln\left(|S(j\omega)|\right)$ is negative. The Bode's integral theorem indicates that the *average sensitivity improvement* at low frequencies thanks to the feedback control is obtained through the *average sensitivity deterioration* at high frequencies.

If the plant is unstable the situation is worse since the right part of (2.15) is becoming positive.

There is an analogous result for the complementary sensitivity function $T(s) = 1 - S(s) = \frac{L(s)}{1+L(s)}$. Denoting by z_k the zeros of $L(s)$ in the open right-half plane it can be shown that the following relation holds:

$$T(s) = \frac{L(s)}{1+L(s)} \Rightarrow \int_0^\infty ln\left(\left|T(\frac{1}{j\omega})\right|\right) d\omega = \pi \sum_k \frac{1}{z_k} \tag{2.17}$$

2.4 Requirements on sensitivity function

Consider Figure 2.2 where controller $C(s)$ is laid in the direct path. Then it can be shown that for this controller configuration the transfer function $\frac{\epsilon(s)}{R(s)}$ has the same expression than the sensitivity function $S(s)$:

$$\frac{\epsilon(s)}{R(s)} = \frac{1}{1+F(s)\,C(s)} = S(s) \tag{2.18}$$

[1] Multivariable Gain-Phase and Sensitivity Integral Relations and Design Tradeoffs, Jie Chen, IEEE Transactions On Automatic Control, Vol. 43, No. 3, March 1998

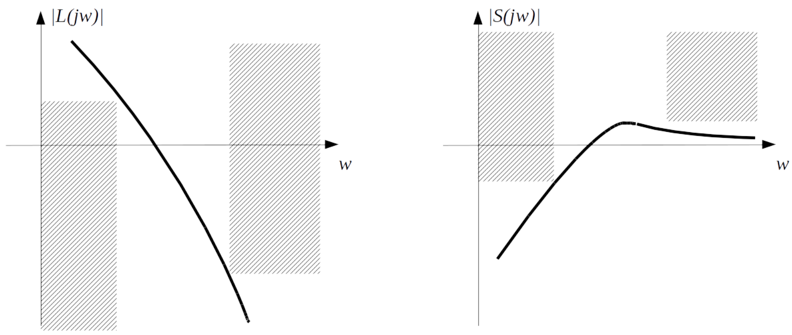

Figure 2.4: Requirements on loop gain $L(s)$ and sensitivity function $S(s)$

Thus, having in mind the final value theorem which states that $lim_{t\to\infty}\epsilon(t) = \lim_{s\to 0} s\,\epsilon(s)$, we conclude that for good tracking performances the sensitivity function $S(s)$ shall be close to zero (in fact much smaller than 1) at low frequency, meaning that loop gain $L(s) = F(s)\,C(s)$ shall be sufficiently high at low frequency. Customary performance specifications require the sensitivity function to be small at low frequencies and to level off to one at high frequencies.

Indeed, the loop gain $L(s) = F(s)\,C(s)$ shall be sufficiently small at high frequency to be robust to high frequency noise and unmodelled dynamics, meaning that sensitivity function $S(s)$ shall be close to 1 at high frequency.

Those requirements are illustrated in Figure 2.4. Bode's integral theorem states that those requirements are conflicting because rolling-off the magnitude of the sensitivity function at high frequencies results in an increase at low frequencies, and vice versa. Sensitivity reduction at low frequencies inevitably leads to sensitivity increase at higher frequencies.

2.5 MIMO systems

In the following $\mathbf{F}_n(s)$ will represent the *nominal* transfer function, that is the known part of the actual transfer function $\mathbf{F}(s)$.

We have seen that sensitivity function plays an important role in robust control. For MIMO systems, the (output) sensitivity matrix $\mathbf{S}(s)$ is defined as follows:

$$\mathbf{S}(s) = \left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1} \tag{2.19}$$

The (output) complementary sensitivity matrix $\mathbf{T}(s)$ reads:

$$\mathbf{T}(s) = \left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1} \mathbf{F}_n(s)\mathbf{C}(s) \tag{2.20}$$

Notice that $\mathbf{S}(s)$ and $\mathbf{T}(s)$ are related according to:

$$\mathbf{S}(s) + \mathbf{T}(s) = \mathbb{I} \tag{2.21}$$

Indeed:

$$\begin{aligned}\mathbb{I}-\mathbf{S}(s) &= \mathbb{I}-(\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))^{-1}\\ &= (\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))^{-1}(\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))-(\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))^{-1}\\ &= (\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))^{-1}((\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))-\mathbb{I})\\ &= (\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))^{-1}\mathbf{F}_n(s)\mathbf{C}(s)\\ &= \mathbf{T}(s)\end{aligned} \tag{2.22}$$

Furthermore the (output) complementary sensitivity matrix $\mathbf{T}(s)$ can be expressed as follows:

$$\mathbf{T}(s)=\mathbf{S}(s)\mathbf{F}_n(s)\mathbf{C}(s)=\mathbf{F}_n(s)\mathbf{C}(s)\mathbf{S}(s) \tag{2.23}$$

Indeed:

$$\begin{aligned}&\mathbf{S}(s)\mathbf{F}_n(s)\mathbf{C}(s)=\mathbf{F}_n(s)\mathbf{C}(s)\mathbf{S}(s)\\ &\Leftrightarrow (\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))^{-1}\mathbf{F}_n(s)\mathbf{C}(s)=\mathbf{F}_n(s)\mathbf{C}(s)(\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))^{-1}\\ &\Leftrightarrow \mathbf{F}_n(s)\mathbf{C}(s)=(\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))\mathbf{F}_n(s)\mathbf{C}(s)(\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))^{-1}\\ &\Leftrightarrow \mathbf{F}_n(s)\mathbf{C}(s)(\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))=(\mathbb{I}+\mathbf{F}_n(s)\mathbf{C}(s))\mathbf{F}_n(s)\mathbf{C}(s)\\ &\Leftrightarrow \mathbf{F}_n(s)\mathbf{C}(s)+(\mathbf{F}_n(s)\mathbf{C}(s))^2=\mathbf{F}_n(s)\mathbf{C}(s)+(\mathbf{F}_n(s)\mathbf{C}(s))^2\end{aligned} \tag{2.24}$$

Chapter 3

Modelling uncertainty

3.1 Introduction

A control system is robust if it is insensitive to differences between the actual system and the model of the system which is used to design the controller.

There are typically two sources of uncertainties: unstructured uncertainties and parametric uncertainties:

– Unstructured uncertainties come from unmodelled or neglected system dynamics (usually high-frequency) or neglected nonlinearities when modelling the plant as a linear time invariant system.

– Parametric uncertainties come from an inaccurate description of component characteristics. They can be described by variations of certain system parameters over some set of possible values. They affect the low-frequency range performance.

These modelling uncertainties may adversely affect the stability and performance of a control system. In this chapter we will discuss how dynamic perturbations are usually described so that they can be used in the framework of robust system design.

Furthermore we will illustrate that uncertainty can be modelled through the block diagram in Figure 3.1 where:

– $\underline{w}$ is the exogenous input vector such as disturbances or reference input

– $\underline{z}$ is the performance output vector, that is the vector that allows to characterize the performance of the closed loop system. This is a *virtual* output used only for design that we wish to maintain as small as possible

– $\mathbf{M}(s)$ is the transfer matrix of the *known* generalized plant.

– $\mathbf{\Delta}(s)$ is an unknown transfer matrix which represents uncertainty. $\mathbf{\Delta}(s)$ is taken in the set of all stable and proper rational transfer matrices that map the imaginary axis into the open disk of radius γ:

$$\mathbf{\Delta}(s) \in RH_\infty \text{ s.t. } \|\mathbf{\Delta}(s)\|_\infty < \gamma \tag{3.1}$$

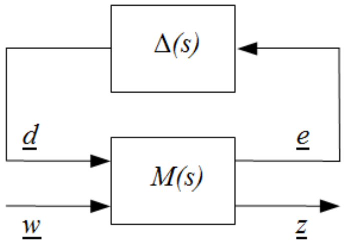

Figure 3.1: $M\Delta$ structure

– The output of $\mathbf{\Delta}(s)$ is $\underline{d}(s)$ and its input is $\underline{e}(s)$.

The transfer matrix of the interconnected system depicted in Figure 3.1 is the following:

$$\begin{bmatrix} \underline{e}(s) \\ \underline{z}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{11}(s) & \mathbf{M}_{12}(s) \\ \mathbf{M}_{21}(s) & \mathbf{M}_{22}(s) \end{bmatrix} \begin{bmatrix} \underline{d}(s) \\ \underline{w}(s) \end{bmatrix} \tag{3.2}$$

3.2 Unstructured uncertainty

We recall that unstructured uncertainties come from unmodelled or neglected system dynamics (usually high-frequency) or neglected nonlinearities when modelling the plant as a linear time invariant system. As far as unstructured uncertainty is concerned, we will consider additive uncertainty, input multiplicative uncertainty and numerator-denominator perturbations. Those uncertainties can easily be plotted in the Bode magnitude plot.

3.2.1 Additive uncertainty

If the plant is subject to additive uncertainty then its input-output relation can be written as follows where $\mathbf{F}_n$ is the transfer matrix of the *nominal* plant:

$$\underline{y}_p(s) = \left(\mathbf{F}_n(s) + \mathbf{\Delta}(s)\mathbf{W}_u(s)\right)\underline{u}(s) \tag{3.3}$$

This equation indicates that the deviation of the actual transfer matrix from the *nominal* transfer matrix $\mathbf{F}_n(s)$ is located inside a circle of radius $\|\mathbf{\Delta}(s)\mathbf{W}_u(s)\|_\infty \leq \|\mathbf{\Delta}(s)\|_\infty\|\mathbf{W}_u(s)\|_\infty \leq \gamma\|\mathbf{W}_u(s)\|_\infty$ which varies with frequency ω when setting $s = j\omega$. Thus the weighting function $\mathbf{W}_u(s)$ indicates how the size of the uncertainties depends upon frequency. Typically $\mathbf{W}_u(s)$ is a high-pass filter because linear time invariant models are, usually, not very accurate at high frequency.

Plant with additive uncertainty is shown on Figure 3.2.

To be compliant with the block diagram of Figure 3.1 we write:

$$\begin{cases} \underline{y}_p(s) = \mathbf{F}_n(s)\underline{u}(s) + \underline{d}(s) \\ \underline{d}(s) = \mathbf{\Delta}(s)\mathbf{W}_u(s)\underline{u}(s) := \mathbf{\Delta}(s)\underline{e}(s) \Rightarrow \underline{e}(s) = \mathbf{W}_u(s)\underline{u}(s) \end{cases} \tag{3.4}$$

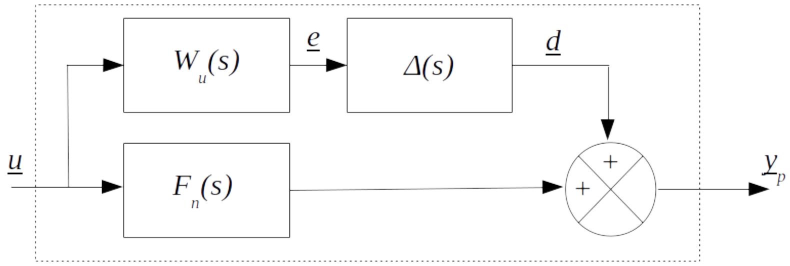

Figure 3.2: Plant with additive uncertainty

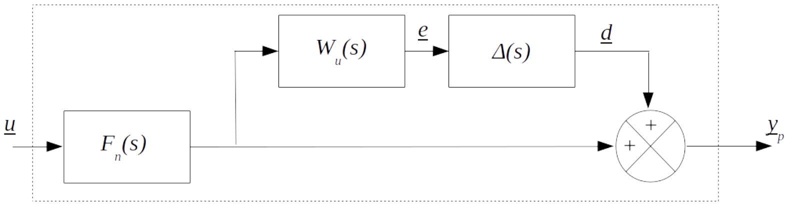

Figure 3.3: Plant with multiplicative uncertainty

That is:

$$\begin{bmatrix} \underline{e}(s) \\ \underline{y}_p(s) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{W}_u(s) \\ \mathbb{I} & \mathbf{F}_n(s) \end{bmatrix} \begin{bmatrix} \underline{d}(s) \\ \underline{u}(s) \end{bmatrix} := \mathbf{M}(s) \begin{bmatrix} \underline{d}(s) \\ \underline{u}(s) \end{bmatrix} \tag{3.5}$$

We can make the identification $\underline{y}_p(s) := \underline{z}(s)$ and $\underline{u}(s) := \underline{w}(s)$.

3.2.2 Output multiplicative uncertainty

Now if the plant is subject to output multiplicative uncertainty then its input-output relation can be written as follows:

$$\underline{y}_p(s) = (\mathbb{I} + \mathbf{\Delta}(s)\mathbf{W}_u(s))\,\mathbf{F}_n(s)\underline{u}(s) \tag{3.6}$$

Plant with multiplicative uncertainty is shown on Figure 3.3.

To be compliant with the block diagram of Figure 3.1 we define $\mathbf{\Delta}(s)$ as follows:

$$\begin{cases} \underline{y}_p(s) = \mathbf{F}_n(s)\underline{u}(s) + \underline{d}(s) \\ \underline{d}(s) = \mathbf{\Delta}(s)\mathbf{W}_u(s)\mathbf{F}_n(s)\underline{u}(s) := \mathbf{\Delta}(s)\underline{e}(s) \end{cases} \tag{3.7}$$

That is:

$$\begin{bmatrix} \underline{e}(s) \\ \underline{y}_p(s) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{W}_u(s)\mathbf{F}_n(s) \\ \mathbb{I} & \mathbf{F}_n(s) \end{bmatrix} \begin{bmatrix} \underline{d}(s) \\ \underline{u}(s) \end{bmatrix} := \mathbf{M}(s) \begin{bmatrix} \underline{d}(s) \\ \underline{u}(s) \end{bmatrix} \tag{3.8}$$

We can make the identification $\underline{y}_p(s) := \underline{z}(s)$ and $\underline{u}(s) := \underline{w}(s)$.

3.2.3 Input-output multiplicative uncertainty

Finally if the plant is subject to input-output multiplicative uncertainty then its input-output relation can be written as follows:

$$\underline{y}_p(s) = (\mathbb{I} + \mathbf{\Delta}_1(s)\mathbf{W}_1(s))\,\mathbf{F}_n(s)\,(\mathbb{I} + \mathbf{\Delta}_2(s)\mathbf{W}_2(s))\,\underline{u}(s) \tag{3.9}$$

To be compliant with the block diagram of Figure 3.1 we first define $\mathbf{\Delta}_1(s)$ as follows:

$$\left\{ \begin{array}{l} \underline{z}(s) := \underline{y}_p(s) = \mathbf{F}_n(s)\,(\mathbb{I} + \mathbf{\Delta}_2(s)\mathbf{W}_2(s))\,\underline{u}(s) + \underline{d}_1(s) \\ \underline{d}_1(s) = \mathbf{\Delta}_1(s)\mathbf{W}_1(s)\mathbf{F}_n(s)\,(\mathbb{I} + \mathbf{\Delta}_2(s)\mathbf{W}_2(s))\,\underline{u}(s) := \mathbf{\Delta}_1(s)\underline{e}_1(s) \end{array} \right. \tag{3.10}$$

In a second step, we do the same with $\mathbf{\Delta}_2(s)$:

$$\left\{ \begin{array}{l} \underline{z}(s) := \underline{y}_p(s) = \mathbf{F}_n(s)\underline{u}(s) + \mathbf{F}_n(s)\underline{d}_2(s) + \underline{d}_1(s) \\ \underline{d}_2(s) = \mathbf{\Delta}_2(s)\mathbf{W}_2(s)\underline{u}(s) := \mathbf{\Delta}_2(s)\underline{e}_2(s) \end{array} \right. \tag{3.11}$$

Using the expressions of $\underline{d}_2(s)$ the expression of $\underline{e}_1(s)$ becomes:

$$\begin{array}{l} \underline{d}_2(s) = \mathbf{\Delta}_2(s)\mathbf{W}_2(s)\underline{u}(s) \\ \Rightarrow \underline{e}_1(s) = \mathbf{W}_1(s)\mathbf{F}_n(s)\underline{u}(s) + \mathbf{W}_1(s)\mathbf{F}_n(s)\underline{d}_2(s) \end{array} \tag{3.12}$$

We finally get:

$$\left[\begin{array}{c} \underline{e}_1(s) \\ \underline{e}_2(s) \\ \hline \underline{z}(s) \end{array} \right] = \left[\begin{array}{cc|c} \mathbf{0} & \mathbf{W}_1(s)\mathbf{F}_n(s) & \mathbf{W}_1(s)\mathbf{F}_n(s) \\ \mathbf{0} & \mathbf{0} & \mathbf{W}_2(s) \\ \hline \mathbb{I} & \mathbf{F}_n(s) & \mathbf{F}_n(s) \end{array} \right] \left[\begin{array}{c} \underline{d}_1(s) \\ \underline{d}_2(s) \\ \hline \underline{u}(s) \end{array} \right] \tag{3.13}$$

That is:

$$\left[\begin{array}{c} \underline{e}(s) \\ \underline{z}(s) \end{array} \right] := \mathbf{M}(s) \left[\begin{array}{c} \underline{d}(s) \\ \underline{u}(s) \end{array} \right] \tag{3.14}$$

Where:

$$\underline{e}(s) = \left[\begin{array}{c} \underline{e}_1(s) \\ \underline{e}_2(s) \end{array} \right] \text{ and } \underline{d}(s) = \left[\begin{array}{c} \underline{d}_1(s) \\ \underline{d}_2(s) \end{array} \right] \tag{3.15}$$

And:

$$\underline{d}(s) = \left[\begin{array}{c} \underline{d}_1(s) \\ \underline{d}_2(s) \end{array} \right] = \mathbf{\Delta}(s) \left[\begin{array}{c} \underline{e}_1(s) \\ \underline{e}_2(s) \end{array} \right] := \mathbf{\Delta}(s)\underline{e}(s) \tag{3.16}$$

Where:

$$\mathbf{\Delta}(s) = \left[\begin{array}{cc} \mathbf{\Delta}_1(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Delta}_2(s) \end{array} \right] \tag{3.17}$$

3.2.4 Numerator-denominator perturbations

Let the plant transfer function $F(s)$ by represented as follows, where $N_n(s)$ and $D_n(s)$ represent the nominal plant numerator and denominator, respectively[1]:

$$F(s) := \frac{y_p(s)}{u(s)} = \frac{N_n(s) + \delta_n(s)\mathbf{W}_n(s)}{D_n(s) + \delta_d(s)\mathbf{W}_d(s)} \tag{3.18}$$

[1]Huibert Kwakernaak, Robust control and H_∞-optimization – Tutorial paper, Automatica Volume 29, Issue 2, March 1993, Pages 255-273

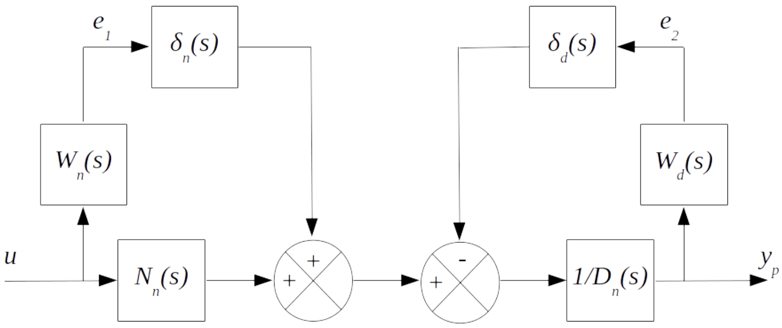

Figure 3.4: Numerator-denominator perturbations model

Frequency dependent functions $\mathbf{W}_n(s)$ and $\mathbf{W}_d(s)$ represent the largest possible perturbations of the denominator and numerator, respectively, whereas $\delta_n(s)$ and $\delta_d(s)$ are frequency dependent functions representing the uncertainty of magnitude not greater and satisfying:

$$\left\| \begin{bmatrix} \delta_n(s) & \delta_d(s) \end{bmatrix} \right\|_\infty < 1 \tag{3.19}$$

It can be checked that the plant transfer function $F(s)$ can be represented by the block diagram of Figure 3.4. Note that the perturbation $\delta_d(s)$ appears in the feedback loop:

$$\begin{aligned} y_p(s) &= \tfrac{1}{D_n(s)} \left(-\delta_d(s)\mathbf{W}_d(s)y_p(s) + (N_n(s) + \delta_n(s)\mathbf{W}_n(s))\, u(s)\right) \\ \Leftrightarrow \tfrac{y_p(s)}{u(s)} &= \frac{N_n(s) + \delta_n(s)\mathbf{W}_n(s)}{D_n(s) + \delta_d(s)\mathbf{W}_d(s)} \end{aligned} \tag{3.20}$$

To be compliant with the block diagram of Figure 3.1 we define $\mathbf{\Delta}(s)$ as follows:

$$\begin{aligned} y_p(s) &= \tfrac{1}{D_n(s)} \left(d(s) + N_n(s)\underline{u}(s)\right) \\ d(s) &= \delta_n(s)\mathbf{W}_n(s)u(s) - \delta_d(s)\mathbf{W}_d(s)y_p(s) \\ &= \begin{bmatrix} \delta_n(s) & -\delta_d(s) \end{bmatrix} \begin{bmatrix} e_1(s) \\ e_2(s) \end{bmatrix} \\ &:= \mathbf{\Delta}(s)\underline{e}(s) \end{aligned} \tag{3.21}$$

For MIMO plant, the left normalized coprime factorization of the nominal plant transfer function $\mathbf{F}_n(s)$ shall be used[2]:

$$\begin{aligned} &\mathbf{F}_n(s) := \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right) \\ &\Leftrightarrow \mathbf{F}_n(s) = \mathbf{D}_n^{-1}(s)\mathbf{N}_n(s) \text{ where } \left\{ \begin{array}{l} \mathbf{D}_n(s) = \left(\begin{array}{c|c} \mathbf{A}+\mathbf{HC} & \mathbf{H} \\ \hline \mathbf{R}^{-0.5}\mathbf{C} & \mathbf{R}^{-0.5} \end{array}\right) \\ \mathbf{N}_n(s) = \left(\begin{array}{c|c} \mathbf{A}+\mathbf{HC} & \mathbf{B}+\mathbf{HD} \\ \hline \mathbf{R}^{-0.5}\mathbf{C} & \mathbf{R}^{-0.5}\mathbf{D} \end{array}\right) \end{array}\right. \end{aligned} \tag{3.22}$$

[2]D. J. Hoyle, R. A. Hyde and D. J. N. Limebeer, An H_∞ approach to two degree of freedom design, [1991] Proceedings of the 30th IEEE Conference on Decision and Control, Brighton, UK, 1991, pp. 1581-1585 vol.2, doi: 10.1109/CDC.1991.261671.

In the preceding relation matrices $\mathbf{R}$ and $\mathbf{H}$ are defined as follows:

$$\begin{cases} \mathbf{R} := \mathbb{I} + \mathbf{D}\mathbf{D}^T \\ \mathbf{H} := -\left(\mathbf{B}\mathbf{D}^T + \mathbf{Z}\mathbf{C}^T\right)\mathbf{R}^{-1} \end{cases} \tag{3.23}$$

Assuming that $\mathbf{R}^{-1}$ is a positive definite matrix, it can be factored as follows where matrix $\mathbf{R}_{chol}$ is an upper triangular matrix with positive diagonal elements:

$$\mathbf{R}^{-1} = \mathbf{R}_{chol}^T \mathbf{R}_{chol} \tag{3.24}$$

$\mathbf{R}_{chol}$ is called the Cholesky factor of $\mathbf{R}^{-1}$ and can be interpreted as the square root of the positive definite matrix $\mathbf{R}^{-1}$:

$$\mathbf{R}^{-0.5} := \mathbf{R}_{chol} \tag{3.25}$$

Matrix $\mathbf{Z} = \mathbf{Z}^T \geq 0$ is the unique stabilizing solution to the following Riccati algebraic equation:

$$\left(\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{D}^T\mathbf{C}\right)\mathbf{Z} + \mathbf{Z}\left(\mathbf{A} - \mathbf{B}\mathbf{S}^{-1}\mathbf{D}^T\mathbf{C}\right)^T - \mathbf{Z}\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\mathbf{Z} + \mathbf{B}\mathbf{S}^{-1}\mathbf{B}^T = \mathbf{0} \tag{3.26}$$

with:

$$\mathbf{S} := \mathbb{I} + \mathbf{D}^T\mathbf{D} \tag{3.27}$$

3.3 Parametric uncertainty

We recall that parametric uncertainties come from an inaccurate description of component characteristics. We will consider in that section some examples of parametric uncertainty.

Example 3.1. *Let's consider a second order system with parametric uncertainty on damping ratio c:*

$$\ddot{y}_p(t) + c\dot{y}_p(t) + y_p(t) = u(t) \quad \textit{where} \quad \underline{c} < c < \overline{c} \tag{3.28}$$

The nominal value of the damping ratio c is denoted c_0:

$$c_0 = \frac{\overline{c} + \underline{c}}{2} \tag{3.29}$$

Introducing weight $\mathbf{W}$ defined as follows:

$$\mathbf{W} = \frac{\overline{c} - \underline{c}}{2c_0} \tag{3.30}$$

We get:

$$c = c_0\left(1 + \mathbf{W}\Delta\right) \tag{3.31}$$

Where:

$$-1 < \Delta < 1 \tag{3.32}$$

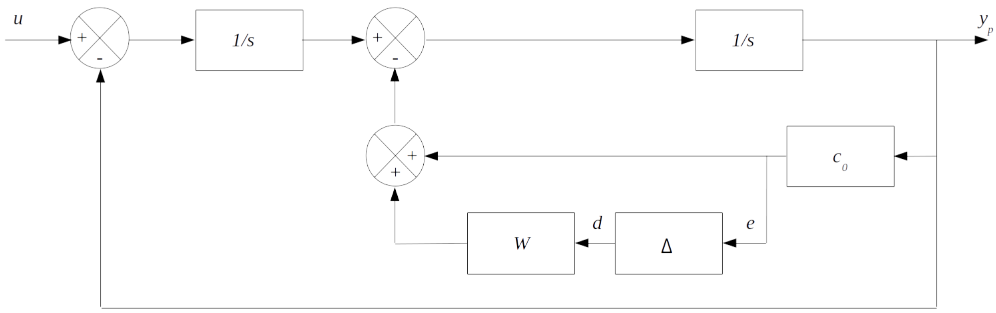

Figure 3.5: Second order system with parametric uncertainty

The transfer function of the system is obtained by taking the Laplace transform of (3.28) *without any initial condition on $y_p(0)$ and $\dot{y}_p(0)$:*

$$\begin{array}{l} s^2 y_p(s) + c_0 \left(1 + \mathbf{W}\Delta\right) s y_p(s) + y_p(s) = u(s) \\ \Leftrightarrow y_p(s) = \frac{u(s) - y_p(s)}{s^2} - c_0 \left(1 + \mathbf{W}\Delta\right) \frac{y_p(s)}{s} \end{array} \tag{3.33}$$

In order to obtain the block diagram in Figure 3.1 where $\mathbf{M}(s)$ is the transfer function of the known generalized plant and $\mathbf{\Delta}(s)$ the transfer function of the uncertainty let's draw the block diagram associated with (3.33) *which is shown in Figure 3.5.*

We choose to identify output $d(s)$ with $\Delta\, c_0\, y_p(s)$ (others choices are possible, for example $d(s) = \Delta\, y_p(s)$). Thus equation (3.33) *can be rewritten as follows:*

$$\begin{array}{l} d(s) := \Delta\, c_0\, y_p(s) \\ \Rightarrow y_p(s) = \frac{u(s)}{s^2} - c_0 \frac{y_p(s)}{s} - \mathbf{W} \frac{d(s)}{s} - \frac{y_p(s)}{s^2} \\ \Leftrightarrow y_p(s) \left(1 + \frac{c_0}{s} + \frac{1}{s^2}\right) = \frac{u(s)}{s^2} - \mathbf{W} \frac{d(s)}{s} \\ \Leftrightarrow y_p(s) = \frac{1}{s^2 + c_0 s + 1} u(s) - \frac{\mathbf{W} s}{s^2 + c_0 s + 1} d(s) \end{array} \tag{3.34}$$

Then identifying input $e(s)$ with $c_0 y_p(s)$ we get:

$$e(s) := c_0 y_p(s) \Rightarrow e(s) = \frac{c_0}{s^2 + c_0 s + 1} u(s) - \frac{c_0 \mathbf{W} s}{s^2 + c_0 s + 1} d(s) \tag{3.35}$$

Finally we obtain the matrix formalism of (3.2)*:*

$$\begin{bmatrix} e(s) \\ y_p(s) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{11}(s) & \mathbf{M}_{12}(s) \\ \mathbf{M}_{21}(s) & \mathbf{M}_{22}(s) \end{bmatrix} \begin{bmatrix} d(s) \\ u(s) \end{bmatrix} \tag{3.36}$$

Where:

$$\left\{ \begin{array}{l} \mathbf{M}_{11}(s) = -\frac{c_0 \mathbf{W} s}{s^2 + c_0 s + 1} \\ \mathbf{M}_{12}(s) = \frac{c_0}{s^2 + c_0 s + 1} \\ \mathbf{M}_{21}(s) = -\frac{s \mathbf{W}}{s^2 + c_0 s + 1} \\ \mathbf{M}_{22}(s) = \frac{1}{s^2 + c_0 s + 1} \end{array} \right. \tag{3.37}$$

Moreover:

$$d(s) = \mathbf{\Delta}(s) e(s) \text{ where } \mathbf{\Delta}(s) = \Delta \tag{3.38}$$

And:

$$-1 < \Delta < +1 \Rightarrow \|\Delta(s)\|_\infty = 1 \tag{3.39}$$

■

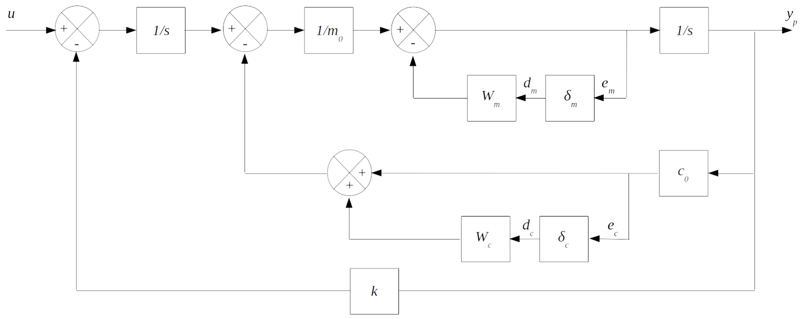

Figure 3.6: Mass-spring damper with parametric uncertainty

Example 3.2. *Let's consider a mass-spring damper whose dynamical equation of motion is the following:*

$$m\ddot{y}_p(t) + c\dot{y}_p(t) + ky_p(t) = u(t) \tag{3.40}$$

We will assume parametric uncertainty on damping ratio c and mass m whereas the spring stiffness k is perfectly known:

$$\left\{ \begin{array}{l} \underline{c} < c < \overline{c} \\ \underline{m} < m < \overline{m} \end{array} \right. \tag{3.41}$$

The nominal value of the damping ratio c is c_0 and the nominal value of the mass m is m_0:

$$\left\{ \begin{array}{l} c_0 = \frac{\overline{c}+\underline{c}}{2} \\ m_0 = \frac{\overline{m}+\underline{m}}{2} \end{array} \right. \tag{3.42}$$

Introducing weight $\mathbf{W}_c$ on the damping ratio and weight $\mathbf{W}_m$ on the mass we get:

$$\left\{ \begin{array}{l} \mathbf{W}_c = \frac{\overline{c}-\underline{c}}{2c_0} \\ \mathbf{W}_m = \frac{\overline{m}-\underline{m}}{2m_0} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} c = c_0\left(1 + \mathbf{W}_c\Delta_c\right) \\ m = m_0\left(1 + \mathbf{W}_m\Delta_m\right) \end{array} \right. \tag{3.43}$$

Where:

$$\left\{ \begin{array}{l} -1 < \Delta_c < 1 \\ -1 < \Delta_m < 1 \end{array} \right. \tag{3.44}$$

The transfer function of the system is obtained by taking the Laplace transform of (3.40) *without any initial condition on $y_p(0)$ and $\dot{y}_p(0)$:*

$$\begin{array}{l} m_0\left(1 + \mathbf{W}_m\Delta_m\right)s^2 y_p(s) + c_0\left(1 + \mathbf{W}_c\Delta_c\right)sy_p(s) + ky_p(s) = u(s) \\ \Leftrightarrow y_p(s) = \frac{1}{m_0(1+\mathbf{W}_m\Delta_m)}\frac{u(s)-ky_p(s)}{s^2} - \frac{c_0(1+\mathbf{W}_c\Delta_c)}{m_0(1+\mathbf{W}_m\Delta_m)}\frac{y_p(s)}{s} \end{array} \tag{3.45}$$

In order to obtain the block diagram in Figure 3.1 where $\mathbf{M}(s)$ is the transfer function of the known generalized plant and $\mathbf{\Delta}(s)$ the transfer function of the uncertainty let's draw the block diagram associated with (3.45) *which is shown in Figure 3.6.*

We choose to identify output $\underline{d}(s)$ with $\left[\begin{array}{cc} d_m & d_c \end{array}\right]^T$ where $d_m = \Delta_m s y_p(s)$ and $d_c = \Delta_c c_0 y_p(s)$. With this choice the first equation of (3.45) *can be rewritten* *as follows:*

$$\begin{array}{l}\underline{d}(s) := \begin{bmatrix} d_m \\ d_c \end{bmatrix} = \begin{bmatrix} \Delta_m s y_p(s) \\ \Delta_c c_0 y_p(s) \end{bmatrix} \\ \Rightarrow m_0 s^2 y_p(s) + m_0 \mathbf{W}_m s d_m(s) + c_0 s y_p(s) + \mathbf{W}_c s d_c(s) + k y_p(s) = u(s) \\ \Leftrightarrow \left(m_0 s^2 + c_0 s + k\right) y_p(s) = u(s) - m_0 \mathbf{W}_m s d_m(s) - \mathbf{W}_c s d_c(s) \\ \Leftrightarrow y_p(s) = \frac{1}{m_0 s^2 + c_0 s + k} u(s) - \frac{m_0 \mathbf{W}_m s}{m_0 s^2 + c_0 s + k} d_m(s) - \frac{\mathbf{W}_c s}{m_0 s^2 + c_0 s + k} d_c(s) \end{array} \tag{3.46}$$

Then identifying input $\underline{e}(s)$ *with* $\begin{bmatrix} e_m(s) & e_c(s) \end{bmatrix}^T$ *where* $e_m(s) = s y_p(s)$ *and* $e_c(s) = c_0 y_p(s)$ *we finally obtain the matrix formalism of* (3.2)*:*

$$\underline{e}(s) := \begin{bmatrix} e_m(s) \\ e_c(s) \end{bmatrix} = \begin{bmatrix} s y_p(s) \\ c_0 y_p(s) \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{e}(s) \\ y_p(s) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{11}(s) & \mathbf{M}_{12}(s) \\ \mathbf{M}_{21}(s) & \mathbf{M}_{22}(s) \end{bmatrix} \begin{bmatrix} \underline{d}(s) \\ u(s) \end{bmatrix} \tag{3.47}$$

Where:

$$\left\{ \begin{array}{l} \mathbf{M}_{11}(s) = \begin{bmatrix} -\frac{m_0 \mathbf{W}_m s^2}{m_0 s^2 + c_0 s + k} & -\frac{\mathbf{W}_c s^2}{m_0 s^2 + c_0 s + k} \\ -\frac{c_0 m_0 \mathbf{W}_m s}{m_0 s^2 + c_0 s + k} & -\frac{c_0 \mathbf{W}_c s}{m_0 s^2 + c_0 s + k} \end{bmatrix} \\ \mathbf{M}_{12}(s) = \begin{bmatrix} \frac{s}{m_0 s^2 + c_0 s + k} \\ \frac{c_0}{m_0 s^2 + c_0 s + k} \end{bmatrix} \\ \mathbf{M}_{21}(s) = \begin{bmatrix} -\frac{m_0 \mathbf{W}_m s}{m_0 s^2 + c_0 s + k} & -\frac{\mathbf{W}_c s}{m_0 s^2 + c_0 s + k} \end{bmatrix} \\ \mathbf{M}_{22}(s) = \frac{1}{m_0 s^2 + c_0 s + k} \end{array} \right. \tag{3.48}$$

Moreover:

$$\underline{d}(s) = \mathbf{\Delta}(s) \underline{e}(s) \text{ where } \mathbf{\Delta}(s) = \begin{bmatrix} \Delta_m & 0 \\ 0 & \Delta_c \end{bmatrix} \tag{3.49}$$

And:

$$\left\{ \begin{array}{l} -1 < \Delta_c < +1 \\ -1 < \Delta_m < +1 \end{array} \right. \Rightarrow \|\Delta(s)\|_\infty = 1 \tag{3.50}$$

■

Chapter 4

Stability Analysis of Uncertain Systems

4.1 Nyquist stability criterion

4.1.1 MIMO (Generalized) case

Let's consider the closed-loop in Figure 4.1 where $\mathbf{L}(s)$ denotes the loop transfer function.

The generalized (MIMO) Nyquist stability criterion[1] states that the number of unstable closed-loop poles (that are the roots of $\det(\mathbb{I} + \mathbf{L}(s))$) is equal to the number of unstable open-loop poles of the loop transfer function $\mathbf{L}(s)$ plus the number of encirclements of the critical point $(0, 0)$ by the Nyquist plot of $\det(\mathbb{I}+\mathbf{L}(s))$; the encirclement is counted positive in the clockwise direction and negative otherwise.

We remind that the Nyquist plot of $\det(\mathbb{I} + \mathbf{L}(s))$ is the image of $\det(\mathbb{I} + \mathbf{L}(s))$ as s goes clockwise around the Nyquist contour: this includes the entire imaginary axis ($s = jw$) and an infinite semi-circle around the right half plane as shown in Figure 4.2.

It is worth noticing that if the loop transfer function $\mathbf{L}(s)$ is stable without zeros in the right half plane (that is minimum phase), the Nyquist plot of $\det(\mathbb{I}+$

[1] Multivariable Feedback Control: Analysis and Design, Sigurd Skogestad & Ian Postlethwaite, Wiley-Interscience 2005

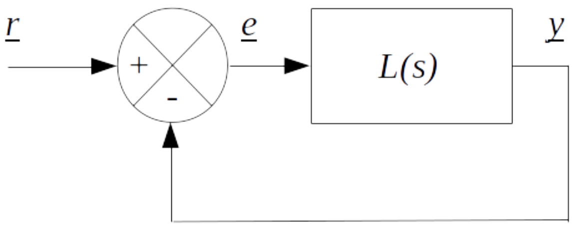

Figure 4.1: Unity feedback loop

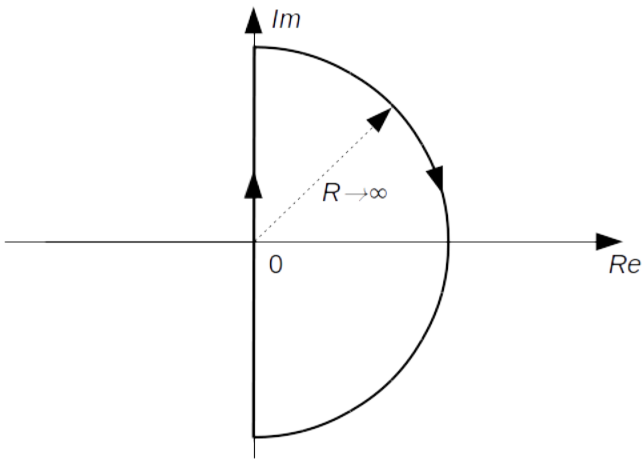

Figure 4.2: Nyquist contour

$\mathbf{L}(s)$) must *not* encircle the critical point $(0, 0)$ for the closed-loop to be stable.

4.1.2 SISO case

For Single-Input Single-Output (SISO) systems the loop transfer function $\mathbf{L}(s)$ is a scalar and we have:

$$\det(\mathbb{I} + \mathbf{L}(s)) = \det(1 + L(s)) = 1 + L(s) \tag{4.1}$$

Thus for Single-Input Single-Output (SISO) systems the number of encirclements of the critical point $(0, 0)$ by the Nyquist plot of $1 + L(s)$ is equivalent to the number of encirclements of the critical point $(-1, 0)$ by the Nyquist plot of $L(s)$.

4.1.3 Phase margin, gain margin and modulus margin

We assume in that section that the loop transfer function $\mathbf{L}(s)$ is stable without zeros in the right half plane (that is minimum phase transfer function).

We recall that the gain crossover frequency is the frequency ω_c at which the magnitude $\|L(j\omega)\|$ of the loop transfer function $L(s)$ is unity. The gain crossover frequency is closely related to the rise time and thus to the bandwidth of the closed-loop system.

At the gain crossover frequency ω_c the phase angle $\Phi(\omega_c) = \arg(L(j\omega_c))$ of the loop transfer function reads as follows where M_Φ represents the phase margin of the closed-loop system:

$$\begin{array}{c} \|L(j\omega_c)\| = 1 \Rightarrow M_\Phi = 180^\circ + \arg(L(j\omega_c)) \quad (\textit{modulo } 360^\circ) \\ \Leftrightarrow \Phi(\omega_c) = \arg(L(j\omega_c)) = -180^\circ + M_\Phi \quad (\textit{modulo } 360^\circ) \end{array} \tag{4.2}$$

For closed-loop system to be stable the phase margin M_Φ must be positive; it indicates the amount of phase lag which is required to bring the closed-loop system unstable.

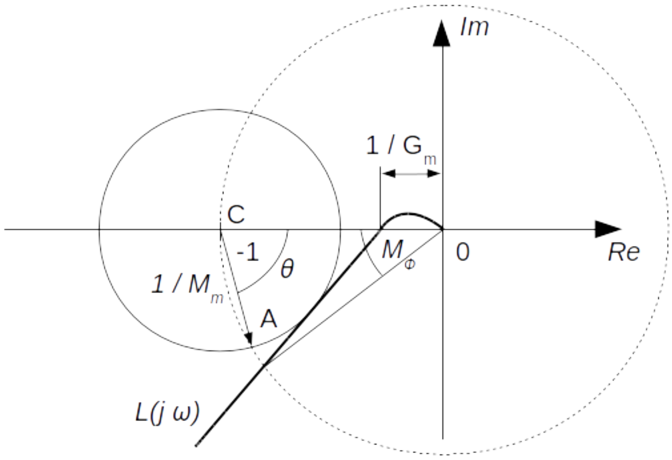

Figure 4.3: Nyquist plot and modulus margin M_m

The phase crossover frequency is the frequency ω_π at which the phase angle of the loop transfer function $L(s)$ is equal to $-180°$. At the phase crossover frequency ω_π the magnitude $\|L(j\omega)\|$ of the loop transfer function is the reciprocal of the gain margin Gm:

$$\arg\left(L(j\omega_\pi)\right) = -180° \quad (modulo\ 360°) \Rightarrow Gm = \frac{1}{\|L(j\omega_\pi)\|} \tag{4.3}$$

The modulus margin M_m is defined as the inverse of the infinity-norm of the sensitivity function, that is the reciprocal of the peak of the sensitivity function $S(s)$:

$$M_m = \|S(s)\|_\infty^{-1} = \frac{1}{\|S(s)\|_\infty} \text{ where } S(s) = \frac{1}{1+L(s)} \tag{4.4}$$

In other words, and assuming that SISO loop transfer function $L(s)$ is stable without zeros in the right half plane (that is minimum phase), the reciprocal of the modulus margin M_m is the shortest distance from the Nyquist plot of the loop transfer function $L(s)$ to the critical point $(-1, 0)$. This is obtained as the tangent of the Nyquist plot of $L(j\omega)$ with the circle of centre $(-1, 0)$ as shown in Figure 4.3. Typical values of $\frac{1}{M_m}$ are in the range of $[0.5, 0.75]$.

For the MIMO case, and assuming that loop transfer function $\mathbf{L}(s)$ is stable without zeros in the right half plane (that is minimum phase), modulus margin M_m represents the minimum distance between the Nyquist plot of $\det(\mathbb{I}+\mathbf{L}(s))$ and the critical point $(0, 0)$:

$$\begin{aligned} M_m &= \min_{\omega\geq 0} |\det(\mathbb{I}+\mathbf{L}(j\omega))| \\ &= \inf_{\omega\geq 0} \underline{\sigma}\left(\mathbb{I}+\mathbf{L}(j\omega)\right) \\ &= \frac{1}{\sup_{\omega\geq 0} \overline{\sigma}\left((\mathbb{I}+\mathbf{L}(j\omega))^{-1}\right)} \end{aligned} \tag{4.5}$$

where $\overline{\sigma}\left(\mathbf{G}\left(j\omega\right)\right)$ is the largest singular-value of matrix $\mathbf{G}(j\omega)$, that is the ω valued upper bound of the set of singular-values $\{\sigma_i\left(\mathbf{G}(j\omega)\right)\}$ and $\underline{\sigma}\left(\mathbf{G}\left(j\omega\right)\right)$ is the smallest singular-value of matrix $\mathbf{G}(j\omega)$, that is the ω valued lower bound of the set of singular-values $\{\sigma_i\left(\mathbf{G}(j\omega)\right)\}$.

Using the fact that $\vec{OC} + \vec{CA} = \vec{OA}$, we can write the following relations when identifying the real part and the imaginary part of the equation[2]:

$$\begin{array}{l} \vec{OC} + \vec{CA} = \vec{OA} \\ \Leftrightarrow (-1 + j \cdot 0) + (\frac{1}{M_m}\, e^{-j\theta}) = 1 \cdot e^{j(\pi + \widehat{AOC})} \\ \Leftrightarrow \left\{ \begin{array}{l} -1 + \frac{1}{M_m} \cos(\theta) = -\cos(\widehat{AOC}) \\ -\frac{1}{M_m} \sin(\theta) = -\sin(\widehat{AOC}) \end{array} \right. \end{array} \tag{4.6}$$

Using the fact that triangle (AOC) is an isocele triangle we have:

$$2\,\theta + \widehat{AOC} = \pi \tag{4.7}$$

Thus, from the fact that $\cos(2\,\theta) = 2\,\cos^2(\theta) - 1$ and $\sin(2\,\theta) = 2\cos(\theta)\sin(\theta)$ we get:

$$\left\{ \begin{array}{l} -1 + \frac{1}{M_m} \cos(\theta) = \cos(2\,\theta) \\ \frac{1}{M_m} \sin(\theta) = \sin(2\,\theta) \end{array} \right. \Rightarrow \frac{1}{M_m} = 2\,\cos(\theta) = 2\,\sin\left(\frac{\widehat{AOC}}{2}\right) \tag{4.8}$$

From the preceding relation and Figure 4.3 phase margin M_Φ can be related to modulus margin M_m through the following inequality:

$$M_\Phi \geq \widehat{AOC} = 2\,\mathrm{asin}\left(\frac{1}{2\,M_m}\right) \tag{4.9}$$

We can also notice from Figure 4.3 that gain margin G_m can be related to modulus margin M_m through the following inequality:

$$\frac{1}{M_m} + \frac{1}{G_m} \leq 1 \Leftrightarrow G_m \geq \frac{M_m}{M_m - 1} \tag{4.10}$$

4.2 Kharitonov theorem

Let's consider the following polynomial where each coefficient a_i is uncertain:

$$P(s) = a_0 + a_1 s^1 + a_2 s^2 + \ldots + a_n s^n \tag{4.11}$$

We assume that each coefficient $a_i \in \mathbb{R}$ can take any value in a specified interval:

$$l_i \leq a_i \leq u_i \tag{4.12}$$

It is also assumed that the leading coefficient cannot be zero: $0 \notin [l_n, u_n]$.

Kharitonov theorem states that polynomial $P(s)$ is stable (i.e. all members of the family are stable) if and only if the four so-called Kharitonov polynomials are stable [3]:

$$\left\{ \begin{array}{l} k_1(s) = l_0 + l_1 s^1 + u_2 s^2 + u_3 s^3 + l_4 s^4 + l_5 s^5 + \cdots \\ k_2(s) = u_0 + u_1 s^1 + l_2 s^2 + l_3 s^3 + u_4 s^4 + u_5 s^5 + \cdots \\ k_3(s) = l_0 + u_1 s^1 + u_2 s^2 + l_3 s^3 + l_4 s^4 + u_5 s^5 + \cdots \\ k_4(s) = u_0 + l_1 s^1 + l_2 s^2 + u_3 s^3 + u_4 s^4 + l_5 s^5 + \cdots \end{array} \right. \tag{4.13}$$

[2]PID controller design with an H_∞ criterion, Sangjin Han, Lee H.Keel, Shankar P.Bhattacharyya, IFAC-PapersOnLine Volume 51, Issue 4, 2018, Pages 400-405

[3]https://en.wikipedia.org/wiki/Kharitonov's_theorem

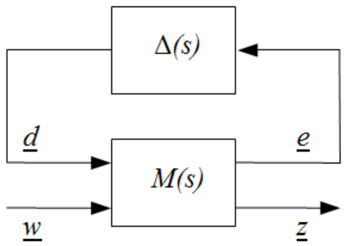

Figure 4.4: $M\Delta$ structure

4.3 Small gain theorem

Small gain theorem deals with the stability of the interconnection of two systems as shown in Figure 4.4.

The input-output relation is written as follows:

$$\begin{bmatrix} \underline{e}(s) \\ \underline{z}(s) \end{bmatrix} = \mathbf{M}(s) \begin{bmatrix} \underline{d}(s) \\ \underline{w}(s) \end{bmatrix} \tag{4.14}$$

Where:

$$\mathbf{M}(s) = \begin{bmatrix} \mathbf{M}_{11}(s) & \mathbf{M}_{12}(s) \\ \mathbf{M}_{21}(s) & \mathbf{M}_{22}(s) \end{bmatrix} \tag{4.15}$$

We will assume that there is no internal Right Half Plane (RHP) pole cancellation in the open loop transfer matrix $\mathbf{M}_{11}(s)\mathbf{\Delta}(s)$ (i.e. $\mathbf{M}_{11}(s)\mathbf{\Delta}(s)$ contains no hidden unstable mode).

The $M\Delta$ structure in Figure 3.1 is stable for all allowed perturbations if and only if all the roots of the characteristic equation $\det(\mathbb{I} - \mathbf{M}_{11}(s)\mathbf{\Delta}(s)) = 0$ (we assume positive feedback) are situated in the Left Half Plane (LHP) [4].

The generalized (MIMO) Nyquist theorem [1] indicates that if N denotes the number of open loop unstable poles in $\mathbf{M}_{11}(s)\mathbf{\Delta}(s)$ then the $M\Delta$ structure is stable if and only if the Nyquist plot of $\det(\mathbb{I} - \mathbf{M}_{11}(j\omega)\mathbf{\Delta}(j\omega))$ makes N anti-clockwise encirclements of the origin and does not pass through the origin.

The Routh criterion may also be applied to the characteristic equation $\det(\mathbb{I} - \mathbf{M}_{11}(s)\mathbf{\Delta}(s))$ to check the stability as soon as $\mathbf{\Delta}(s)$ represents unstructured uncertainty for which transfer matrix $\mathbf{\Delta}(s)$ is known. On the other hand we may use the Kharitonov theorem in the case of parametric uncertainty.

Let's consider the case where $\mathbf{M}_{11}(s) \in RH_\infty$, which means that $\mathbf{M}_{11}(s)$ is taken in the set of all stable and proper rational transfer matrices. Then the closed-loop system in Figure 4.4 is internally stable for all $\mathbf{\Delta}(s) \in RH_\infty$ with $\|\mathbf{\Delta}(s)\|_\infty \leq \gamma$ if $\|\mathbf{M}_{11}(s)\|_\infty \leq 1/\gamma$. This can be summarized by the so-called *small gain theorem*:

$$\begin{array}{c} \|\mathbf{M}_{11}(s)\|_\infty \|\mathbf{\Delta}(s)\|_\infty < 1 \Rightarrow \text{ closed-loop stability} \\ \text{assuming} \begin{cases} \mathbf{M}_{11}(s) \in RH_\infty \\ \mathbf{\Delta}(s) \in RH_\infty \end{cases} \end{array} \tag{4.16}$$

[4]Essentials of Robust Control, Kemin Zhou & John C. Doyle, Prentice Hall 1997

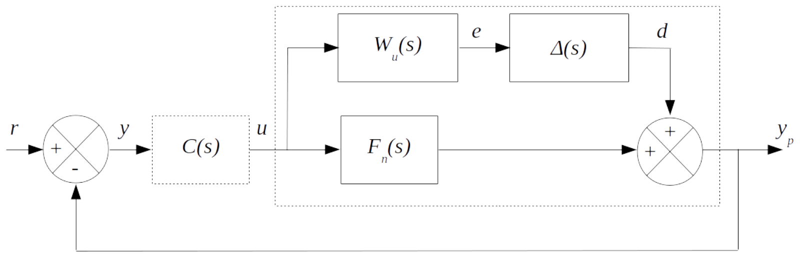

Figure 4.5: Feedback loop with additive uncertainty

From a practical point of view, transfer matrix $\mathbf{M}_{11}(s)$ can be computed directly by setting the exogenous input vector $\underline{w}$ to zero.

It can be also shown that the small gain theorem provides a sufficient, but not necessary, condition to guarantee the internal stability of the loop even if $\mathbf{\Delta}(s)$ is a nonlinear and/or time-varying *stable* operator with an appropriate definition of stability. Thus the small gain theorem provides a conservative stability condition.

Finally, the small gain theorem is usually slightly adapted as follows: the interconnection is stable for all $\mathbf{\Delta}(s)$ such that $\|\mathbf{\Delta}(s)/W_a(s)\|_\infty \leq 1/\gamma$ if $\|W_a(s)\mathbf{M}_{11}(s)\|_\infty < \gamma$:

- if $\gamma < 1$ stability is proved for a larger uncertainty set than required ;
- if $\gamma > 1$ stability is proved for a subset of the admissible uncertainties.

4.4 Applications

4.4.1 Additive uncertainties

Let's consider the feedback control loop in Figure 4.5 where $\mathbf{F}_n(s)$ represents the transfer function of the nominal plant and $\mathbf{W}_u(s)\mathbf{\Delta}(s)$ is the additive model uncertainty. The transfer function $\mathbf{F}(s)$ of the actual plant reads:

$$\mathbf{F}(s) = \mathbf{F}_n(s) + \mathbf{\Delta}(s)\mathbf{W}_u(s) \tag{4.17}$$

- $\mathbf{W}_u(s)$ is a given stable uncertainty weighting function;
- $\mathbf{\Delta}(s)$ is the uncertainty itself. This is any stable transfer function which is here assumed to be such that $\|\mathbf{\Delta}(s)\|_\infty \leq 1$.

We will denote:

- The reference input by r
- The output vector of the controller $\mathbf{C}(s)$ by u. In the feedback loop under study, this is also the control vector of the *actual* plant;
- The input vector of the controller by y

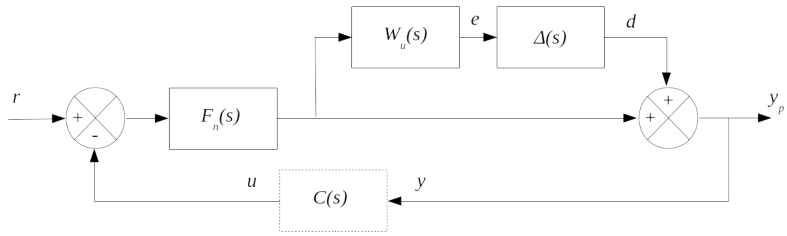

Figure 4.6: Feedback loop with multiplicative uncertainty

We are interested in finding a conservative condition leading to a stable closed-loop despite the additive uncertainty on the plant.

By setting the exogenous input vector $\underline{w} := r$ to zero, transfer matrix $\mathbf{M}_{11}(s)$ can be computed directly:

$$\begin{array}{l} \underline{w} := r = 0 \Rightarrow e(s) = -\mathbf{W}_u(s)\mathbf{C}(s)\left(d(s) + \mathbf{F}_n(s)\mathbf{W}_u^{-1}(s)e(s)\right) \\ \Rightarrow e(s)\left(1 + \mathbf{F}_n(s)\mathbf{C}(s)\right) = -\mathbf{W}_u(s)\mathbf{C}(s)d(s) \end{array} \tag{4.18}$$

We finally get:

$$e(s) = \mathbf{M}_{11}(s)d(s) \text{ where } \mathbf{M}_{11}(s) = \frac{-\mathbf{W}_u(s)\mathbf{C}(s)}{1 + \mathbf{F}_n(s)\mathbf{C}(s)} \tag{4.19}$$

Therefore using the assumption that $\|\mathbf{\Delta}(s)\|_\infty \leq 1$, the small gain theorem indicates that the closed-loop system in Figure 4.5 is internally stable if :

$$\|\mathbf{M}_{11}(s)\|_\infty \|\mathbf{\Delta}(s)\|_\infty < 1 \Rightarrow \left\| \mathbf{W}_u(s) \frac{\mathbf{C}(s)}{1 + \mathbf{F}_n(s)\mathbf{C}(s)} \right\|_\infty < 1 \tag{4.20}$$

It is worth noticing that $\mathbf{S}(s) = \frac{1}{1+\mathbf{F}_n(s)\mathbf{C}(s)}$ defines the sensitivity function. Thus a sufficient condition for the closed-loop to be stable is:

$$\|\mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s)\|_\infty < 1 \tag{4.21}$$

4.4.2 Multiplicative uncertainties

Let's consider the feedback control loop in Figure 4.6 where $\mathbf{F}_n(s)$ represents the transfer function of the nominal plant and $\mathbf{W}_u(s)\mathbf{\Delta}(s)$ is the multiplicative model uncertainty. The transfer function $\mathbf{F}(s)$ of the actual plant reads:

$$\mathbf{F}(s) = \left(\mathbb{I} + \mathbf{\Delta}(s)\mathbf{W}_u(s)\right)\mathbf{F}_n(s) \tag{4.22}$$

- $\mathbf{W}_u(s)$ is a given stable uncertainty weighting function;
- $\mathbf{\Delta}(s)$ is the uncertainty itself. This is any stable transfer function which is here assumed to be such that $\|\mathbf{\Delta}(s)\|_\infty \leq 1$.

We will denote:

- The reference input by r

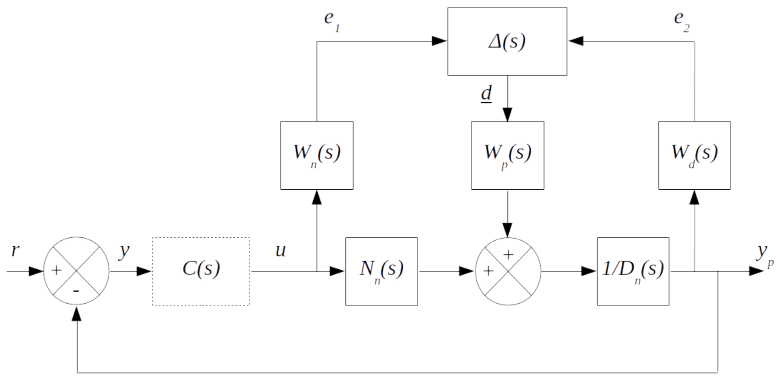

Figure 4.7: Feedback loop with numerator-denominator perturbation model

- The output vector of the controller $\mathbf{C}(s)$ by u. In the feedback loop under study, this is *not* the control vector of the *actual* plant;
- The input vector of the controller by y

We are interested in finding a conservative condition leading to a stable closed-loop despite the multiplicative uncertainty on the plant.

By setting the exogenous input vector $\underline{w} := r$ to zero, transfer matrix $\mathbf{M}_{11}(s)$ can be computed directly:

$$\begin{array}{l} \underline{w} := r = 0 \Rightarrow e(s) = -\mathbf{W}_u(s)\mathbf{F}_n(s)\mathbf{C}(s)\left(d(s) + \mathbf{W}_u^{-1}(s)e(s)\right) \\ \Rightarrow e(s)\left(1 + \mathbf{C}(s)\mathbf{F}_n(s)\right) = -\mathbf{W}_u(s)\mathbf{F}_n(s)\mathbf{C}(s)d(s) \end{array} \tag{4.23}$$

We finally get:

$$e(s) = \mathbf{M}_{11}(s)d(s) \text{ where } \mathbf{M}_{11}(s) = \frac{-\mathbf{W}_u(s)\mathbf{F}_n(s)\mathbf{C}(s)}{1 + \mathbf{F}_n(s)\mathbf{C}(s)} \tag{4.24}$$

Therefore using the assumption that $\|\mathbf{\Delta}(s)\|_\infty \leq 1$, the small gain theorem indicates that the closed-loop system in Figure 4.6 is internally stable if :

$$\|\mathbf{M}_{11}(s)\|_\infty \|\mathbf{\Delta}(s)\|_\infty < 1 \Rightarrow \left\| \mathbf{W}_u(s)\frac{\mathbf{F}_n(s)\mathbf{C}(s)}{1 + \mathbf{F}_n(s)\mathbf{C}(s)} \right\|_\infty < 1 \tag{4.25}$$

It is worth noticing that $\mathbf{T}(s) = \frac{\mathbf{F}_n(s)\mathbf{C}(s)}{1+\mathbf{F}_n(s)\mathbf{C}(s)}$ defines the complementary sensitivity function (the sensitivity function is defined as $\mathbf{S}(s) = \frac{1}{1+\mathbf{F}_n(s)\mathbf{C}(s)}$). Thus a sufficient condition for the closed-loop to be stable is:

$$\|\mathbf{W}_u(s)\mathbf{T}(s)\|_\infty < 1 \tag{4.26}$$

4.4.3 Numerator-denominator perturbation of SISO plant

Let's consider the feedback control loop in Figure 4.7.

The SISO plant transfer function $F(s)$ is represented as follows, where $N_n(s)$ and $D_n(s)$ represent the nominal plant numerator and denominator, respectively[5]:

$$F(s) := \frac{y_p(s)}{u(s)} = \frac{N_n(s) + \delta_n(s)\mathbf{W}_n(s)}{D_n(s) + \delta_d(s)\mathbf{W}_d(s)} \tag{4.27}$$

Frequency dependent functions $\mathbf{W}_n(s)$ and $\mathbf{W}_d(s)$ represent the largest possible perturbations of the denominator and numerator, respectively, whereas $\delta_n(s)$ and $\delta_d(s)$ are frequency dependent functions representing the uncertainty of magnitude not greater and satisfying:

$$\left\| \begin{bmatrix} \delta_n(s) & \delta_d(s) \end{bmatrix} \right\|_\infty < 1 \tag{4.28}$$

The nominal plant transfer function $\mathbf{F}_n(s)$ is defined as follows:

$$\mathbf{F}_n(s) := \frac{N_n(s)}{D_n(s)} \tag{4.29}$$

We have seen in section 3.2.4 that $\mathbf{\Delta}(s) := \begin{bmatrix} \delta_n(s) & -\delta_d(s) \end{bmatrix}$. We have added function $\mathbf{W}_p(s) \neq \mathbb{I}$ which will allow for partial pole placement[5].

We will denote:

- The reference input by r
- The output vector of the controller $\mathbf{C}(s)$ by u
- The input vector of the controller by y

We are interested in finding a conservative condition leading to a stable closed-loop despite the Numerator-denominator uncertainties on the plant.

By setting the exogenous input vector $\underline{w} := r$ to zero, transfer matrix $\mathbf{M}_{11}(s)$ can be computed directly:

- Firstly the transfer from $e_1(s)$ to $\underline{d}(s)$ is obtained as follows:

$$\begin{array}{l} \underline{w} := r = 0 \Rightarrow e_1(s) = -\mathbf{W}_n(s)\mathbf{C}(s)\frac{1}{D_n(s)}\left(\mathbf{W}_p(s)\underline{d}(s) + N_n(s)\mathbf{W}_n^{-1}(s)e_1(s)\right) \\ \Rightarrow e_1(s)\left(1 + \mathbf{C}(s)\frac{N_n(s)}{D_n(s)}\right) = -\mathbf{W}_n(s)\mathbf{C}(s)\frac{1}{D_n(s)}\mathbf{W}_p(s)\underline{d}(s) \end{array} \tag{4.30}$$

  Using the expression of the nominal plant transfer function $\mathbf{F}_n(s) := \frac{N_n(s)}{D_n(s)}$ we finally get:

$$e_1(s) = \frac{-\mathbf{W}_n(s)\mathbf{C}(s)\frac{1}{D_n(s)}\mathbf{W}_p(s)}{1 + \mathbf{C}(s)\frac{N_n(s)}{D_n(s)}}\underline{d}(s) = \frac{-\mathbf{W}_n(s)\mathbf{C}(s)\frac{\mathbf{W}_p(s)}{D_n(s)}}{1 + \mathbf{C}(s)\mathbf{F}_n(s)}\underline{d}(s) \tag{4.31}$$

- Secondly the transfer from $e_2(s)$ to $\underline{d}(s)$ is obtained as follows:

[5] Huibert Kwakernaak, Robust control and H_∞-optimization – Tutorial paper, Automatica Volume 29, Issue 2, March 1993, Pages 255-273

$$
\begin{array}{l}
\underline{w} := r = 0 \Rightarrow e_2(s) = \mathbf{W}_d(s)\frac{1}{D_n(s)}\left(\mathbf{W}_p(s)\underline{d}(s) - N_n(s)\mathbf{C}(s)\mathbf{W}_d^{-1}(s)e_2(s)\right) \\
\Rightarrow e_2(s)\left(1 + \frac{N_n(s)}{D_n(s)}\mathbf{C}(s)\right) = \mathbf{W}_d(s)\frac{1}{D_n(s)}\mathbf{W}_p(s)\underline{d}(s)
\end{array}
\tag{4.32}
$$

Using the expression of the nominal plant transfer function $\mathbf{F}_n(s) := \frac{N_n(s)}{D_n(s)}$ we finally get:

$$
e_2(s) = \frac{\mathbf{W}_d(s)\frac{1}{D_n(s)}\mathbf{W}_p(s)}{1 + \frac{N_n(s)}{D_n(s)}\mathbf{C}(s)}\underline{d}(s) = \frac{\mathbf{W}_d(s)\frac{\mathbf{W}_p(s)}{D_n(s)}}{1 + \mathbf{F}_n(s)\mathbf{C}(s)}\underline{d}(s) \tag{4.33}
$$

We finally get:

$$
\begin{bmatrix} e_1(s) \\ e_2(s) \end{bmatrix} := \underline{e}(s) = \mathbf{M}_{11}(s)\underline{d}(s) \text{ where } \mathbf{M}_{11}(s) = \begin{bmatrix} \frac{-\mathbf{W}_n(s)\mathbf{C}(s)\frac{\mathbf{W}_p(s)}{D_n(s)}}{1+\mathbf{C}(s)\mathbf{F}_n(s)} \\ \frac{\mathbf{W}_d(s)\frac{\mathbf{W}_p(s)}{D_n(s)}}{1+\mathbf{F}_n(s)\mathbf{C}(s)} \end{bmatrix} \tag{4.34}
$$

It is worth noticing that $\mathbf{S}(s) = \frac{1}{1+\mathbf{F}_n(s)\mathbf{C}(s)}$ defines the sensitivity function. Thus a sufficient condition for the closed-loop to be stable is:

$$
\left\| \begin{bmatrix} \mathbf{W}_n(s)\frac{\mathbf{W}_p(s)}{D_n(s)}\mathbf{C}(s)\mathbf{S}(s) \\ \mathbf{W}_d(s)\frac{\mathbf{W}_p(s)}{D_n(s)}\mathbf{S}(s) \end{bmatrix} \right\|_\infty < 1 \tag{4.35}
$$

Kwakernaak[5] have shown that when the degree of polynomial $\mathbf{W}_p(s)$ is chosen to be equal to the degree of polynomial $D_n(s)$, then the roots of $\mathbf{W}_p(s)$ belong to the set of the closed-loop poles. Furthermore by suitably choosing $\mathbf{W}_n(s)$ and $\mathbf{W}_d(s)$ these roots may often be arranged to be dominant poles.

4.4.4 Time-delay systems

Let's consider the following state space representation where τ represents a delay in the system:

$$
\begin{cases} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{A}_d\underline{x}(t-\tau) + \mathbf{B}u(t-\tau) \\ y(t) = \mathbf{C}\underline{x}(t) \end{cases} \tag{4.36}
$$

The state space representation (4.36) can be rewritten as follows:

$$
\begin{cases} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbb{I}\underline{w}(t) \\ y(t) = \mathbf{C}\underline{x}(t) \\ \underline{z}(t) = \mathbf{A}_d\underline{x}(t) + \mathbf{B}u(t) \\ \underline{w}(t) = \underline{z}(t-\tau) \end{cases} \tag{4.37}
$$

More generally the state space representation of a time-delay systems is the following:

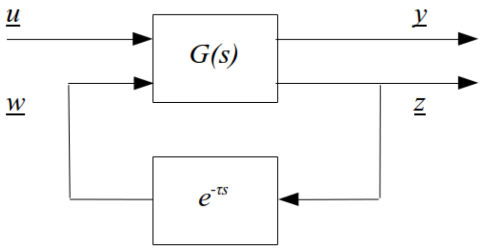

Figure 4.8: General structure for time-delay systems

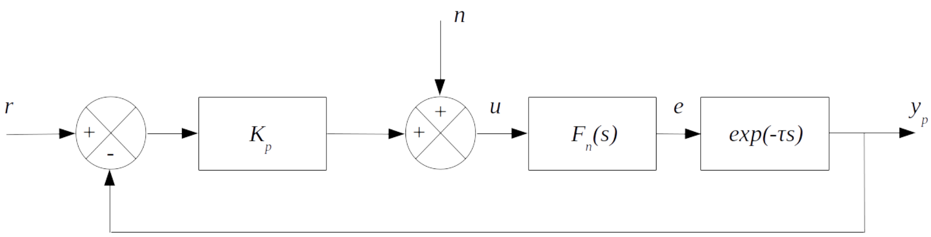

Figure 4.9: System with delay

$$\begin{cases} \underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}_1 u(t) + \mathbf{B}_2 \underline{w}(t) \\ y(t) = \mathbf{C}_1 \underline{x}(t) + \mathbf{D}_{11} u(t) + \mathbf{D}_{12} \underline{w}(t) \\ \underline{z}(t) = \mathbf{C}_2 \underline{x}(t) + \mathbf{D}_{21} u(t) + \mathbf{D}_{22} \underline{w}(t) \\ \underline{w}(t) = \underline{z}(t-\tau) \end{cases} \tag{4.38}$$

Having in mind that the Laplace transform of $\underline{x}(t-\tau)$ is $\underline{X}(s)e^{-\tau s}$ where $\underline{X}(s) = \mathcal{L}[\underline{x}(t)]$ this leads to the general structure for time-delay systems in Figure 4.8.

Example 4.1. *Let's consider the feedback control loop in Figure 4.9 where we will assume a pure unknown delay in the open loop, denoted by τ ($\tau > 0$).*

We will denote:

- *The reference input by r*
- *The control vector of the plant by u*
- *The vector of system disturbances by n*
- *The vector of measured outputs by y_p*
- *Constant $K_p > 0$ is a proportional gain*

The transfer function of the plant is denoted $F_n(s)$:

$$F_n(s) = \frac{1}{s+4} \tag{4.39}$$

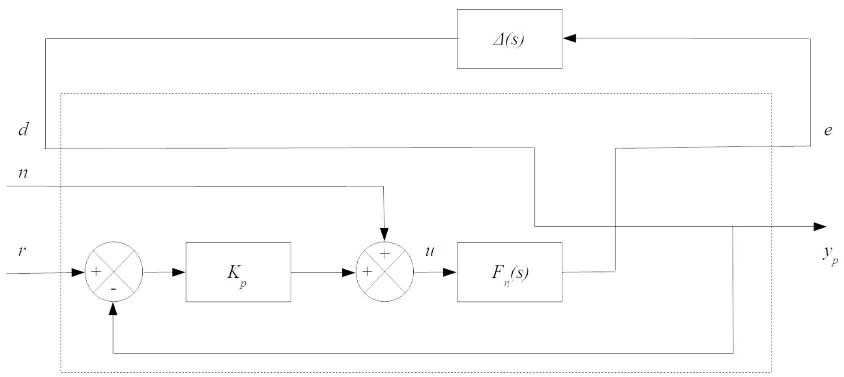

Figure 4.10: First $M\Delta$ structure for a system with delay

We are interested in finding a conservative condition on the gain K_p such that the closed-loop is stable. Stability of the feedback control loop is equivalent to negativity of the real part of all solutions $s \in \mathbb{C}$ of the transcendental *equation where the delay τ is unknown:*

$$1 + \exp(-\tau s) K_p F_n(s) = 0 \tag{4.40}$$

The Small Gain Theorem provides an easy sufficient criterion of stability analysis for this configuration.

$M\Delta$ structure

Because the uncertainty is related to the unknown delay τ we set $\mathbf{\Delta}(s)$ as follows:

$$\mathbf{\Delta}(s) := \exp(-\tau s) \tag{4.41}$$

It is clear that $\mathbf{\Delta}(s) \in RH_\infty$ (a bounded input leads to a bounded output). Furthermore we have seen that for a stable SISO linear system with transfer function $\mathbf{\Delta}(s)$ the H_∞ norm is defined as follows:

$$\|\mathbf{\Delta}(s)\|_\infty = \max_\omega |\mathbf{\Delta}(j\omega)| = \max_\omega |\exp(-j\tau\omega)| = 1 \tag{4.42}$$

Accordingly, using the relation $y_p(s) = \exp(-\tau s)e(s) = \mathbf{\Delta}(s)e(s)$, the block diagram in Figure 4.9 can be redrawn in the $M\Delta$ structure form as shown in Figure 4.10.

In order to compute $\mathbf{M}_{11}(s)$, let's compute the relation between $e(s)$ and $d(s)$ assuming that $r(s) = 0$ and $n(s) = 0$. Starting from $e(s)$ and reading Figure 4.10 back the arrows, *we get using that fact that $d(s) = y_p(s)$:*

$$\begin{cases} r(s) = 0 \\ n(s) = 0 \\ d(s) = y_p(s) \end{cases} \Rightarrow e(s) = F_n(s) K_p \left(0 - y_p(s)\right) = -K_p F_n(s) d(s) \tag{4.43}$$

Substituting $F_n(s)$ by its actual expression leads to the following expression of $\mathbf{M}_{11}(s)$:

$$\begin{cases} \mathbf{M}_{11}(s) = -K_p F_n(s) \\ F_n(s) = \frac{1}{s+4} \end{cases} \Rightarrow \mathbf{M}_{11}(s) = \frac{-K_p}{s+4} \tag{4.44}$$

Transfer function $\mathbf{M}_{11}(s) \in RH_\infty$, *which means that* $\mathbf{M}_{11}(s)$ *belongs to the set of all stable and proper rational transfer functions. Thus the* H_∞ *norm of* $\mathbf{M}_{11}(s)$ *exists and is computed as follows:*

$$\|\mathbf{M}_{11}(s)\|_\infty = \max_\omega \left| \frac{-K_p}{j\omega + 4} \right| = \max_\omega \frac{|K_p|}{\sqrt{\omega^2 + 16}} = \frac{|K_p|}{4} \tag{4.45}$$

Therefore using the assumption that $\|\mathbf{\Delta}(s)\|_\infty = 1$, *the small gain theorem indicates that a sufficient condition ensuring the stability of the feedback loop in Figure 4.9 whatever the value of delay* τ *is the following:*

$$\|\mathbf{M}_{11}(s)\|_\infty < 1 \Leftrightarrow |K_p| < 4 \tag{4.46}$$

Routh criterion

We can retrieve this result through the Routh criterion by approximating $\exp(-\tau s)$ *using the Padé approximation:*

$$\exp(-\tau s) = \frac{\exp(-\tau s/2)}{\exp(\tau s/2)} \approx \frac{1 - \tau s/2}{1 + \tau s/2} = \frac{2 - \tau s}{2 + \tau s} \tag{4.47}$$

The transcendental *equation of the loop can now be written as follows:*

$$0 = 1 + \exp(-\tau s) K_p F_n(s) \approx 1 + K_p F_n(s) \frac{2 - \tau s}{2 + \tau s} \tag{4.48}$$

Using the fact that $F_n(s) = \frac{1}{s+4}$ *we get:*

$$0 = 1 + \frac{K_p}{s+4} \times \frac{2 - \tau s}{2 + \tau s} = \frac{(s+4)(2+\tau s) + K_p(2 - \tau s)}{(s+4)(2+\tau s)} \tag{4.49}$$

Thus the roots of the preceding equation are the roots of $(s+4)(2+\tau s) + K_p(2-\tau s)$. *We get:*

$$(s+4)(2+\tau s) + K_p(2-\tau s) = \tau s^2 + (2 + 4\tau - \tau K_p)s + 8 + 2K_p \tag{4.50}$$

We can now apply the Routh criterion to check for which values of K_p *all the roots remain in the left half plane (of the complex plane):*

$$\begin{array}{|c||c|c|} \hline s^2 & \tau & 8 + 2K_p \\ \hline s^1 & 2 + 4\tau - \tau K_p & \\ \hline s^0 & 8 + 2K_p & \\ \hline \end{array} \tag{4.51}$$

Using the fact that $\tau > 0$ *and applying the Routh criterion leads to the following condition for all the poles to have negative real part:*

$$\left\{ \begin{array}{c} 2 + 4\tau - \tau K_p > 0 \\ 8 + 2K_p > 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} K_p < 4 + \frac{2}{\tau} \\ K_p > -4 \end{array} \right. \tag{4.52}$$

Assuming $K_p > 0$ *and* $\tau \to \infty$ *(which is the worst case) leads to the condition obtained with the small gain theorem:*

$$K_p < \lim_{\tau \to \infty} 4 + \frac{2}{\tau} = 4 \tag{4.53}$$

■

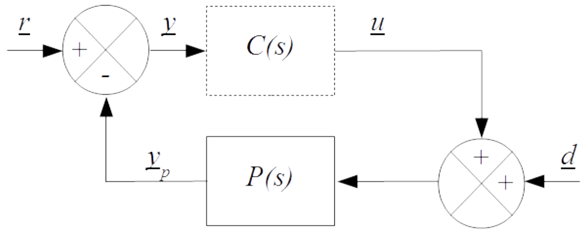

Figure 4.11: Block diagram to check internal stability of feedback loop

4.5 Well-posedness of feedback loop and internal stability

In this section we consider Figure 4.11. Well-posedness of feedback loop indicates that all closed-loop transfer matrices are well-defined and proper. The feedback system in Figure 4.11 is said to be well-posed if and only if $(\mathbb{I}+\mathbf{P}(\infty)\mathbf{C}(\infty))$ is invertible. Well-posedness condition is equivalent to the invertibility of matrix $\begin{bmatrix} \mathbb{I} & \mathbf{C}(\infty) \\ -\mathbf{P}(\infty) & \mathbb{I} \end{bmatrix}$.

Let $\mathbf{H}(s)$ be the following transfer matrix from external signal $\begin{bmatrix} \underline{r}^T & \underline{d}^T \end{bmatrix}^T$ to internal signals $\begin{bmatrix} \underline{y}_p^T & \underline{u}^T \end{bmatrix}^T$ in Figure 4.11:

$$\begin{cases} \begin{bmatrix} \underline{y}_p(s) \\ \underline{u}(s) \end{bmatrix} = \mathbf{H}(s) \begin{bmatrix} \underline{r}(s) \\ \underline{d}(s) \end{bmatrix} \\ \mathbf{H}(s) = \begin{bmatrix} \mathbf{P}(s)\mathbf{C}(s)\left(\mathbb{I}+\mathbf{P}(s)\mathbf{C}(s)\right)^{-1} & \mathbf{P}(s)\left(\mathbb{I}+\mathbf{C}(s)\mathbf{P}(s)\right)^{-1} \\ \mathbf{C}(s)\left(\mathbb{I}+\mathbf{P}(s)\mathbf{C}(s)\right)^{-1} & -\mathbf{C}(s)\mathbf{P}(s)\left(\mathbb{I}+\mathbf{C}(s)\mathbf{P}(s)\right)^{-1} \end{bmatrix} \end{cases} \tag{4.54}$$

To get $\mathbf{H}(s)$, the following push through rule has been used, which is a straightforward result when multiplying by $(\mathbb{I}+\mathbf{P}(s)\mathbf{C}(s))$ on the left side and by $(\mathbb{I}+\mathbf{C}(s)\mathbf{P}(s))$ on the right side:

$$\left(\mathbb{I}+\mathbf{P}(s)\mathbf{C}(s)\right)^{-1}\mathbf{P}(s) = \mathbf{P}(s)\left(\mathbb{I}+\mathbf{C}(s)\mathbf{P}(s)\right)^{-1} \tag{4.55}$$

When transfer matrix $\mathbf{H}(s)$ is stable, the closed loop in Figure 4.11 is said to be internally stable. Internal stability indicates that the zero-input solution converges to zero, for any initial states. This is a basic requirement for feedback system. Indeed interconnected systems may be subject to some nonzero initial conditions and some (possibly small) errors and it is not acceptable from a practical point of view that such nonzero initial conditions lead to unbounded signals in the closed-loop system. Internal stability guarantees that all signals in a system are bounded provided that the input signals (at any locations) are bounded.

Transfer matrix $\mathbf{H}(s)$ is sometimes called the *gang of four*. It can be shown that internal stability of the closed-loop is equivalent to asymptotical stability of either $(\mathbb{I} + \mathbf{P}(s)\mathbf{C}(s))^{-1}\,\mathbf{P}(s)$ or the following matrix $\mathbf{M}(s)$[6]:

$$\mathbf{M}(s) = \begin{bmatrix} \mathbb{I} & \mathbf{C}(s) \\ -\mathbf{P}(s) & \mathbb{I} \end{bmatrix}^{-1} \tag{4.56}$$

When the $M\Delta$ structure in Figure 4.4 is considered, it can be shown that the small gain condition is sufficient to guarantee internal stability[4].

[6]Multivariable Feedback Control: Analysis and Design, Sigurd Skogestad & Ian Postlethwaite, Wiley-Interscience 2005

Chapter 5

Robust control problem

5.1 Generalized plant

5.1.1 Realization

Robust control problems are solved in a dedicated framework previously presented in Figure 5.1 where:

- $\mathbf{G}(s)$ is the transfer matrix of the *generalized* Linear Time Invariant (LTI) plant. We will assume that $\mathbf{G}(s)$ is a proper and real rational transfer matrices describing some linear time invariant systems. From Figure 5.1 we have:

$$\begin{bmatrix} \underline{z}(s) \\ \underline{y}(s) \end{bmatrix} = \mathbf{G}(s) \begin{bmatrix} \underline{w}(s) \\ \underline{u}(s) \end{bmatrix} \tag{5.1}$$

  As far as robust control problems are most conveniently solved in the time domain we will consider the minimal realization of transfer matrix $\mathbf{G}(s)$ of the generalized plant shown in Figure 5.1:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{D}_{22} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{5.2}$$

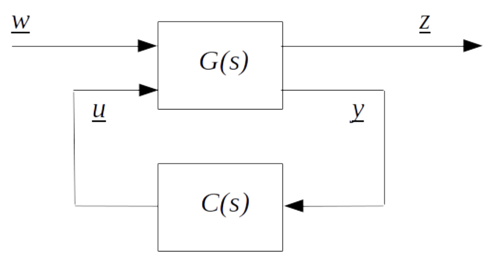

Figure 5.1: Generalized plant $\mathbf{G}(s)$ with controller $\mathbf{C}(s)$

We recall that the realization of $\mathbf{G}(s) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ is a minimal realization if and only if $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{C}, \mathbf{A})$ is observable. Then $\mathbf{A}$ has the smallest possible dimension. Here $\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}$.

- $\mathbf{C}(s)$ is the transfer matrix of the controller. We will also assume that $\mathbf{C}(s)$ is a proper and real rational transfer matrix describing the linear time invariant controller;

- $\underline{u}$ is the vector of outputs computed by the controller $\mathbf{C}(s)$. According to the feedback loop configuration, this could be the control vector of the *actual* plant, for example when there is no comparison with a reference signal;

- $\underline{y}$ is the vector of inputs available for the controller $\mathbf{C}(s)$. According to the feedback loop configuration, this could be the output vector of the *actual* plant;

- $\underline{w}$ is the input vector formed by *exogenous inputs* such as reference inputs, disturbances or noise;

- $\underline{z}$ is the performance output vector, that is the vector that allows to characterize the performance of the closed-loop system, such as errors for example. This is a *virtual* output used only for design that we wish to maintain as *small* as possible.

The objective of the robust control is to minimize the impact of exogenous inputs vector $\underline{w}$ on the performance output vector $\underline{z}$ through control vector $\underline{u}$.

5.1.2 Block diagram of the generalized plant

Assume that the linear time invariant generalized plant shown in Figure 5.1 has the following state-space representation where $\underline{x}(t)$ is the state vector, $\underline{w}(t)$ is a exogenous input vector, $\underline{u}(t)$ is the control vector, $\underline{z}(t)$ is an error vector and $\underline{y}(t)$ is the observation vector:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{5.3}$$

Usually this state-space representation includes a plant augmented with frequency-dependent weights.

Taking the Laplace transform of the first equation of (5.3) and assuming no initial conditions we get:

$$\begin{array}{c} s\underline{x}(s) = \mathbf{A}\underline{x}(s) + \mathbf{B}_1\underline{w}(s) + \mathbf{B}_2\underline{u}(s) \\ \Leftrightarrow \underline{x}(s) = (s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}_1\underline{w}(s) + (s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}_2\underline{u}(s) \end{array} \tag{5.4}$$

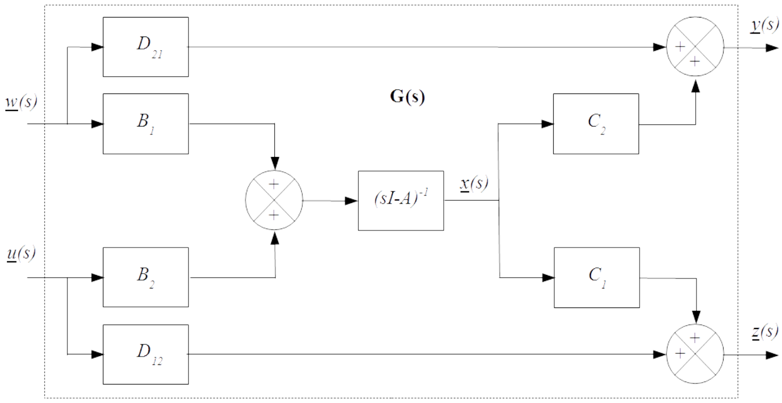

Figure 5.2: Block diagram of the generalized plant

Thus we can write the Laplace transform of the two last equations of (5.2) as follows:

$$\left[\begin{array}{c} \underline{z}(s) \\ \underline{y}(s) \end{array}\right] = \mathbf{G}(s) \left[\begin{array}{c} \underline{w}(s) \\ \underline{u}(s) \end{array}\right] \tag{5.5}$$

Where:

$$\begin{array}{rl} \mathbf{G}(s) & = \left[\begin{array}{cc} \mathbf{G}_{11}(s) & \mathbf{G}_{12}(s) \\ \mathbf{G}_{21}(s) & \mathbf{G}_{22}(s) \end{array}\right] \\ & = \left[\begin{array}{c} \mathbf{C}_1 \\ \mathbf{C}_2 \end{array}\right] (s\mathbb{I} - \mathbf{A})^{-1} \left[\begin{array}{cc} \mathbf{B}_1 & \mathbf{B}_2 \end{array}\right] + \left[\begin{array}{cc} \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \end{array} \tag{5.6}$$

The block diagram of the transfer matrix $\mathbf{G}(s)$ is shown in Figure 5.2.

5.1.3 Adding integral action

We consider the following state-space realization of the nominal plant where the state vector $\underline{x}_p$ is of dimension n (that is the size of state matrix $\mathbf{A}_p$). In addition $\underline{y}_p(t)$ denotes the output vector of the nominal plant and $\underline{u}(t)$ the input vector. We will assume that the feedforward gain matrix $\mathbf{D}$ is zero ($\mathbf{D} = \mathbf{0}$):

$$\left\{\begin{array}{l} \dot{\underline{x}}_p(t) = \mathbf{A}_p \underline{x}_p(t) + \mathbf{B}_p \underline{u}(t) \\ \underline{y}_p(t) = \mathbf{C}_p \underline{x}_p(t) \end{array}\right. \tag{5.7}$$

In some circumstances it may be helpful to use integral action in the controller design. This can be achieved by adding to the state vector of the state-space realization (5.7) the integral of the tracking error $\underline{e}_I(t)$ which is defined as follows:

$$\underline{e}_I(t) = \int_0^t \left(\underline{r}(\tau) - \underline{y}_p(\tau)\right) \, d\tau \tag{5.8}$$

where $\underline{r}(t)$ is the reference input signal.

Assuming that the feedforward gain matrix $\mathbf{D}$ is zero ($\mathbf{D} = \mathbf{0}$) we get:

$$\dot{\underline{e}}_I(t) = \underline{r}(t) - \underline{y}_p(t) = \underline{r}(t) - \mathbf{C}_p \underline{x}_p(t) \tag{5.9}$$

This leads to the following augmented state-space realization:

$$\begin{bmatrix} \underline{\dot{x}}_p(t) \\ \underline{\dot{e}}_I(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_p & \mathbf{0} \\ -\mathbf{C}_p & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x}_p(t) \\ \underline{e}_I(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_p \\ \mathbf{0} \end{bmatrix} \underline{u}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbb{I} \end{bmatrix} \underline{r}(t) \tag{5.10}$$

The output vector of the augmented system is the following:

$$\begin{bmatrix} \underline{y}_p(t) \\ \underline{e}_I(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_p & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} \begin{bmatrix} \underline{x}_p(t) \\ \underline{e}_I(t) \end{bmatrix} \tag{5.11}$$

The regulation problem deals with the case where $\underline{r}(t) = \underline{0}$. In that situation the preceding augmented state-space realization has the same structure than the state-space realization (5.7).

On the other hand the tracking problem deals with the case where $\underline{r}(t) \neq \underline{0}$. Let's denote $\underline{x}_e(t)$ the augmented state-space vector defined as follows::

$$\underline{x}_e(t) = \begin{bmatrix} \underline{x}_p(t) \\ \underline{e}_I(t) \end{bmatrix} \tag{5.12}$$

Thus the augmented state-space realization (5.10) reads as follows where the tracking error $\underline{r}(t) - \underline{y}_p(t)$ may be defined as the output vector $\underline{y}(t)$ of the generalized plant:

$$\begin{cases} \underline{\dot{x}}_e(t) = \begin{bmatrix} \mathbf{A}_p & \mathbf{0} \\ -\mathbf{C}_p & \mathbf{0} \end{bmatrix} \underline{x}_e(t) + \begin{bmatrix} \mathbf{0} \\ \mathbb{I} \end{bmatrix} \underline{r}(t) + \begin{bmatrix} \mathbf{B}_p \\ \mathbf{0} \end{bmatrix} \underline{u}(t) \\ \underline{y}(t) = \underline{r}(t) - \underline{y}_p(t) = \underline{r}(t) - \begin{bmatrix} \mathbf{C}_p & \mathbf{0} \end{bmatrix} \underline{x}_e(t) \end{cases} \tag{5.13}$$

We finally get a state space realization similar to the formalism (5.2) by defining $\underline{w}(t) := \underline{r}(t)$. Nevertheless the performance output vector $\underline{z}(t)$ has still to be defined:

$$\begin{bmatrix} \underline{\dot{x}}_e(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}_e(t) \\ \underline{r}(t) \\ \underline{u}(t) \end{bmatrix} \tag{5.14}$$

5.2 Linear Fractional Transformation (LFT)

The transfer matrix of the generalized plant $\mathbf{G}(s)$ is split as follows:

$$\begin{bmatrix} \underline{z}(s) \\ \underline{y}(s) \end{bmatrix} = \mathbf{G}(s) \begin{bmatrix} \underline{w}(s) \\ \underline{u}(s) \end{bmatrix} := \begin{bmatrix} \mathbf{G}_{11}(s) & \mathbf{G}_{12}(s) \\ \mathbf{G}_{21}(s) & \mathbf{G}_{22}(s) \end{bmatrix} \begin{bmatrix} \underline{w}(s) \\ \underline{u}(s) \end{bmatrix} \tag{5.15}$$

Setting $\underline{u}(s) = \mathbf{C}(s)\underline{y}(s)$ leads to the performance transfer matrix $\mathbf{T}_{zw}(s)$, which is actually the transfer matrix between the performance output vector $\underline{z}$ and the exogenous input vector $\underline{w}$ of the closed-loop system:

$$\underline{u}(s) = \mathbf{C}(s)\underline{y}(s) \Rightarrow \underline{z}(s) = \mathbf{T}_{zw}(s)\underline{w}(s) \tag{5.16}$$

Now let's compute the expression of the performance transfer matrix $\mathbf{T}_{zw}(s)$. First using (5.15) we get the relation between the vector of output $\underline{y}$ and the exogenous input vector $\underline{w}$:

$$\begin{array}{l} \left\{ \begin{array}{l} \underline{z}(s) = \mathbf{G}_{11}(s)\underline{w}(s) + \mathbf{G}_{12}(s)\underline{u}(s) \\ \underline{y}(s) = \mathbf{G}_{21}(s)\underline{w}(s) + \mathbf{G}_{22}(s)\underline{u}(s) \\ \underline{u}(s) = \mathbf{C}(s)\underline{y}(s) \end{array} \right. \\ \Rightarrow \left\{ \begin{array}{l} \underline{z}(s) = \mathbf{G}_{11}(s)\underline{w}(s) + \mathbf{G}_{12}(s)\mathbf{C}(s)\underline{y}(s) \\ \underline{y}(s) = \mathbf{G}_{21}(s)\underline{w}(s) + \mathbf{G}_{22}(s)\mathbf{C}(s)\underline{y}(s) \end{array} \right. \\ \Rightarrow \underline{y}(s) = \left(\mathbb{I} - \mathbf{G}_{22}(s)\mathbf{C}(s)\right)^{-1}\mathbf{G}_{21}(s)\underline{w}(s) \end{array} \tag{5.17}$$

We finally get:

$$\underline{z}(s) = \mathbf{T}_{zw}(s)\underline{w}(s) \tag{5.18}$$

Where the performance transfer matrix $\mathbf{T}_{zw}(s)$ is a Linear Fractional Transformation (LFT) with respect to $\mathbf{G}$:

$$\begin{array}{ll} \mathbf{T}_{zw}(s) & = \mathbf{G}_{11}(s) + \mathbf{G}_{12}(s)\mathbf{C}(s)\left(\mathbb{I} - \mathbf{G}_{22}(s)\mathbf{C}(s)\right)^{-1}\mathbf{G}_{21}(s) \\ & = \mathbf{G}_{11}(s) + \mathbf{G}_{12}(s)\left(\mathbb{I} - \mathbf{C}(s)\mathbf{G}_{22}(s)\right)^{-1}\mathbf{C}(s)\mathbf{G}_{21}(s) \\ & := \mathcal{F}_l\left(\mathbf{G}(s), \mathbf{C}(s)\right) \end{array} \tag{5.19}$$

The H_2 control problem consists in finding a controller $\mathbf{C}(s)$ which minimizes $\|\mathbf{T}_{zw}(s)\|_2 = \|\mathcal{F}_l\left(\mathbf{G}(s), \mathbf{C}(s)\right)\|_2$. Similarly the H_∞ control problem consists in finding a controller $\mathbf{C}(s)$ which minimizes $\|\mathbf{T}_{zw}(s)\|_\infty = \|\mathcal{F}_l\left(\mathbf{G}(s), \mathbf{C}(s)\right)\|_\infty$.

5.3 Observer-based controller

The controller $\mathbf{C}(s)$ is obtained by gathering two subsystems:

– A state estimator (observer) which provides an estimate $\hat{\underline{x}}(t)$ of the actual state vector $\underline{x}(t)$ from the input $\underline{u}(t)$ and output $\underline{y}(t)$ of the system. The so-called Luenberger observer is obtained through the following equation where $\mathbf{L}$ is the observer gain :

$$\dot{\hat{\underline{x}}}(t) = \mathbf{A}\hat{\underline{x}}(t) + \mathbf{B}_2\underline{u}(t) + \mathbf{L}\left(\underline{y}(t) - \mathbf{C}_2\hat{\underline{x}}(t)\right) \tag{5.20}$$

It is worth noticing that matrices $\mathbf{A}$, $\mathbf{B}_2$ and $\mathbf{C}_2$ are known and come from the state space realization of the generalized plant (5.2).

– A state-feedback control law based on the the estimate $\hat{\underline{x}}(t)$ and the controller gain $\mathbf{K}$:

$$\underline{u}(t) = -\mathbf{K}\hat{\underline{x}}(t) \tag{5.21}$$

Thus the dynamics of the controller $\mathbf{C}(s)$ is driven by the following equations:

$$\left\{ \begin{array}{l} \dot{\hat{\underline{x}}}(t) = \left(\mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{L}\mathbf{C}_2\right)\hat{\underline{x}}(t) + \mathbf{L}\underline{y}(t) \\ \underline{u}(t) = -\mathbf{K}\hat{\underline{x}}(t) \end{array} \right. \tag{5.22}$$

From the preceding relation it is clear that the input of the controller $\mathbf{C}(s)$ is $\underline{y}(t)$ whereas its output is $\underline{u}(t)$. By taking the Laplace transform of the preceding relation we get the transfer function $\mathbf{C}(s)$ of the controller:

$$\begin{array}{l} s\,\hat{\underline{x}}(s) = (\mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{L}\mathbf{C}_2)\,\hat{\underline{x}}(s) + \mathbf{L}\underline{y}(s) \\ \Leftrightarrow (s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} + \mathbf{L}\mathbf{C}_2)\,\hat{\underline{x}}(s) = \mathbf{L}\underline{y}(s) \\ \Leftrightarrow \hat{\underline{x}}(s) = (s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} + \mathbf{L}\mathbf{C}_2)^{-1}\mathbf{L}\underline{y}(s) \end{array} \tag{5.23}$$

We finally get:

$$\underline{u}(s) = -\mathbf{K}\hat{\underline{x}}(s) := \mathbf{C}(s)\underline{y}(s) \tag{5.24}$$

where:

$$\begin{array}{rl} \mathbf{C}(s) & = -\mathbf{K}\,(s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} + \mathbf{L}\mathbf{C}_2)^{-1}\,\mathbf{L} \\ & := \left[\begin{array}{c|c} \mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{L}\mathbf{C}_2 & \mathbf{L} \\ \hline -\mathbf{K} & \mathbf{0} \end{array}\right] \end{array} \tag{5.25}$$

In other words the realization of the controller $\mathbf{C}(s)$ which solves the robust control problem is the following, where matrices $\mathbf{K}$ and $\mathbf{L}$ are the parameters to be found:

$$\begin{array}{l} \left\{ \begin{array}{c} \underline{u}(s) = \mathbf{C}(s)\underline{y}(s) \\ \mathbf{C}(s) = -\mathbf{K}\,(s\mathbb{I} - (\mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{L}\mathbf{C}_2))^{-1}\,\mathbf{L} \end{array} \right. \\ \Leftrightarrow \left[\begin{array}{c} \dot{\hat{\underline{x}}}(t) \\ \underline{u}(t) \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{L}\mathbf{C}_2 & \mathbf{L} \\ \hline -\mathbf{K} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \hat{\underline{x}}(t) \\ \underline{y}(t) \end{array}\right] \end{array} \tag{5.26}$$

It is worth noticing that it is the state space realization of the *generalized* plant which is used here, through matrices $\mathbf{A}$, $\mathbf{B}_2$ and $\mathbf{C}_2$, not state space realization of the *actual* plant.

5.4 H_2 and H_∞ robust control problems

5.4.1 H_2 control problem

The H_2 control problem consists in finding a proper real rational controller $\mathbf{C}(s)$ which minimizes the effect of the exogenous inputs $\underline{w}(s)$ on the performance output vector $\underline{z}(s)$ with respect to the H_2-norm. In other words it consists in finding an admissible controller $\mathbf{C}(s)$ such that the transfer $\|\mathbf{T}_{zw}(s)\|_2 = \|\mathcal{F}_l\,(\mathbf{G}(s), \mathbf{C}(s))\,\|_2$ is minimized.

5.4.2 H_∞ control problem

The worst case of the ratio $\frac{\|\underline{z}(t)\|_2}{\|\underline{w}(t)\|_2}$ is given by the H_∞-norm of the Linear Fractional Transformation (LFT) $\mathcal{F}_l\,(\mathbf{G}(s), \mathbf{C}(s))$:

$$\max_{\underline{w}(t) \neq \underline{0}} \frac{\|\underline{z}(t)\|_2}{\|\underline{w}(t)\|_2} = \|\mathbf{T}_{zw}(s)\|_\infty := \|\mathcal{F}_l\,(\mathbf{G}(s), \mathbf{C}(s))\,\|_\infty \tag{5.27}$$

The H_∞ control problem consists in finding a proper real rational controller $\mathbf{C}(s)$ which minimizes the effect of the exogenous inputs $\underline{w}(s)$ on the performance output vector $\underline{z}(s)$ with respect to the H_∞-norm. In other words it consists in finding an admissible controller $\mathbf{C}(s)$ such that the transfer $\|\mathbf{T}_{zw}(s)\|_\infty :=$ $\|\mathcal{F}_l\left(\mathbf{G}(s), \mathbf{C}(s)\right)\|_\infty$ is minimized. The minimum value of $\|\mathcal{F}_l\left(\mathbf{G}(s), \mathbf{C}(s)\right)\|_\infty$ is denoted γ_{opt}.

In practice the minimum bound γ_{opt} is often difficult to find. Consequently a simpler problem, also called suboptimal H_∞ control problem, is defined as follows: given $\gamma \geq \gamma_{opt}$ find an admissible controller $\mathbf{C}(s)$ such that:

$$\|\mathbf{T}_{zw}(s)\|_\infty := \|\mathcal{F}_l\left(\mathbf{G}(s), \mathbf{C}(s)\right)\|_\infty \leq \gamma \tag{5.28}$$

5.4.3 Algebraic Riccati Equation

In order to solve the H_2 and H_∞ robust control problems we recall some facts about Algebraic Riccati Equation.

First we recall that the Algebraic Riccati Equation is the following matrix equation:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{R}\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{5.29}$$

Matrices $\mathbf{A}$, $\mathbf{P}$, $\mathbf{R}$ and $\mathbf{Q}$ are real matrices. Furthermore we will assume that $\mathbf{R}$ and $\mathbf{Q}$ are symmetric but not necessarily positive definite matrices :

$$\left\{\begin{array}{l} \mathbf{R} = \mathbf{R}^T \\ \mathbf{Q} = \mathbf{Q}^T \end{array}\right. \tag{5.30}$$

The solution $\mathbf{P} = \mathbf{P}^T$ of the Algebraic Riccati Equation is said to be a stabilizing solution if $\mathbf{A}-\mathbf{R}\mathbf{P}$ is stable (that is all the eigenvalues of $\mathbf{A}-\mathbf{R}\mathbf{P}$ have negative real part). The stabilizing solution of an Algebraic Riccati Equation is unique.

In order to solve the Algebraic Riccati Equation the following Hamiltonian matrix $\mathbf{H}$ is associated to equation (5.29) :

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & -\mathbf{R} \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix} \tag{5.31}$$

Thus if $\mathbf{A}$ is a $n \times n$ matrix then $\mathbf{H}$ is a $2n \times 2n$ matrix.

Matrix $\mathbf{H}$ is called s called Hamiltonian because it has the following property:

$$\mathbf{H}\mathbf{J} = (\mathbf{H}\mathbf{J})^T \text{ where } \mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbb{I} \\ -\mathbb{I} & \mathbf{0} \end{bmatrix} \tag{5.32}$$

It can be shown that the eigenvalues of $\mathbf{H}$ are symmetric with respect to the imaginary axis. Assuming that $\mathbf{H}$ has no pure imaginary eigenvalues (this is the *stability* condition) the computation of the solution $\mathbf{P} = \mathbf{P}^T$ of the Algebraic Riccati Equation (5.29) which renders $\mathbf{A} - \mathbf{R}\mathbf{P}$ stable involves the eigenvectors corresponding to the eigenvalues with negative real part of the Hamiltonian matrix $\mathbf{H}$. More precisely form the $2n \times n$ matrix $\begin{bmatrix} \underline{v}_1 & \cdots & \underline{v}_n \end{bmatrix} := \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$ by putting into column the n eigenvectors $\underline{v}_i$ corresponding to the n eigenvalues of the $2n \times 2n$ matrix $\mathbf{H}$ with negative real part. Assuming that $\mathbf{X}_1$ is nonsingular (this is the *complementary* condition) the positive semi definite solution $\mathbf{P}$ of the Algebraic Riccati Equation (5.29) can be computed as follows:

$$\begin{bmatrix} \underline{v}_1 & \cdots & \underline{v}_n \end{bmatrix} := \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \Rightarrow \mathbf{P} = \mathbf{X}_2 \mathbf{X}_1^{-1} \tag{5.33}$$

Furthermore all the eigenvalues of the stable matrix $\mathbf{A} - \mathbf{RP}$ are equal to the eigenvalues with negative real part of the Hamiltonian matrix $\mathbf{H}$.

Example 5.1. *Let's solve the following Algebraic Riccati Equation where α is assumed to be a positive constant ($\alpha > 0$):*

$$1 - \alpha P^2 = 0 \tag{5.34}$$

Identifying the Algebraic Riccati Equation to be solved with Equation (5.29) *leads to the following expressions of $\mathbf{A}$, $\mathbf{R}$ and $\mathbf{Q}$:*

$$\alpha P^2 - 1 := \mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PRP} + \mathbf{Q} \Rightarrow \left\{ \begin{array}{l} \mathbf{A} = 0 \\ \mathbf{R} = \alpha \\ \mathbf{Q} = 1 \end{array} \right. \tag{5.35}$$

It is clear that the stabilizing solution is the positive solution of $\alpha P^2 - 1 = 0$:

$$P = \frac{1}{\sqrt{\alpha}} \tag{5.36}$$

Indeed in that case the eigenvalue of $\mathbf{A} - \mathbf{RP} = -\sqrt{\alpha}$ will be with negative real part.

Let's us retrieve the stabilizing solution through the Hamiltonian matrix $\mathbf{H}$. The following Hamiltonian matrix $\mathbf{H}$ is associated to equation (5.29) *:*

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & -\mathbf{R} \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ -1 & 0 \end{bmatrix} \tag{5.37}$$

Let's compute the eigenvalues of $\mathbf{H}$:

$$s\mathbb{I} - \mathbf{H} = \begin{bmatrix} s & \alpha \\ 1 & s \end{bmatrix} \Rightarrow \det(s\mathbb{I} - \mathbf{H}) = s^2 - \alpha \tag{5.38}$$

There are two eigenvalues:

$$\left\{ \begin{array}{l} \lambda_1 = -\sqrt{\alpha} \\ \lambda_2 = +\sqrt{\alpha} \end{array} \right. \tag{5.39}$$

The eigenvalue with negative real part is $\lambda_1 = -\sqrt{\alpha}$. The associated eigenvector is computed as follows:

$$\begin{array}{c} \mathbf{H}\underline{v}_1 = \lambda_1 \underline{v}_1 \\ \Leftrightarrow \begin{bmatrix} 0 & -\alpha \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = -\sqrt{\alpha} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \\ \Leftrightarrow \left\{ \begin{array}{l} \alpha v_{12} = \sqrt{\alpha} v_{11} \\ v_{11} = \sqrt{\alpha} v_{12} \end{array} \right. \Rightarrow v_{11} = \sqrt{\alpha} v_{12} \end{array} \tag{5.40}$$

Thus the eigenvector associated with the eigenvalue with negative real part is $\lambda_1 = -\sqrt{\alpha}$ *can be expressed as:*

$$\underline{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha} v_{12} \\ v_{12} \end{bmatrix} \tag{5.41}$$

Identifying this expression with $\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$ *we finally get:*

$$\underline{v}_1 = \begin{bmatrix} \sqrt{\alpha} v_{12} \\ v_{12} \end{bmatrix} := \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \Rightarrow P = \mathbf{X}_2 \mathbf{X}_1^{-1} = \frac{1}{\sqrt{\alpha}} \tag{5.42}$$

We can check that the eigenvalue of the stable matrix $\mathbf{A} - \mathbf{RP} = -\sqrt{\alpha}$ *is equal to the eigenvalue with negative real part of the Hamiltonian matrix* $\mathbf{H}$.

■

5.5 Closed-loop transfer function and separation principle

5.5.1 Closed-loop transfer

We will assume the following form of the realization of the generalized plant $\mathbf{G}(s)$:

$$\mathbf{G}(s) : \begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{5.43}$$

We are looking for the following dynamical output feedback controller $\mathbf{C}(s)$ that meets some specifications for the closed-loop system:

$$\mathbf{C}(s) : \begin{bmatrix} \dot{\underline{x}}_K(t) \\ \underline{u}(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A}_K & \mathbf{B}_K \\ \hline \mathbf{C}_K & \mathbf{D}_K \end{array}\right] \begin{bmatrix} \underline{x}_K(t) \\ \underline{y}(t) \end{bmatrix} \tag{5.44}$$

For an observer-based controller, expression of matrices $\mathbf{A}_K$, $\mathbf{B}_K$, $\mathbf{C}_K$ and $\mathbf{D}_K$ are provided in (5.25).

With the plant $\mathbf{G}(s)$ and the controller $\mathbf{C}(s)$ defined as above a realization of the closed-loop system $\mathcal{P}_{cl}$ is the following:

$$\mathcal{P}_{cl} : \begin{bmatrix} \dot{\underline{x}}_{cl}(t) \\ \underline{z}(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A}_{cl} & \mathbf{B}_{cl} \\ \hline \mathbf{C}_{cl} & \mathbf{D}_{cl} \end{array}\right] \begin{bmatrix} \underline{x}_{cl}(t) \\ \underline{w}(t) \end{bmatrix} \tag{5.45}$$

where:

$$\begin{cases} \underline{x}_{cl}(t) = \begin{bmatrix} \underline{x}(t) \\ \underline{x}_K(t) \end{bmatrix} \\ \mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A} + \mathbf{B}_2 \mathbf{D}_K \mathbf{C}_2 & \mathbf{B}_2 \mathbf{C}_K \\ \mathbf{B}_K \mathbf{C}_2 & \mathbf{A}_K \end{bmatrix} \\ \mathbf{B}_{cl} = \begin{bmatrix} \mathbf{B}_1 + \mathbf{B}_2 \mathbf{D}_K \mathbf{D}_{21} \\ \mathbf{B}_K \mathbf{D}_{21} \end{bmatrix} \\ \mathbf{C}_{cl} = \begin{bmatrix} \mathbf{C}_1 + \mathbf{D}_{12} \mathbf{D}_K \mathbf{C}_2 & \mathbf{D}_{12} \mathbf{C}_K \end{bmatrix} \\ \mathbf{D}_{cl} = \mathbf{D}_{12} \mathbf{D}_K \mathbf{D}_{21} \end{cases} \tag{5.46}$$

Once the loop is closed, transfer function $\mathbf{T}_{zw}(s)$ from $\underline{w}(s)$ to $\underline{z}(s)$ reads:

$$\begin{cases} \underline{z}(s) = \mathbf{T}_{zw}(s)\underline{w}(s) \\ \mathbf{T}_{zw}(s) = \mathbf{C}_{cl}\left(s\mathbb{I} - \mathbf{A}_{cl}\right)^{-1}\mathbf{B}_{cl} + \mathbf{D}_{cl} \end{cases} \tag{5.47}$$

In the particular case of state feedback control where $\underline{u}(t) = -\mathbf{K}\underline{x}(t)$ the realization the dynamical output feedback controller $\mathbf{C}(s)$ in (5.44) reduce as follows:

$$\mathbf{C}(s) : \left[\begin{array}{c|c} \mathbf{A}_K & \mathbf{B}_K \\ \hline \mathbf{C}_K & \mathbf{D}_K \end{array}\right] = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{K} \end{array}\right] \tag{5.48}$$

whereas the realization of the generalized plant $\mathbf{G}(s)$ in (5.43) reads:

$$\begin{cases} \mathbf{C}_2 = \mathbb{I} \\ \mathbf{D}_{21} = \mathbf{0} \end{cases} \Rightarrow \mathbf{G}(s) : \begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbb{I} & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{5.49}$$

5.5.2 Separation principle

For an observed-based controller, we have seen in (5.25) that $\mathbf{D}_K = \mathbf{0}$. Consequently the state matrix $\mathbf{A}_{cl}$ of the closed-loop transfer function in (5.46) reduces as follows:

$$\mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_2\mathbf{C}_K \\ \mathbf{B}_K\mathbf{C}_2 & \mathbf{A}_K \end{bmatrix} \tag{5.50}$$

where according to (5.25):

$$\begin{cases} \mathbf{A}_K = \mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{L}\mathbf{C}_2 \\ \mathbf{B}_K = \mathbf{L} \\ \mathbf{C}_K = -\mathbf{K} \end{cases} \tag{5.51}$$

Thus the closed-loop state matrix $\mathbf{A}_{cl}$ reads:

$$\mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}_2\mathbf{K} \\ \mathbf{L}\mathbf{C}_2 & \mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{L}\mathbf{C}_2 \end{bmatrix} \tag{5.52}$$

The closed-loop eigenvalues are the roots of the characteristic polynomial $\chi_{A_{cl}}(s)$ defined as follows:

$$\chi_{A_{cl}}(s) = \det\left(s\mathbb{I} - \mathbf{A}_{cl}\right) = \det\left(\begin{bmatrix} s\mathbb{I} - \mathbf{A} & \mathbf{B}_2\mathbf{K} \\ -\mathbf{L}\mathbf{C}_2 & s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} + \mathbf{L}\mathbf{C}_2 \end{bmatrix}\right) \tag{5.53}$$

Now we will use the fact that adding one column / row to another column / row does not change the value of the determinant. Thus adding the first column to the second column of $\begin{bmatrix} s\mathbb{I} - \mathbf{A} & \mathbf{B}_2\mathbf{K} \\ -\mathbf{L}\mathbf{C}_2 & s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} + \mathbf{L}\mathbf{C}_2 \end{bmatrix}$ leads to the following expression of $\chi_{A_{cl}}(s)$:

$$\chi_{A_{cl}}(s) = \det\left(\begin{bmatrix} s\mathbb{I} - \mathbf{A} & s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} \\ -\mathbf{L}\mathbf{C}_2 & s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} \end{bmatrix}\right) \tag{5.54}$$

Now subtracting the second row to the first row of $\begin{bmatrix} s\mathbb{I} - \mathbf{A} & s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} \\ -\mathbf{L}\mathbf{C}_2 & s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} \end{bmatrix}$ leads to the following expression of $\chi_{A_{cl}}(s)$:

$$\chi_{A_{cl}}(s) = \det\left(\begin{bmatrix} s\mathbb{I} - \mathbf{A} + \mathbf{L}\mathbf{C}_2 & \mathbf{0} \\ -\mathbf{L}\mathbf{C}_2 & s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K} \end{bmatrix}\right) \tag{5.55}$$

It is worth noticing that matrix is block triangular. Consequently we can write:

$$\chi_{A_{cl}}(s) = \det\left(s\mathbb{I} - \mathbf{A}_{cl}\right) = \det\left(s\mathbb{I} - \mathbf{A} + \mathbf{L}\mathbf{C}_2\right)\det\left(s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K}\right) \tag{5.56}$$

In other words the eigenvalues of the closed-loop system are obtained by the union of the eigenvalues of matrix $\mathbf{A} - \mathbf{B}_2\mathbf{K}$, that is the state matrix of the closed-loop system without the observer, and the eigenvalues of matrix $\mathbf{A} - \mathbf{L}\mathbf{C}_2$, that is the state matrix of the closed-loop system without the controller. This result is known as the separation principle. As a consequence the observer and the controller can be designed separately: the eigenvalues obtained thanks to the controller gain $\mathbf{K}$ assuming full state feedback are independent of the eigenvalues obtained thanks to the observer gain $\mathbf{L}$ assuming no controller.

Usually observer gain $\mathbf{L}$ is chosen such that the eigenvalues of matrix $\mathbf{A} - \mathbf{L}\mathbf{C}_2$ are around 5 to 10 times faster than the eigenvalues of matrix $\mathbf{A} - \mathbf{B}_2\mathbf{K}$, so that the state estimation moves towards the actual state as early as possible.

5.6 One degree of freedom control loop

5.6.1 Feedback loop

In the following, we will consider the control of an uncertain system as represented in Figure 5.3 where:

- $\mathbf{F}_n(s)$ is the transfer matrix of the *nominal* plant (that is the plant without uncertainty) to be controlled;
- $\mathbf{F}(s)$ is the transfer matrix of the *actual* plant;
- $\mathbf{W}_u(s)$ is a given stable uncertainty weighting function;
- $\mathbf{\Delta}(s)$ is an additive uncertainty which is assumed to be bounded;

$$\begin{cases} \mathbf{F}(s) = \mathbf{F}_n(s) + \mathbf{W}_u(s)\mathbf{\Delta}(s) \\ \|\mathbf{\Delta}(s)\|_\infty = 1 \end{cases} \tag{5.57}$$

- $\mathbf{C}(s)$ is the transfer matrix of the controller;
- $\underline{u}$ is the output vector of the controller $\mathbf{C}(s)$. In the feedback loop under study, this is also the control vector of the *actual* plant;
- $\underline{y}_p$ is the output vector of the *actual* plant;
- $\underline{r}$ is the reference input;

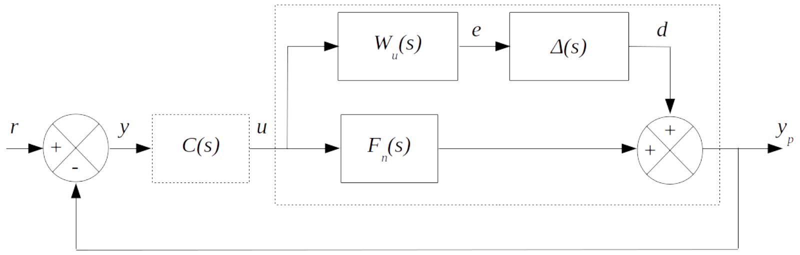

Figure 5.3: One degree of freedom feedback control of uncertain system

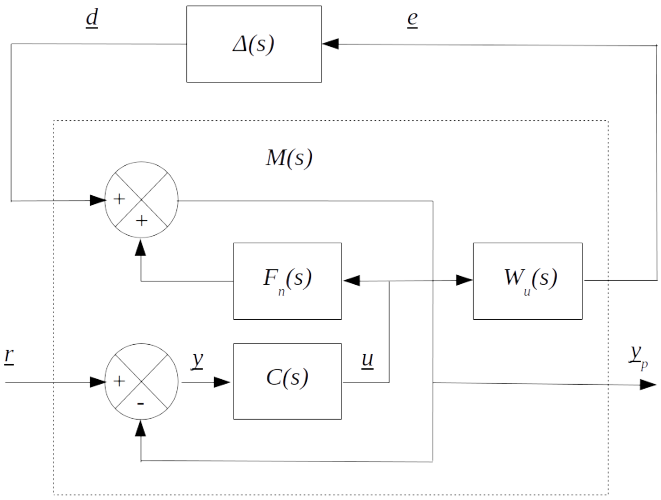

Figure 5.4: $M\Delta$ structure corresponding to additive uncertainty

– $\underline{y}$ is the input vector of the controller $\mathbf{C}(s)$: $\underline{y} = \underline{r} - \underline{y}_p$.

The robust control problem consists in finding the controller $\mathbf{C}(s)$ such that the feedback loop in Figure 5.3 is stable for all uncertainties $\mathbf{\Delta}(s)$ while minimizing the norm (either the H_2 or the H_∞ norm) of a performance transfer function denoted $\mathbf{T}_{zw}(s)$ which will be defined later.

5.6.2 Robust stability

In order to assess the stability of the closed-loop uncertain plant represented in Figure 5.3 we have to build the corresponding $M\Delta$ structure as shown in Figure 5.4.

The transfer matrix of the interconnected system depicted in Figure 5.4 is the following:

$$\begin{bmatrix} \underline{e}(s) \\ \underline{y}_p(s) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{11}(s) & \mathbf{M}_{12}(s) \\ \mathbf{M}_{21}(s) & \mathbf{M}_{22}(s) \end{bmatrix} \begin{bmatrix} \underline{d}(s) \\ \underline{r}(s) \end{bmatrix} \tag{5.58}$$

To apply the small gain theorem we have to compute matrix $\mathbf{M}_{11}(s)$. Matrix $\mathbf{M}_{11}(s)$ corresponds to the transfer matrix $\mathbf{T}_{ed}(s)$ between $\underline{e}(s)$ which *feeds* the uncertainty block $\mathbf{\Delta}(s)$ its output $\underline{d}(s)$ assuming that reference input $r(s)$ is set to zero:

$$\mathbf{M}_{11}(s) := \mathbf{T}_{ed}(s) \tag{5.59}$$

We get from Figure 5.4:

$$\begin{cases} \underline{e}(s) = \mathbf{W}_u(s)\underline{u}(s) \\ \underline{u}(s) = \mathbf{C}(s)\left(\underline{r}(s) - \underline{y}_p(s)\right) \\ \underline{y}_p(s) = \underline{d}(s) + \mathbf{F}_n(s)\underline{u}(s) \end{cases} \tag{5.60}$$

Thus:

$$\begin{array}{c} \underline{y}_p(s) = \underline{d}(s) + \mathbf{F}_n(s)\mathbf{C}(s)\left(\underline{r}(s) - \underline{y}_p(s)\right) \\ \Leftrightarrow \underline{y}_p(s) = \left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1}\left(\underline{d}(s) + \mathbf{F}_n(s)\mathbf{C}(s)\underline{r}(s)\right) \end{array} \tag{5.61}$$

To get $\mathbf{T}_{ed}(s)$ the reference input $\underline{r}(s)$ is set to zero. We get:

$$\begin{array}{rl} \underline{r}(s) = \underline{0} & \Rightarrow \begin{cases} \underline{y}_p(s) = \left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1}\underline{d}(s) \\ \underline{u}(s) = -\mathbf{C}(s)\underline{y}_p(s) \end{cases} \\ \Rightarrow \underline{e}(s) & = \mathbf{W}_u(s)\underline{u}(s) \\ & = -\mathbf{W}_u(s)\mathbf{C}(s)\underline{y}_p(s) \\ & = -\mathbf{W}_u(s)\mathbf{C}(s)\left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1}\underline{d}(s) \\ & = \mathbf{T}_{ed}(s)\underline{d}(s) \end{array} \tag{5.62}$$

Thus transfer matrix $\mathbf{T}_{ed}(s)$ is defined by:

$$\mathbf{T}_{ed}(s) = -\mathbf{W}_u(s)\mathbf{C}(s)\left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1} \tag{5.63}$$

We recognize in $\left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1}$ the (output) sensitivity matrix $\mathbf{S}(s)$:

$$\mathbf{S}(s) = \left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1} \tag{5.64}$$

Consequently:

$$\mathbf{T}_{ed}(s) = -\mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s) \tag{5.65}$$

When the plant is subject to additive uncertainty then the small gain theorem may be applied on relation (5.65) to get a conservative stability condition for the closed-loop system to remain stable despite the uncertainty:

$$\begin{array}{c} \|\mathbf{\Delta}(s)\|_\infty \|\mathbf{T}_{ed}(s)\|_\infty < 1 \\ \Leftrightarrow \|\mathbf{\Delta}(s)\|_\infty \| - \mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s)\|_\infty < 1 \\ \Leftrightarrow \|\mathbf{\Delta}(s)\|_\infty \|\mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s)\|_\infty < 1 \end{array} \tag{5.66}$$

As far as we assume that $\|\mathbf{\Delta}(s)\|_\infty = 1$ the preceding relation reads:

$$\|\mathbf{\Delta}(s)\|_\infty = 1 \Rightarrow \|\mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s)\|_\infty < 1 \tag{5.67}$$

This relation indicates that for good robustness, the norm of the product $\mathbf{C}(s)\mathbf{S}(s)$ is required to be *small*.

The same kind of result exists for a plant with output multiplicative uncertainty. As a summary we have:

– For additive uncertainty the robust stability condition reads:

$$\mathbf{F}(s) = \mathbf{F}_n(s) + \mathbf{W}_u(s)\boldsymbol{\Delta}(s) \Rightarrow \|\mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s)\|_\infty < 1 \tag{5.68}$$

– For output multiplicative uncertainty the robust stability condition reads:

$$\mathbf{F}(s) = \left(\mathbb{I} + \mathbf{W}_u(s)\boldsymbol{\Delta}(s)\right)\mathbf{F}_n(s) \Rightarrow \|\mathbf{W}_u(s)\mathbf{T}(s)\|_\infty < 1 \tag{5.69}$$

Where $\mathbf{T}(s)$ is the complementary sensitivity function:

$$\mathbf{T}(s) = \mathbf{S}(s)\mathbf{F}_n(s)\mathbf{C}(s) \tag{5.70}$$

Chapter 6

H_2 robust control design

6.1 Introduction

In this chapter the H_2 observer-based controller design is considered. We recall in Figure 6.1 the general framewok for such design.

We have seen that the general form of the realization of the generalized plant $\mathbf{G}(s)$ is the following:

$$\begin{bmatrix} \underline{\dot{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{6.1}$$

The H_2 control problem consists in finding the controller $\mathbf{C}(s)$ which minimizes $\|\mathbf{T}_{zw}(s)\|_2 = \|\mathcal{F}_l(\mathbf{G}, \mathbf{K})\|_2$. The usual technique to compute $\|\mathbf{T}_{zw}(s)\|_2$ is to compute the H_2-norm of $\underline{z}(t)$ assuming that the exogenous disturbance input vector $\underline{w}(t)$ is the Dirac delta function $\delta(t)$:

$$\|\mathbf{T}_{zw}(s)\|_2^2 = \int_0^\infty \underline{z}^T(t)\underline{z}(t)dt \text{ assuming that } \underline{w}(t) = \mathbb{I}\delta(t) \tag{6.2}$$

We will first tackle the control problem where the full state of the system is assumed to be available for control. Then the optimal state estimator problem is tackled: it consists in estimating the full state of the system thanks to the

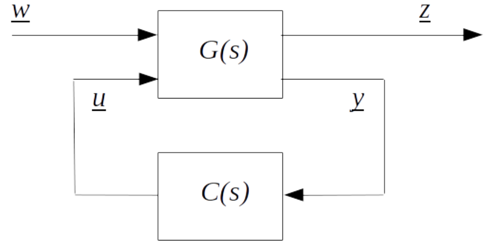

Figure 6.1: Generalized plant $\mathbf{G}(s)$ with controller $\mathbf{C}(s)$

measurement vector $\underline{y}(t)$. Finally both results are gathered to get the H_2 observer-based controller law.

The relation with LQG control is also stressed. Indeed the H_2 control problem can easily be extended to the stochastic case where the exogenous disturbance input vector $\underline{w}(t)$ is a white noise:

$$\|\mathbf{T}_{zw}(s)\|_2^2 = E\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \underline{z}^T(t)\underline{z}(t)dt\right)$$
$$\text{assuming that } E\left(\underline{w}(t)\underline{w}^T(\tau)\right) = \mathbb{I}\delta(t-\tau) \quad (6.3)$$

where $E()$ denotes the mathematical expectation.

6.2 State feedback problem

6.2.1 Problem to be solved

In that section we present the H_2 state feedback problem. We will assume that the state vector $\underline{x}(t)$ is fully available for control and we are looking for a state feedback control $\mathbf{K}$ such that:

$$\underline{u}(t) = -\mathbf{K}\underline{x}(t) \tag{6.4}$$

As far as the state vector $\underline{x}(t)$ is fully available for control the realization of the generalized plant is the following:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbb{I} & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{6.5}$$

In addition the following assumptions are made:

- $(\mathbf{A}, \mathbf{C}_1)$ has no unobservable modes on the imaginary axis
- $(\mathbf{A}, \mathbf{B}_2)$ is stabilizable
- $\mathbf{D}_{12}^T\mathbf{D}_{12} > 0$ (thus the product is invertible)
- $\mathbf{D}_{12}^T\mathbf{C}_1 = \mathbf{0}$

The two last assumptions are not very restrictive and can be relaxed but they are convenient because they lead to simplifications in the solution of the problem.

By taking the Laplace transform (without initial condition) of (6.5) and using the fact that $\underline{u} = -\mathbf{K}\underline{x}$ we get:

$$\begin{array}{l} \left\{\begin{array}{l} s\underline{x}(s) = \mathbf{A}\underline{x}(s) + \mathbf{B}_1\underline{w}(s) - \mathbf{B}_2\mathbf{K}\underline{x}(s) \\ \underline{z}(s) = \mathbf{C}_1\underline{x}(s) - \mathbf{D}_{12}\mathbf{K}\underline{x}(s) \end{array}\right. \\ \Rightarrow \left\{\begin{array}{l} \underline{x}(s) = \left(s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K}\right)^{-1}\mathbf{B}_1\underline{w}(s) \\ \underline{z}(s) = \left(\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}\right)\underline{x}(s) \end{array}\right. \\ \Rightarrow \underline{z}(s) = \left(\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}\right)\left(s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K}\right)^{-1}\mathbf{B}_1\underline{w}(s) \end{array} \tag{6.6}$$

Thus:

$$\mathbf{T}_{zw}(s) = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})\,(s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K})^{-1}\,\mathbf{B}_1 \tag{6.7}$$

From the results of section 1.7.2, minimizing the H_2-norm of $\mathbf{T}_{zw}(s)$ consists in solving the following semi-definite program where $\mathbf{Y}$ stands for the controllability grammian of $\mathbf{T}_{zw}(s)$:

$$\begin{array}{ll} min & \operatorname{tr}\left((\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})\,\mathbf{Y}\,(\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})^T\right) \\ s.t. & \left\{ \begin{array}{l} (\mathbf{A} - \mathbf{B}_2\mathbf{K})\,\mathbf{Y} + \mathbf{Y}\,(\mathbf{A} - \mathbf{B}_2\mathbf{K})^T + \mathbf{B}_1\mathbf{B}_1^T = \mathbf{0} \\ \mathbf{Y} = \mathbf{Y}^T > \mathbf{0} \end{array} \right. \end{array} \tag{6.8}$$

A similar semi-definite program to be solved can also be formulated with the observability grammian of $\mathbf{T}_{zw}(s)$. Denoting $\mathbf{X}$ the observability grammian of $\mathbf{T}_{zw}(s)$ we get from the results of section 1.7.2:

$$\begin{array}{ll} min & \operatorname{tr}\left(\mathbf{B}_1^T\mathbf{X}\mathbf{B}_1\right) \\ s.t. & \left\{ \begin{array}{l} (\mathbf{A} - \mathbf{B}_2\mathbf{K})^T\,\mathbf{X} + \mathbf{X}\,(\mathbf{A} - \mathbf{B}_2\mathbf{K}) + (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})^T\,(\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}) = \mathbf{0} \\ \mathbf{X} = \mathbf{X}^T > \mathbf{0} \end{array} \right. \end{array} \tag{6.9}$$

6.2.2 Corresponding LQR problem to be solved

As in LQR design approach, we may use the Lagrange multiplier approach to solve the H_2 state feedback problem. Let define the (scalar) Hamiltonian H as follows where $\mathbf{\Lambda} = \mathbf{\Lambda}^T$ is a $n \times n$ diagonal matrix of Lagrange multipliers to be determined. On the basis on (6.9) involving the observability grammian we get:

$$\begin{array}{l} H = \operatorname{tr}\left(\mathbf{B}_1^T\mathbf{X}\mathbf{B}_1\right) \\ + \operatorname{tr}\left(\mathbf{\Lambda}\left((\mathbf{A} - \mathbf{B}_2\mathbf{K})^T\,\mathbf{X} + \mathbf{X}\,(\mathbf{A} - \mathbf{B}_2\mathbf{K}) + (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})^T\,(\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})\right)\right) \end{array} \tag{6.10}$$

Let $\mathbf{YXZ}$ be a square matrix where matrices $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are of appropriate dimension. Then the following properties hold[1]:

$$\left\{ \begin{array}{l} \frac{\partial \operatorname{tr}(\mathbf{XYZ})}{\partial \mathbf{Y}} = \mathbf{X}^T\mathbf{Z}^T \\ \frac{\partial \operatorname{tr}\left(\mathbf{X}\mathbf{Y}^T\mathbf{Z}\right)}{\partial \mathbf{Y}} = \frac{\partial \operatorname{tr}\left(\mathbf{Z}^T\mathbf{Y}\mathbf{X}^T\right)}{\partial \mathbf{Y}} = \mathbf{ZX} \end{array} \right. \tag{6.11}$$

Thus the necessary conditions to solve this optimization problem with respect to matrices $\mathbf{K}$, $\mathbf{X}$ and $\mathbf{\Lambda}$ read as follows:

$$\left\{ \begin{array}{ll} \frac{\partial H}{\partial \mathbf{K}} = \mathbf{0} & \Rightarrow 2\left(-\mathbf{B}_2^T\mathbf{X} - \mathbf{D}_{12}^T\mathbf{C}_1 + \mathbf{D}_{12}^T\mathbf{D}_{12}\mathbf{K}\right)\mathbf{\Lambda} = \mathbf{0} \\ \frac{\partial H}{\partial \mathbf{X}} = \mathbf{0} & \Rightarrow \mathbf{B}_1\mathbf{B}_1^T + \left(\mathbf{\Lambda}\,(\mathbf{A} - \mathbf{B}_2\mathbf{K})^T\right)^T + \mathbf{\Lambda}\,(\mathbf{A} - \mathbf{B}_2\mathbf{K}) = \mathbf{0} \\ \frac{\partial H}{\partial \mathbf{\Lambda}} = \mathbf{0} & \Rightarrow (\mathbf{A} - \mathbf{B}_2\mathbf{K})^T\,\mathbf{X} + \mathbf{X}\,(\mathbf{A} - \mathbf{B}_2\mathbf{K}) \\ & \qquad + (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})^T\,(\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}) = \mathbf{0} \end{array} \right. \tag{6.12}$$

[1]Lewis F., Vrabie D., Syrmos V., Optimal Control, John Wiley & Sons, 3rd Edition, 2012

From the first equation, and assuming that matrices $\mathbf{\Lambda}$ and $\mathbf{D}_{12}^T\mathbf{D}_{12}$ are nonsingular, the static *output* feedback gain $\mathbf{K}$ can be computed as a function of matrix $\mathbf{X}$:

$$\begin{array}{ll} \frac{\partial H}{\partial \mathbf{K}} = \mathbf{0} & \Rightarrow -\mathbf{B}_2^T\mathbf{X} - \mathbf{D}_{12}^T\mathbf{C}_1 + \mathbf{D}_{12}^T\mathbf{D}_{12}\mathbf{K} = \mathbf{0} \\ & \Rightarrow \mathbf{K} = \left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\left(\mathbf{B}_2^T\mathbf{X} + \mathbf{D}_{12}^T\mathbf{C}_1\right) \end{array} \tag{6.13}$$

Assuming that $\mathbf{D}_{12}^T\mathbf{C}_1 = \mathbf{0}$, the expression of $\mathbf{K}$ simplifies as follows:

$$\mathbf{D}_{12}^T\mathbf{C}_1 = \mathbf{0} \Rightarrow \mathbf{K} = \left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\mathbf{X} \tag{6.14}$$

Moreover, assuming that $\mathbf{D}_{12}^T\mathbf{C}_1 = \mathbf{0} \Leftrightarrow \left(\mathbf{D}_{12}^T\mathbf{C}_1\right)^T = \mathbf{C}_1^T\mathbf{D}_{12} = \mathbf{0}$, the last equation of (6.12) reads:

$$\begin{array}{l} (\mathbf{A} - \mathbf{B}_2\mathbf{K})^T\mathbf{X} + \mathbf{X}(\mathbf{A} - \mathbf{B}_2\mathbf{K}) + \mathbf{C}_1^T\mathbf{C}_1 + \mathbf{K}^T\mathbf{D}_{12}^T\mathbf{D}_{12}\mathbf{K} = \mathbf{0} \\ \Leftrightarrow \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{C}_1^T\mathbf{C}_1 - \mathbf{K}^T\mathbf{B}_2^T\mathbf{X} - \mathbf{X}\mathbf{B}_2\mathbf{K} + \mathbf{K}^T\mathbf{D}_{12}^T\mathbf{D}_{12}\mathbf{K} = \mathbf{0} \end{array} \tag{6.15}$$

Finally, assuming that $\mathbf{D}_{12}^T\mathbf{C}_1 = \mathbf{0}$ and using the relation $\mathbf{K} = \left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\mathbf{X}$ we conclude that matrix $\mathbf{X}$ is the positive semi-definite solution of the following algebraic Riccati equation (ARE):

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\mathbf{X} + \mathbf{C}_1^T\mathbf{C}_1 = \mathbf{0} \tag{6.16}$$

The following Hamiltonian matrix $\mathbf{H}_Y$ is associated to equation (6.16) :

$$\mathbf{H}_Y = \begin{bmatrix} \mathbf{A} & -\mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T \\ -\mathbf{C}_1^T\mathbf{C}_1 & -\mathbf{A}^T \end{bmatrix} \tag{6.17}$$

It is worth noticing that the H_2 state feedback problem reduces to a linear quadratic (LQ) optimal control problem where the performance index to be minimized reads as follows:

$$J_{LQ} = \int_0^\infty \left(\underline{x}^T(t)\mathbf{Q}\underline{x}(t) + \underline{u}^T(t)\mathbf{R}\underline{u}(t)\right) dt \tag{6.18}$$

Matrix $\mathbf{Q}$ is a symmetric positive semi-definite matrix and matrix $\mathbf{R}$ is a symmetric positive definite matrix. They are defined as follows:

$$\left\{ \begin{array}{l} \mathbf{Q} = \mathbf{Q}^T := \mathbf{C}_1^T\mathbf{C}_1 \geq 0 \\ \mathbf{R} = \mathbf{R}^T := \mathbf{D}_{12}^T\mathbf{D}_{12} > 0 \end{array} \right. \tag{6.19}$$

Moreover, the minimum value of the performance index achieved by the state feedback (6.14) is:

$$\|\mathbf{T}_{zw}(s)\|_2^2 = \mathrm{tr}\left(\mathbf{B}_1^T\mathbf{X}\mathbf{B}_1\right) \tag{6.20}$$

Furthermore, comparing (6.20) and (6.9), we conclude that the positive semi-definite solution $\mathbf{X}$ of the algebraic Riccati equation (ARE) (6.16) is the observability grammian of $\mathbf{T}_{zw}(s)$.

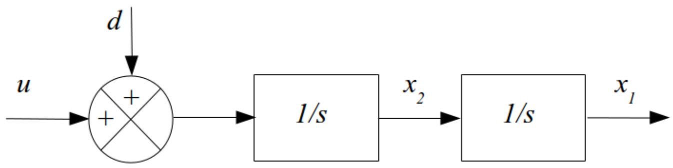

Figure 6.2: Double integrator affected by disturbance on input $u(t)$

6.2.3 Example

Consider the double integrator in Figure 6.2 where the input control $u(t)$ is affected by a disturbance $d(t)$.

The state space form is the following:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) + d(t) \end{cases} \tag{6.21}$$

Here the exogenous input vector is:

$$w(t) = d(t) \tag{6.22}$$

We get:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{6.23}$$

Furthermore we will assume that the state vector $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ is available for control.

The *virtual* output $\underline{z}(t)$ used only for design that we wish to maintain as small as possible is the actual position $x_1(t)$ and the control $u(t)$ (we include control signal $u(t)$ in order to bound it):

$$\underline{z}(t) = \begin{bmatrix} x_1(t) \\ u(t) \end{bmatrix} \tag{6.24}$$

Using the standard state space equations (6.5) for state feedback H_2 control we get:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbb{I} & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{6.25}$$

Where:

$$\begin{cases} \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathbf{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \mathbf{D}_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \tag{6.26}$$

It is clear that the following assumptions are satisfied:

- $(\mathbf{A}, \mathbf{C}_1)$ as no unobservable modes on the imaginary axis
- $(\mathbf{A}, \mathbf{B}_2)$ is stabilizable
- $\mathbf{D}_{12}^T \mathbf{D}_{12} > 0$ (here $\mathbf{D}_{12}^T \mathbf{D}_{12} = 1$)
- $\mathbf{D}_{12}^T \mathbf{C}_1 = \mathbf{0}$

Let the symmetric matrix $\mathbf{X}$ be written as follows:

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \tag{6.27}$$

Then expanding (6.16) leads to the following equations:

$$\begin{array}{l} \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}_2 \left(\mathbf{D}_{12}^T \mathbf{D}_{12}\right)^{-1} \mathbf{B}_2^T \mathbf{X} + \mathbf{C}_1^T \mathbf{C}_1 = \mathbf{0} \\ \Leftrightarrow \begin{bmatrix} 1 - x_{12}^2 & x_{11} - x_{12}x_{22} \\ x_{11} - x_{12}x_{22} & 2x_{12} - x_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \tag{6.28}$$

From which we get:

$$\left\{ \begin{array}{l} x_{12} = +1 \text{ (+ sign because of } x_{22} \text{ expression)} \\ x_{22} = \pm\sqrt{2x_{12}} \\ x_{11} = x_{12}x_{22} \end{array} \right. \tag{6.29}$$

It is the positive semi-definite solution $\mathbf{X}$ of the algebraic Riccati equation (ARE) which stabilizes matrix $\mathbf{A} - \mathbf{B}_2\mathbf{K}$. Matrix $\mathbf{X}$ is obtained by taking the positive values of x_{ij}:

$$\left\{ \begin{array}{l} x_{12} = +1 \\ x_{22} = \sqrt{2} \\ x_{11} = \sqrt{2} \end{array} \right. \Rightarrow \mathbf{X} = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \tag{6.30}$$

Thus the control $\underline{u}(t)$ has the following expression:

$$\underline{u}(t) = -\mathbf{K}\underline{x}(t) \tag{6.31}$$

Where:

$$\begin{array}{c} \mathbf{K} = \left(\mathbf{D}_{12}^T \mathbf{D}_{12}\right)^{-1} \mathbf{B}_2^T \mathbf{X} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \\ \Leftrightarrow \mathbf{K} = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \end{array} \tag{6.32}$$

6.3 State estimation

6.3.1 Problem to be solved

In that section we solve the H_2 state estimation problem whose purpose is to asymptotically reconstructs the system's state $\underline{x}(t)$. The state estimation problem consists in finding the matrix gain $\mathbf{L}$ of a observer (also called estimator) which estimates the full state of the system $\hat{\underline{x}}(t)$ thanks to the measurement vector $\underline{y}(t)$ while minimizing the influence of the exogenous disturbance $\underline{w}(t)$.

Let's consider the following state-space realization of the generalized plant:

$$\begin{cases} \underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}_1\underline{w}(t) + \mathbf{B}_2\underline{u}(t) \\ \underline{y}(t) = \mathbf{C}_2\underline{x}(t) + \mathbf{D}_{21}\underline{w}(t) \end{cases} \tag{6.33}$$

The observer for this system is defined as follows where gain $\mathbf{L}$ has to be set:

$$\begin{cases} \underline{\dot{\hat{x}}}(t) = \mathbf{A}\underline{\hat{x}}(t) + \mathbf{B}_2\underline{u}(t) + \mathbf{L}\left(\underline{y}(t) - \underline{\hat{y}}(t)\right) \\ \underline{\hat{y}}(t) = \mathbf{C}_2\underline{\hat{x}}(t) \end{cases} \tag{6.34}$$

The H_2 state estimation problem consists in finding the gain $\mathbf{L}$ which renders $\mathbf{A} - \mathbf{L}\mathbf{C}_2$ Hurwitz while minimizing the H_2-norm of the transfer matrix $\mathbf{T}_{zw}(s)$ from the exogenous disturbance $\underline{w}(t)$ to the estimation error $\underline{z}(t) = \mathbf{C}_1\,\underline{e}(t)$ where the state error $\underline{e}(t)$ reads:

$$\underline{e}(t) = \underline{x}(t) - \underline{\hat{x}}(t) \tag{6.35}$$

First let's compute the dynamics of the state error $\underline{e}(t)$:

$$\begin{array}{ll} \underline{\dot{e}}(t) & = \underline{\dot{x}}(t) - \underline{\dot{\hat{x}}}(t) \\ & = \mathbf{A}\underline{e}(t) + \mathbf{B}_1\underline{w}(t) - \mathbf{L}\left(\underline{y}(t) - \underline{\hat{y}}(t)\right) \\ & = \mathbf{A}\underline{e}(t) + \mathbf{B}_1\underline{w}(t) - \mathbf{L}\left(\mathbf{C}_2\underline{e}(t) + \mathbf{D}_{21}\underline{w}(t)\right) \\ & = \left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)\underline{e}(t) + \left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)\underline{w}(t) \end{array} \tag{6.36}$$

By taking the Laplace transform (without initial condition) of this relation we obtain:

$$\underline{e}(s) = \left(s\mathbb{I} - \mathbf{A} + \mathbf{L}\mathbf{C}_2\right)^{-1}\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)\underline{w}(s) \tag{6.37}$$

Thus the transfer matrix $\mathbf{T}_{zw}(s)$ from the exogenous disturbance $\underline{w}(t)$ to the estimation error $\underline{z}(t) = \mathbf{C}_1\,\underline{e}(t)$ reads:

$$\mathbf{T}_{zw}(s) = \mathbf{C}_1\left(s\mathbb{I} - \mathbf{A} + \mathbf{L}\mathbf{C}_2\right)^{-1}\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right) \tag{6.38}$$

From the results of section 1.7.2, minimizing the H_2-norm of $\mathbf{T}_{zw}(s)$ consists in solving the following semi-definite program where $\mathbf{Y}$ stands for the observability grammian of $\mathbf{T}_{zw}(s)$:

$$\begin{array}{ll} min & \operatorname{tr}\left(\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)\mathbf{Y}\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)^T\right) \\ s.t. & \begin{cases} \left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)^T\mathbf{Y} + \mathbf{Y}\left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right) + \mathbf{C}_1^T\mathbf{C}_1 \leq \mathbf{0} \\ \mathbf{Y} = \mathbf{Y}^T > \mathbf{0} \end{cases} \end{array} \tag{6.39}$$

A similar semi-definite program to be solved can also be formulated with the controllability grammian of $\mathbf{T}_{zw}(s)$. Denoting $\mathbf{X}$ the controllability grammian of $\mathbf{T}_{zw}(s)$ we get from the results of section 1.7.2:

$$\begin{array}{ll} min & \operatorname{tr}\left(\mathbf{C}_1\mathbf{X}\mathbf{C}_1^T\right) \\ s.t. & \begin{cases} \left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)\mathbf{X} + \mathbf{X}\left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)^T + \left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)^T \leq \mathbf{0} \\ \mathbf{X} = \mathbf{X}^T > \mathbf{0} \end{cases} \end{array} \tag{6.40}$$

State feedback	State estimation
$\mathbf{A}$	$\mathbf{A}^T$
$\mathbf{B}_1$	$\mathbf{C}_1^T$
$\mathbf{B}_2$	$\mathbf{C}_2^T$
$\mathbf{C}_1$	$\mathbf{B}_1^T$
$\mathbf{D}_{12}$	$\mathbf{D}_{21}^T$
$\mathbf{K}$	$\mathbf{L}^T$

Table 6.1: Duality principle for state feedback and state estimation matrices

6.3.2 Duality principle

As underlined by Professor Hannu T. Toivonen [2], when comparing (6.38) and (6.7) it is worth noticing that transfer matrices $\mathbf{T}_{zw}(s)$ whose H_2-norm has to be minimized can be obtained by transpose of each other and by using the equivalence in Table 6.1, which is called the duality principle.

Indeed by taking the transpose of (6.38) we get:

$$\begin{array}{ll} (\mathbf{T}_{zw}(s))^T & = \left(\mathbf{B}_1^T - \mathbf{D}_{21}^T\mathbf{L}^T\right)\left(s\mathbb{I} - \mathbf{A}^T + \mathbf{C}_2^T\mathbf{L}^T\right)^{-1}\mathbf{C}_1^T \\ & := \left(\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}\right)\left(s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K}\right)^{-1}\mathbf{B}_1 \end{array} \tag{6.41}$$

Applying the duality principle on the realization of the generalized plant (6.5) dedicated to state feedback, the realization of the generalized plant for state estimation is the following:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left(\left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbb{I} & \mathbf{0} & \mathbf{0} \end{array} \right] \right)^T \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A}^T & \mathbf{B}_1 & \mathbb{I} \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array} \right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{6.42}$$

In addition the following assumptions are made:

- $(\mathbf{A}, \mathbf{B}_1)$ has no uncontrollable modes on the imaginary axis
- $(\mathbf{A}, \mathbf{C}_2)$ is detectable
- $\mathbf{D}_{21}\mathbf{D}_{21}^T > 0$ (thus the product is invertible)
- $\mathbf{B}_1\mathbf{D}_{21}^T = \mathbf{0}$

The two last assumptions are not very restrictive and can be relaxed but they are convenient because they lead to simplifications in the solution of the problem.

6.3.3 Observer design

Thanks to the duality principle, we can reuse the results of the section dealing with the H_2 state feedback control problem. This leads to the following conclusion: the H_2 optimal observer (or estimator) which estimates the full

[2] http://users.abo.fi/htoivone/courses/robust/rob3.pdf

state of the system $\underline{\hat{x}}(t)$ thanks to the measurement vector $\underline{y}(t)$ while minimizing the influence of the exogenous disturbance $\underline{w}(t)$ has the following the realization:

$$\begin{cases} \dot{\underline{\hat{x}}}(t) = \mathbf{A}\underline{\hat{x}}(t) + \mathbf{B}_2\underline{u}(t) + \mathbf{L}\left(\underline{y}(t) - \underline{\hat{y}}(t)\right) \\ \underline{\hat{y}}(t) = \mathbf{C}_2\underline{\hat{x}}(t) \end{cases} \tag{6.43}$$

where gain matrix $\mathbf{L}$ is obtained thanks to identification with the matrices of the dual problem and depends on the symmetric positive definite matrix $\mathbf{Y} = \mathbf{Y}^T > \mathbf{0}$:

$$\mathbf{K} = \left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\mathbf{X} \Rightarrow \mathbf{L}^T = \left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2\mathbf{Y} \tag{6.44}$$

That is:

$$\mathbf{L} = \mathbf{Y}\mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1} \tag{6.45}$$

Again applying the identification process with the matrices of the dual problem we conclude that positive semi-definite matrix $\mathbf{Y}$ is the solution of the following algebraic Riccati equation (ARE) which comes from (6.16):

$$\mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^T - \mathbf{Y}\mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2\mathbf{Y} + \mathbf{B}_1\mathbf{B}_1^T = \mathbf{0} \tag{6.46}$$

The algebraic Riccati equation (ARE) (6.46) is associated with the following Hamiltonian matrix $\mathbf{H}_Y$:

$$\mathbf{H}_Y = \begin{bmatrix} \mathbf{A}^T & -\mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2 \\ -\mathbf{B}_1\mathbf{B}_1^T & -\mathbf{A} \end{bmatrix} \tag{6.47}$$

In order to get a stabilizing solution the symmetric matrix $\mathbf{Y}$ which solves the algebraic Riccati equation (ARE) (6.46) is chosen to be positive semi-definite. Such a choice leads to a stable closed loop matrix $\mathbf{A}-\mathbf{L}\mathbf{C}_2$, i.e. all its eigenvalues have negative real parts.

Moreover, the minimum value of the cost achieved by the estimator (6.45) is:

$$\|\mathbf{T}_{zw}(s)\|_2^2 = \frac{1}{2}\operatorname{tr}\left(\mathbf{C}_1\mathbf{Y}\mathbf{C}_1^T\right) \tag{6.48}$$

This is clearly the value (6.20) obtained when the duality principle shown in Table 6.1 is applied.

Furthermore, comparing (6.48) and (6.40), we conclude that the positive semi-definite solution $\mathbf{Y}$ of algebraic Riccati equation (ARE) (6.46) is the controllability grammian of $\mathbf{T}_{zw}(s)$.

6.4 H_2 robust control design

6.4.1 LQR problems to be solved

Linear-Quadratic-Regulator (LQR) tackles the problem of designing a state feedback stabilizing controller which minimizes the following criteria:

$$J_{LQR} = \frac{1}{2}\int_0^\infty \left(\underline{x}^T(t)\mathbf{Q}\underline{x}(t) + \underline{u}^T(t)\mathbf{R}\underline{u}(t)\right)dt \tag{6.49}$$

under the following constraint:

$$\underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \tag{6.50}$$

The solution of this problem is closely related to matrix $\mathbf{P} = \mathbf{P}^T > 0$ which solves the following algebraic Riccati equation:

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{6.51}$$

Equations (6.16) and (6.46) show that the H_2 robust control problem can be seen as two separate Linear-Quadratic-Regulator (LQR) design problems:

- The H_2 state feedback problem can be seen as an LQR design problem where the cost to be minimized is:

$$J_{H_{2f}} = \frac{1}{2}\int_0^\infty \left(\underline{x}^T(t)\mathbf{C}_1^T\mathbf{C}_1\underline{x}(t) + \underline{u}^T(t)\mathbf{D}_{12}^T\mathbf{D}_{12}\underline{u}(t)\right) dt \tag{6.52}$$

  under the constraint:

$$\underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}_2\underline{u}(t) \tag{6.53}$$

- The H_2 state estimation problem can be seen as an LQR design problem where the cost to be minimized is:

$$J_{H_{2e}} = \frac{1}{2}\int_0^\infty \left(\underline{x}^T(t)\mathbf{B}_1\mathbf{B}_1^T\underline{x}(t) + \underline{u}^T(t)\mathbf{D}_{21}\mathbf{D}_{21}^T\underline{u}(t)\right) dt \tag{6.54}$$

  under the constraint:

$$\underline{\dot{x}}(t) = \mathbf{A}^T\underline{x}(t) + \mathbf{C}_2^T\underline{u}(t) \tag{6.55}$$

It is worth noticing that in the H_2 robust control problem matrix $\mathbf{A}$ stands for the state matrix of the generalized plant $\mathbf{G}(s)$ whereas in the LQR problem matrix $\mathbf{A}$ stands for the state matrix of the actual nominal plant $\mathbf{F}_n(s)$.

6.4.2 Observer-based controller

We recall the general form of the realization of a plant:

$$\begin{bmatrix} \underline{\dot{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{6.56}$$

Furthermore we will assume that:

- $(\mathbf{A}, \mathbf{B}_1, \mathbf{C}_1)$ has no uncontrallable and no unobservable modes on the imaginary axis

- $(\mathbf{A}, \mathbf{B}_2, \mathbf{C}_2)$ is stabilizable and detectable

- $\mathbf{D}_{12}^T\mathbf{D}_{12} > 0$ and $\mathbf{D}_{21}\mathbf{D}_{21}^T > 0$ (thus the product is invertible)

– $\mathbf{D}_{12}^T \mathbf{C}_1 = \mathbf{0}$ and $\mathbf{B}_1 \mathbf{D}_{21}^T = \mathbf{0}$

The two last assumptions are not very restrictive and can be relaxed but they are convenient because they lead to simplifications in the solution of the problem.

The realization of the controller $\mathbf{C}(s)$ which minimizes $\|\mathbf{T}_{zw}(s)\|_2$ may be split between a state estimator and a state feedback control law of the estimated state as follows:

$$\begin{cases} \dot{\hat{\underline{x}}}(t) = \mathbf{A}\hat{\underline{x}}(t) + \mathbf{B}_2 \underline{u}(t) + \mathbf{L}\left(\underline{y}(t) - \mathbf{C}_2 \hat{\underline{x}}(t)\right) \\ \underline{u}(t) = -\mathbf{K}\hat{\underline{x}}(t) \end{cases} \tag{6.57}$$

In other words the realization of the controller $\mathbf{C}(s)$ is the following:

$$\begin{array}{l} \begin{cases} \underline{u}(s) = \mathbf{C}(s)\underline{y}(s) \\ \mathbf{C}(s) = -\mathbf{K}\left(s\mathbb{I} - (\mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{L}\mathbf{C}_2)\right)^{-1}\mathbf{L} \end{cases} \\ \Leftrightarrow \begin{bmatrix} \dot{\hat{\underline{x}}}(t) \\ \underline{u}(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{L}\mathbf{C}_2 & \mathbf{L} \\ \hline -\mathbf{K} & \mathbf{0} \end{array}\right] \begin{bmatrix} \hat{\underline{x}}(t) \\ \underline{y}(t) \end{bmatrix} \end{array} \tag{6.58}$$

Furthermore the gains $\mathbf{K}$ and $\mathbf{L}$ of the stabilizing controller $\mathbf{C}(s)$ which minimizes $\|\mathbf{T}_{zw}(s)\|_2$ have the following expression:

$$\begin{cases} \mathbf{K} = \left(\mathbf{D}_{12}^T \mathbf{D}_{12}\right)^{-1} \mathbf{B}_2^T \mathbf{X} \\ \mathbf{L} = \mathbf{Y}\mathbf{C}_2^T \left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1} \end{cases} \tag{6.59}$$

Where the symmetric matrices $\mathbf{X} = \mathbf{X}^T$ and $\mathbf{Y} = \mathbf{Y}^T$ are symmetric positive semi-definite solutions of the two following algebraic Riccati equations (AREs)

$$\begin{cases} \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\mathbf{X} + \mathbf{C}_1^T\mathbf{C}_1 = \mathbf{0} \\ \mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^T - \mathbf{Y}\mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2\mathbf{Y} + \mathbf{B}_1\mathbf{B}_1^T = \mathbf{0} \end{cases} \tag{6.60}$$

Those solutions may equivalently be expressed through the following Hamiltonian matrices:

$$\mathbf{H}_X = \begin{bmatrix} \mathbf{A} & -\mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T \\ -\mathbf{C}_1^T\mathbf{C}_1 & -\mathbf{A}^T \end{bmatrix} \tag{6.61}$$

and:

$$\mathbf{H}_Y = \begin{bmatrix} \mathbf{A}^T & -\mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2 \\ -\mathbf{B}_1\mathbf{B}_1^T & -\mathbf{A} \end{bmatrix} \tag{6.62}$$

In order to get a stabilizing solution:

– The symmetric matrix $\mathbf{X}$ shall be chosen such that the matrix $\mathbf{A} - \mathbf{B}_2\mathbf{K}$ is stable, i.e. all its eigenvalues have negative real parts;

– The symmetric matrix $\mathbf{Y}$ shall be chosen such that the matrix $\mathbf{A} - \mathbf{L}\mathbf{C}_2$ is stable, i.e. all its eigenvalues have negative real parts.

A particular feature of the solution is that the optimal estimator and state feedback can be calculated independently of each other. This feature of H_2 controllers is called the *separation principle*.

Moreover, the minimum value of the cost achieved by the controller (6.58) is:

$$\|\mathbf{T}_{zw}(s)\|_2^2 = \frac{1}{2}\left(\operatorname{tr}\left(\mathbf{D}_{12}\mathbf{K}\mathbf{Y}\mathbf{K}^T\mathbf{D}_{12}^T\right) + \operatorname{tr}\left(\mathbf{B}_1^T\mathbf{X}\mathbf{B}_1\right)\right) \tag{6.63}$$

Now assume $\mathbf{D}_{12}^T\mathbf{C}_1 \neq \mathbf{0}$. Then Doyle & al.[3] have shown that relations (6.58) and (6.59) are still valid but the symmetric and positive semi-definite matrix $\mathbf{X} = \mathbf{X}^T \geq \mathbf{0}$ defined in (6.61) is now solution of the following algebraic Riccati equations (AREs) expressed through the following Hamiltonian matrix:

$$\mathbf{H}_X = \begin{bmatrix} \mathbf{A} - \mathbf{B}_2\widetilde{\mathbf{D}}_{12}\mathbf{D}_{12}^T\mathbf{C}_1 & -\mathbf{B}_2\widetilde{\mathbf{D}}_{12}\mathbf{B}_2^T \\ -\widetilde{\mathbf{C}}_1^T\widetilde{\mathbf{C}}_1 & -\left(\mathbf{A} - \mathbf{B}_2\widetilde{\mathbf{D}}_{12}\mathbf{D}_{12}^T\mathbf{C}_1\right)^T \end{bmatrix} \tag{6.64}$$

where:

$$\widetilde{\mathbf{D}}_{12} = \left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1} \tag{6.65}$$

Furthermore state feedback gain matrix $\mathbf{K}$ now reads as follows:

$$\mathbf{K} = \widetilde{\mathbf{D}}_{12}\left(\mathbf{B}_2^T\mathbf{X} + \mathbf{D}_{12}^T\mathbf{C}_1\right) \tag{6.66}$$

Similarly assume $\mathbf{B}_1\mathbf{D}_{21}^T \neq \mathbf{0}$. Then Doyle & al.[3] have shown that relations (6.58) and (6.59) are still valid but the symmetric and positive semi-definite matrix $\mathbf{Y} = \mathbf{Y}^T \geq \mathbf{0}$ defined in (6.62) is now solution of the following algebraic Riccati equations (AREs) expressed through the following Hamiltonian matrix:

$$\mathbf{H}_Y = \begin{bmatrix} \left(\mathbf{A} - \mathbf{B}_1\mathbf{D}_{21}^T\widetilde{\mathbf{D}}_{21}\mathbf{C}_2\right)^T & -\mathbf{C}_2^T\widetilde{\mathbf{D}}_{21}\mathbf{C}_2 \\ -\widetilde{\mathbf{B}}_1\widetilde{\mathbf{B}}_1^T & -\left(\mathbf{A} - \mathbf{B}_1\mathbf{D}_{21}^T\widetilde{\mathbf{D}}_{21}\mathbf{C}_2\right) \end{bmatrix} \tag{6.67}$$

where:

$$\widetilde{\mathbf{D}}_{21} = \left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1} \tag{6.68}$$

Furthermore observer gain matrix $\mathbf{L}$ now reads as follows:

$$\mathbf{L} = \left(\mathbf{Y}\mathbf{C}_2^T + \mathbf{B}_1\mathbf{D}_{21}^T\right)\widetilde{\mathbf{D}}_{21} \tag{6.69}$$

6.4.3 Linear-Quadratic-Gaussian (LQG) control

Let's consider the following plant realization:

$$\begin{cases} \underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) + \underline{d}(t) \\ \underline{y}_p(t) = \mathbf{C}\underline{x}(t) + \underline{n}(t) \end{cases} \tag{6.70}$$

[3] Zhou K., Doyle J., Essentials of Robust Control (p. 262), Prentice Hall, 1997, ISBN 0-13-525833-2

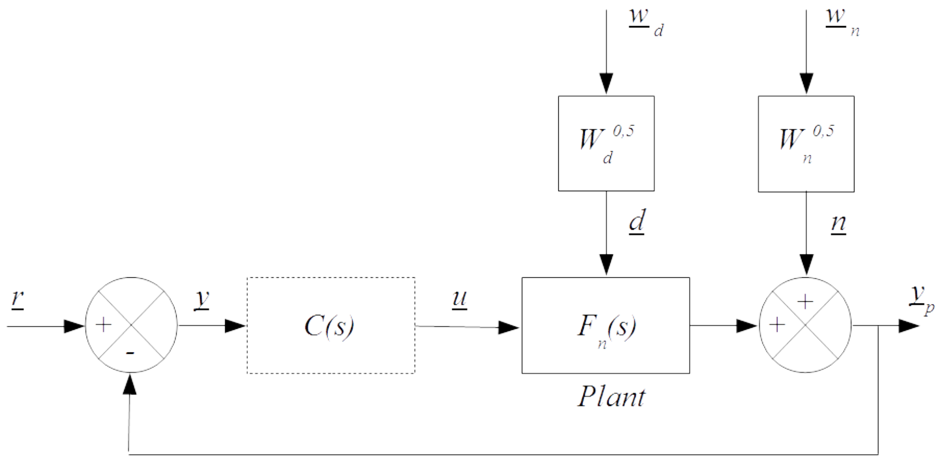

Figure 6.3: Standard form of LQG control problem

Where $\underline{d}(t)$ and $\underline{n}(t)$ are white noise with the intensity of their autocorrelation function equals to $\mathbf{W}_d$ and $\mathbf{W}_n$ respectively, as shown in Figure 6.3. Denoting by $E()$ the mathematical expectation we have:

$$E\left(\begin{bmatrix} \underline{d}(t) \\ \underline{n}(t) \end{bmatrix} \begin{bmatrix} \underline{d}^T(\tau) & \underline{n}^T(\tau) \end{bmatrix}\right) = \begin{bmatrix} \mathbf{W}_d & 0 \\ 0 & \mathbf{W}_n \end{bmatrix} \delta(t-\tau) \tag{6.71}$$

The LQG problem consists in finding a controller $\underline{u}(s) = \mathbf{C}(s)\underline{y}(s)$ such that the following performance index is minimized:

$$J_{LQG} = E\left(\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\underline{x}^T(t)\mathbf{Q}\underline{x}(t) + \underline{u}^T(t)\mathbf{R}\underline{u}(t)\right) dt\right) \tag{6.72}$$

Matrices $\mathbf{Q}$ and $\mathbf{R}$ are symmetric and (semi)-positive definite matrices:

$$\begin{cases} \mathbf{Q} = \mathbf{Q}^T \geq 0 \\ \mathbf{R} = \mathbf{R}^T > 0 \end{cases} \tag{6.73}$$

We will see that Linear-Quadratic-Gaussian (LQG) control is a special case of H_2 control applied to stochastic system and how to set matrices $\mathbf{Q}$ and $\mathbf{R}$. To achieve this goal, we will recast the LQG problem into the H_2 control framework as follows:

– First we define signal $\underline{z}(t)$, whose norm is to be minimized, as follows:

$$\underline{z}(t) = \begin{bmatrix} \mathbf{Q}^{0.5} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{0.5} \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{u}(t) \end{bmatrix} \tag{6.74}$$

– Secondly we write the stochastic inputs $\underline{d}(t)$ and $\underline{n}(t)$ as a function of the vector $\underline{w}(t)$ of exogenous disturbances, which is here a white noise with components $\underline{w}_d(t)$ and $\underline{w}_n(t)$:

$$\begin{array}{l} \begin{bmatrix} \underline{d}(t) \\ \underline{n}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_d^{0.5} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_n^{0.5} \end{bmatrix} \begin{bmatrix} \underline{w}_d(t) \\ \underline{w}_n(t) \end{bmatrix} := \begin{bmatrix} \mathbf{W}_d^{0.5} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_n^{0.5} \end{bmatrix} \underline{w}(t) \\ \Leftrightarrow \underline{w}(t) := \begin{bmatrix} \underline{w}_d(t) \\ \underline{w}_n(t) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_d^{-0.5} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_n^{-0.5} \end{bmatrix} \begin{bmatrix} \underline{d}(t) \\ \underline{n}(t) \end{bmatrix} \end{array} \tag{6.75}$$

Here $\underline{w}(t)$ is a white noise process of unit intensity. Then the LQG cost function reads as follows:

$$J_{LQG} = E\left(\lim_{T\to\infty} \frac{1}{T} \int_0^T \underline{z}^T(t)\underline{z}(t)dt \right)_{E(\underline{w}(t)\underline{w}^T(\tau))=\delta(t-\tau)} := \|\mathbf{T}_{zw}(s)\|_2^2 \quad (6.76)$$

The generalized plant reads as follows:

$$\begin{bmatrix} \underline{\dot{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \quad (6.77)$$

Where we get by identification:

$$\begin{cases} \underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) + \underline{d}(t) \\ \underline{y}_p(t) = \mathbf{C}\underline{x}(t) + \underline{n}(t) \\ \underline{y}(t) = \underline{r}(t) - \underline{y}_p(t) \text{ where } \underline{r}(t) = \underline{0} \\ \begin{bmatrix} \underline{d}(t) \\ \underline{n}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_d^{0.5} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_n^{0.5} \end{bmatrix} \underline{w}(t) \end{cases} \Rightarrow \begin{cases} \mathbf{B}_1 = \begin{bmatrix} \mathbf{W}_d^{0.5} & \mathbf{0} \end{bmatrix} \\ \mathbf{B}_2 = \mathbf{B} \\ \mathbf{C}_2 = -\mathbf{C} \\ \mathbf{D}_{21} = -\begin{bmatrix} \mathbf{0} & \mathbf{W}_n^{0.5} \end{bmatrix} \end{cases} \quad (6.78)$$

Costs (6.72) and (6.76) are equivalent as soon as the following relation holds

$$\begin{aligned} \underline{z}^T(t)\underline{z}(t) &= \left(\mathbf{C}_1\underline{x}(t) + \mathbf{D}_{12}\underline{u}(t)\right)^T \left(\mathbf{C}_1\underline{x}(t) + \mathbf{D}_{12}\underline{u}(t)\right) \\ &:= \underline{x}^T(t)\mathbf{Q}\underline{x}(t) + \underline{u}^T(t)\mathbf{R}\underline{u}(t) \end{aligned} \quad (6.79)$$

This leads to the following identification:

$$\mathbf{C}_1^T\mathbf{D}_{12} = \mathbf{D}_{12}^T\mathbf{C}_1 = \mathbf{0} \Rightarrow \begin{cases} \mathbf{C}_1 := \begin{bmatrix} \mathbf{Q}^{0.5} \\ \mathbf{0} \end{bmatrix} \\ \mathbf{D}_{12} := \begin{bmatrix} \mathbf{0} \\ \mathbf{R}^{0.5} \end{bmatrix} \end{cases} \quad (6.80)$$

We have seen in (6.46) that the solution of the H_2 robust observer design problem is related to the following algebraic Riccati equation:

$$\mathbf{AY} + \mathbf{YA}^T - \mathbf{YC}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2\mathbf{Y} + \mathbf{B}_1\mathbf{B}_1^T = \mathbf{0} \quad (6.81)$$

On the other hand, let's have a look on the Linear-Quadratic-Regulator (LQR) which tackles the problem of designing a state feedback stabilizing controller which minimizes the following criteria:

$$J_{LQR} = \frac{1}{2}\int_0^\infty \left(\underline{x}^T(t)\mathbf{Q}\underline{x}(t) + \underline{u}^T(t)\mathbf{R}\underline{u}(t)\right) dt \quad (6.82)$$

under the following constraint:

$$\underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \quad (6.83)$$

The solution of this problem is closely related to matrix $\mathbf{Y} = \mathbf{Y}^T > 0$ which solves the following algebraic Riccati equation:

$$\mathbf{A}^T\mathbf{Y} + \mathbf{YA} - \mathbf{YBR}^{-1}\mathbf{B}^T\mathbf{Y} + \mathbf{Q} = \mathbf{0} \quad (6.84)$$

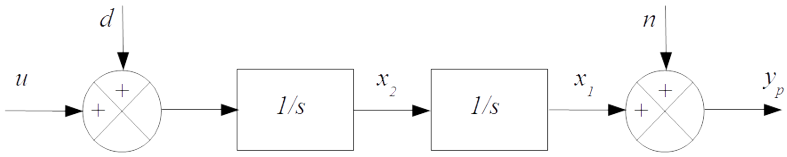

Figure 6.4: Double integrator affected by disturbance on input $u(t)$ and noise on output $y_p(t)$

When the algebraic Riccati equation (6.81) related to the H_2 robust observer design problem is identified to the algebraic Riccati equation (6.84) related the Linear-Quadratic-Regulator (LQR) we get:

$$\begin{array}{ll} \mathbf{0} & = \mathbf{AY} + \mathbf{YA}^T - \mathbf{YC}_2^T \left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1} \mathbf{C}_2\mathbf{Y} + \mathbf{B}_1\mathbf{B}_1^T \\ & := \mathbf{A}^T\mathbf{Y} + \mathbf{YA} - \mathbf{YBR}^{-1}\mathbf{B}^T\mathbf{Y} + \mathbf{Q} \\ \Rightarrow & \left\{ \begin{array}{l} \mathbf{R} := \mathbf{D}_{21}\mathbf{D}_{21}^T = \mathbf{W}_n \geq 0 \\ \mathbf{Q} := \mathbf{B}_1\mathbf{B}_1^T = \mathbf{W}_d > 0 \\ \mathbf{B} \to \mathbf{C}_2^T \\ \mathbf{A} \to \mathbf{A}^T \end{array} \right. \end{array} \tag{6.85}$$

6.4.4 Example

Consider again the double integrator in Figure 6.4 where the input control $u(t)$ is affected by a white noise $d(t)$ and output $y_p(t)$ is affected by a white noise $n(t)$. We consider the problem of finding an H_2 observer-based controller.

The state space form is the following:

$$\left\{ \begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) + d(t) \end{array} \right. \tag{6.86}$$

That is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{6.87}$$

Furthermore we will assume that the following output vector $y_p(t)$ affected by a noise $n(t)$ is available for control:

$$y_p(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + n(t) \tag{6.88}$$

Moreover we will assume that $d(t)$ and $n(t)$ are white noise with the intensity of their autocorrelation function equals to ρ_d and ρ_n respectively. Denoting by $E()$ the mathematical expectation we have:

$$E\left(\begin{bmatrix} d(t) \\ n(t) \end{bmatrix} \begin{bmatrix} d(\tau) & n(\tau) \end{bmatrix} \right) = \begin{bmatrix} \rho_d & 0 \\ 0 & \rho_n \end{bmatrix} \delta(t-\tau) \tag{6.89}$$

The exogenous input vector $\underline{w}(t)$ is defined as follows:

$$\begin{bmatrix} d(t) \\ n(t) \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_d} & \mathbf{0} \\ \mathbf{0} & \sqrt{\rho_n} \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_d} & \mathbf{0} \\ \mathbf{0} & \sqrt{\rho_n} \end{bmatrix} \underline{w}(t) \tag{6.90}$$

The *virtual* output $\underline{z}(t)$ used only for design that we wish to maintain as small as possible is the actual position $x_1(t)$ and the control $u(t)$ (we include control signal $u(t)$ in order to bound it):

$$\underline{z}(t) = \begin{bmatrix} x_1(t) \\ u(t) \end{bmatrix} \tag{6.91}$$

Using the standard state space equations (6.1) for H_2 observer-based controller, and using the feedback control loop of Figure 6.3, we set $y(t) := r(t) - y_p(t)$ and $r(t) = 0$:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ y(t) = -y_p(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ u(t) \end{bmatrix} \tag{6.92}$$

Where:

$$\left\{ \begin{array}{l} \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathbf{C}_2 = -\begin{bmatrix} 1 & 0 \end{bmatrix} \end{array} \right. \quad \textit{and} \quad \left\{ \begin{array}{l} \mathbf{B}_1 = \begin{bmatrix} 0 & 0 \\ \sqrt{\rho_d} & 0 \end{bmatrix} \\ \mathbf{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \mathbf{D}_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathbf{D}_{21} = -\begin{bmatrix} 0 & \sqrt{\rho_n} \end{bmatrix} \end{array} \right. \tag{6.93}$$

It is worth noticing that we have put in $\mathbf{B}_1$ and $\mathbf{D}_{21}$ the root square of the intensity of the autocorrelation function of the white noises which affect the system.

It is clear that the following assumptions are satisfied:

- $(\mathbf{A}, \mathbf{B}_1, \mathbf{C}_1)$ has no uncontrallable and no unobservable modes on the imaginary axis
- $(\mathbf{A}, \mathbf{B}_2, \mathbf{C}_2)$ is stabilizable and detectable
- $\mathbf{D}_{12}^T \mathbf{D}_{12} > 0$ and $\mathbf{D}_{21} \mathbf{D}_{21}^T > 0$ have full rank (thus the product is invertible)
- $\mathbf{D}_{12}^T \mathbf{C}_1 = \mathbf{0}$ and $\mathbf{B}_1 \mathbf{D}_{21}^T = \mathbf{0}$

First we focus on the state feedback control. Let the symmetric matrix $\mathbf{X}$ be written as follows:

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \tag{6.94}$$

Then expanding (6.16) leads to the following equations:

$$\begin{array}{l} \mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} - \mathbf{X} \mathbf{B}_2 \left(\mathbf{D}_{12}^T \mathbf{D}_{12} \right)^{-1} \mathbf{B}_2^T \mathbf{X} + \mathbf{C}_1^T \mathbf{C}_1 = \mathbf{0} \\ \Leftrightarrow \begin{bmatrix} 1 - x_{12}^2 & x_{11} - x_{12} x_{22} \\ x_{11} - x_{12} x_{22} & 2x_{12} - x_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \tag{6.95}$$

From which we get:

$$\left\{\begin{array}{l} x_{12}=+1\,(\,+\text{ sign because of } x_{22} \text{ expression}) \\ x_{22}=\pm\sqrt{2x_{12}} \\ x_{11}=x_{12}x_{22} \end{array}\right. \tag{6.96}$$

It is the positive semi-definite solution $\mathbf{X}$ of the algebraic Riccati equation (ARE) which stabilizes matrix $\mathbf{A}-\mathbf{B}_2\mathbf{K}$. Matrix $\mathbf{X}$ is obtained by taking the positive values of x_{ij}:

$$\left\{\begin{array}{l} x_{12}=+1 \\ x_{22}=\sqrt{2} \\ x_{11}=\sqrt{2} \end{array}\right. \Rightarrow \mathbf{X}=\begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \tag{6.97}$$

Thus the control $\underline{u}(t)$ has the following expression:

$$\underline{u}(t)=-\mathbf{K}\underline{x}(t) \tag{6.98}$$

Where:

$$\begin{array}{c} \mathbf{K}=\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\mathbf{X}=\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \\ \Leftrightarrow \mathbf{K}=\begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \end{array} \tag{6.99}$$

Then we focus on the state estimation problem. Let the symmetric matrix $\mathbf{Y}$ be written as follows:

$$\mathbf{Y}=\begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} \tag{6.100}$$

Expanding (6.46) leads to the following equations:

$$\begin{array}{l} \mathbf{A}\mathbf{Y}+\mathbf{Y}\mathbf{A}^T-\mathbf{Y}\mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2\mathbf{Y}+\mathbf{B}_1\mathbf{B}_1^T=\mathbf{0} \\ \Leftrightarrow \begin{bmatrix} 2y_{12}-\frac{y_{11}^2}{\rho_n} & y_{22}-\frac{y_{11}y_{12}}{\rho_n} \\ y_{22}-\frac{y_{11}y_{12}}{\rho_n} & \rho_d-\frac{y_{12}^2}{\rho_n} \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \tag{6.101}$$

From which we get:

$$\left\{\begin{array}{l} y_{12}=+\sqrt{\rho_d\rho_n}\,(\,+\text{ sign because of } y_{11} \text{ expression}) \\ y_{11}=\pm\sqrt{2\rho_n y_{12}} \\ y_{22}=\frac{y_{11}y_{12}}{\rho_n} \end{array}\right. \tag{6.102}$$

It is the positive semi-definite solution $\mathbf{Y}$ of the algebraic Riccati equation (ARE) which stabilizes matrix $\mathbf{A}-\mathbf{L}\mathbf{C}_2$. Matrix $\mathbf{Y}$ is obtained by taking the positive values of y_{ij}:

$$\left\{\begin{array}{l} y_{12}=+\sqrt{\rho_d\rho_n} \\ y_{11}=\sqrt{2\rho_n\sqrt{\rho_d\rho_n}} \\ y_{22}=\sqrt{2\sqrt{\rho_d\rho_n}} \end{array}\right. \tag{6.103}$$

We get:

$$\begin{array}{rl} \mathbf{Y} & =\begin{bmatrix} \sqrt{2\rho_n\sqrt{\rho_d\rho_n}} & \sqrt{\rho_d\rho_n} \\ \sqrt{\rho_d\rho_n} & \sqrt{2\sqrt{\rho_d\rho_n}} \end{bmatrix} \\ & =\begin{bmatrix} \sqrt{2}\rho_d^{0.25}\rho_n^{0.75} & \rho_d^{0.5}\rho_n^{0.5} \\ \rho_d^{0.5}\rho_n^{0.5} & \sqrt{2}\rho_d^{0.25}\rho_n^{0.25} \end{bmatrix} \end{array} \tag{6.104}$$

Thus the state $\underline{x}(t)$ is estimated as follows:

$$\dot{\hat{\underline{x}}}(t) = \mathbf{A}\hat{\underline{x}}(t) + \mathbf{L}\left(\underline{y}(t) - \mathbf{C}_2\hat{\underline{x}}(t)\right) \tag{6.105}$$

Where:

$$\mathbf{L} = \mathbf{Y}\mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1} = -\begin{bmatrix} \sqrt{2}\left(\frac{\rho_d}{\rho_n}\right)^{0.25} \\ \left(\frac{\rho_d}{\rho_n}\right)^{0.5} \end{bmatrix} \tag{6.106}$$

Chapter 7

H_∞ robust control design

7.1 Introduction

In this chapter the H_∞ observer-based controller design is considered. We recall in Figure 7.1 the general framework for such design.

Furthermore the general form of the realization of the generalized plant $\mathbf{G}(s)$ is the following:

$$\begin{bmatrix} \underline{z}(s) \\ \underline{y}(s) \end{bmatrix} = \mathbf{G}(s) \begin{bmatrix} \underline{w}(s) \\ \underline{u}(s) \end{bmatrix} \Leftrightarrow \begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{7.1}$$

The H_∞ control problem consists in finding the controller $\mathbf{C}(s)$ which achieves the inequality $\|\mathbf{T}_{zw}(s)\|_\infty = \|\mathcal{F}_l\left(\mathbf{G}(s), \mathbf{C}(s)\right)\|_\infty < \gamma$ provided that such a controller exists.

First let's analyze the inequality :

$$\max_{\underline{w}(t) \neq 0} \frac{\|\underline{z}(t)\|_2}{\|\underline{w}(t)\|_2} \leq \gamma \tag{7.2}$$

As far as state vector $\underline{x}(t)$ depends on control vector $\underline{u}(t)$ it is clear that the performance output vector $\underline{z}(t)$ fully depends on control vector $\underline{u}(t)$. Thus the H_∞ state feedback control problem can be envisioned as a zero-sum game

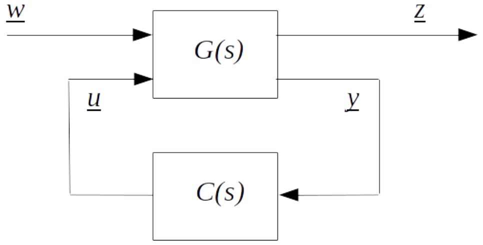

Figure 7.1: Generalized plant $\mathbf{G}(s)$ with controller $\mathbf{C}(s)$

between the controller $\mathbf{C}(s)$ (indeed from Figure 7.1 we have $\underline{u}(s) = \mathbf{C}(s)\underline{y}(s)$) and the disturbance $\underline{w}(t)$:

$$\max_{\underline{w}(t)\neq\underline{0}} \min_{\underline{u}(s)=\mathbf{C}(s)\underline{y}(s)} \|\underline{z}(t)\|_2^2 - \gamma^2\|\underline{w}(t)\|_2^2 = 0 \tag{7.3}$$

This can be rewritten as follows:

$$\max_{\underline{w}(t)\neq\underline{0}} \min_{\underline{u}(s)=\mathbf{C}(s)\underline{y}(s)} J_{H_\infty} = 0 \tag{7.4}$$

Where the performance index J_{H_∞} to be zeroed is the following (1/2 has been added in front of the integral without loss of generality to facilitate the calculus):

$$J_{H_\infty} = \frac{1}{2}\int_0^\infty \left(\underline{z}(t)^T\underline{z}(t) - \gamma^2\underline{w}(t)^T\underline{w}(t)\right) dt \tag{7.5}$$

As underlined by Professor Hannu T. Toivonen [1] the H_∞ control problem can be tackled in two steps. The first steps deals with the computation of the optimal state-feedback control law. The second steps consists in finding the optimal state estimator.

7.2 State feedback control

7.2.1 Problem to be solved

In this section we will assume that the realization of the plant simplifies as follows:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbb{I} & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{7.6}$$

In addition the following assumptions are made:

- $(\mathbf{A}, \mathbf{C}_1)$ has no unobservable modes on the imaginary axis
- $(\mathbf{A}, \mathbf{B}_2)$ is stabilizable
- $\mathbf{D}_{12}^T\mathbf{D}_{12} > 0$ (thus the product is invertible)
- $\mathbf{D}_{12}^T\mathbf{C}_1 = \mathbf{0}$

Taking the Laplace transform (with initial condition set to zero) of the preceding realization leads to the expression of the transfer matrix $\mathbf{G}(s)$:

$$\begin{array}{rl} \underline{x}(s) & = (s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}_1\underline{w}(s) + (s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}_2\underline{u}(s) \\ \Rightarrow \begin{bmatrix} \underline{z}(s) \\ \underline{y}(s) \end{bmatrix} & = \begin{bmatrix} \mathbf{C}_1(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}_1 & \mathbf{C}_1(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}_2 + \mathbf{D}_{12} \\ (s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}_1 & (s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}_2 \end{bmatrix} \begin{bmatrix} \underline{w}(s) \\ \underline{u}(s) \end{bmatrix} \\ & := \mathbf{G}(s)\begin{bmatrix} \underline{w}(s) \\ \underline{u}(s) \end{bmatrix} \end{array} \tag{7.7}$$

[1] http://users.abo.fi/htoivone/courses/robust/rob4.pdf

From (7.6) we get:

$$\begin{aligned} &\underline{z}(t) = \mathbf{C}_1\underline{x}(t) + \mathbf{D}_{12}\underline{u}(t) \\ &\Rightarrow \underline{z}(t)^T\underline{z}(t) = \left(\underline{x}^T(t)\mathbf{C}_1^T + \underline{u}^T(t)\mathbf{D}_{12}^T\right)\left(\mathbf{C}_1\underline{x}(t) + \mathbf{D}_{12}\underline{u}(t)\right) \end{aligned} \tag{7.8}$$

That is:

$$\begin{aligned} \underline{z}(t)^T\underline{z}(t) = \underline{x}^T(t)\mathbf{C}_1^T\mathbf{C}_1\underline{x}(t) + \underline{x}^T(t)\mathbf{C}_1^T\mathbf{D}_{12}\underline{u}(t) \\ + \underline{u}^T(t)\mathbf{D}_{12}^T\mathbf{C}_1\underline{x}(t) + \underline{u}^T(t)\mathbf{D}_{12}^T\mathbf{D}_{12}\underline{u}(t) \end{aligned} \tag{7.9}$$

Thanks to the simplifying assumptions we get:

$$\mathbf{D}_{12}^T\mathbf{C}_1 = \mathbf{0} \Rightarrow \underline{z}(t)^T\underline{z}(t) = \underline{x}^T(t)\mathbf{C}_1^T\mathbf{C}_1\underline{x}(t) + \underline{u}^T(t)\mathbf{D}_{12}^T\mathbf{D}_{12}\underline{u}(t) \tag{7.10}$$

Thus the performance index J_{H_∞} in (7.5) to be minimized reduces as follows:

$$J_{H_\infty} = \frac{1}{2}\int_0^\infty \left(\underline{x}^T(t)\mathbf{C}_1^T\mathbf{C}_1\underline{x}(t) + \underline{u}^T(t)\mathbf{D}_{12}^T\mathbf{D}_{12}\underline{u}(t) - \gamma^2\underline{w}(t)^T\underline{w}(t)\right)dt \tag{7.11}$$

Under the *constraint*:

$$\underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}_1\underline{w}(t) + \mathbf{B}_2\underline{u}(t) \tag{7.12}$$

7.2.2 Controller design

Applying classical results of optimal control theory we form the Hamiltonian $H(\underline{u},\underline{w},\underline{x})$ as the sum of the vector-valued mapping to be minimized and the product between the Lagrange multipliers $\underline{\lambda}(t)$ and the *constraint* to be satisfied:

$$\begin{aligned} H(\underline{u},\underline{w},\underline{x}) = \frac{1}{2}\left(\underline{x}^T(t)\mathbf{C}_1^T\mathbf{C}_1\underline{x}(t) + \underline{u}^T(t)\mathbf{D}_{12}^T\mathbf{D}_{12}\underline{u}(t) - \gamma^2\underline{w}(t)^T\underline{w}(t)\right) + \\ \underline{\lambda}(t)^T\left(\mathbf{A}\underline{x}(t) + \mathbf{B}_1\underline{w}(t) + \mathbf{B}_2\underline{u}(t)\right) \end{aligned} \tag{7.13}$$

The best control signal $\underline{u}(t)$ that minimizes J_{H_∞} is then obtained as the solution of $\frac{\partial H(\underline{u},\underline{w},\underline{x})}{\partial \underline{u}^T} = 0$:

$$\begin{aligned} &\tfrac{\partial H(\underline{u},\underline{w},\underline{x})}{\partial \underline{u}^T} = 0 \Leftrightarrow \mathbf{D}_{12}^T\mathbf{D}_{12}\underline{u}(t) + \mathbf{B}_2^T\underline{\lambda}(t) = 0 \\ &\Leftrightarrow \underline{u}(t) = -\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\underline{\lambda}(t) \end{aligned} \tag{7.14}$$

Similarly the worth disturbance signal $\underline{w}(t)$ that maximizes J_{H_∞} is obtained as the solution of $\frac{\partial H(\underline{u},\underline{w},\underline{x})}{\partial \underline{w}^T} = 0$:

$$\begin{aligned} &\tfrac{\partial H(\underline{u},\underline{w},\underline{x})}{\partial \underline{w}^T} = 0 \Leftrightarrow -\gamma^2\underline{w}(t) + \mathbf{B}_1^T\underline{\lambda}(t) = 0 \\ &\Leftrightarrow \underline{w}(t) = \gamma^{-2}\mathbf{B}_1^T\underline{\lambda}(t) \end{aligned} \tag{7.15}$$

Furthermore the Lagrange multipliers $\underline{\lambda}(t)$ satisfy the following equation:

$$\frac{\partial H(\underline{u},\underline{w},\underline{x})}{\partial \underline{x}^T} = -\underline{\dot{\lambda}}(t) \Leftrightarrow \mathbf{C}_1^T\mathbf{C}_1\underline{x}(t) + \mathbf{A}^T\underline{\lambda}(t) = -\underline{\dot{\lambda}}(t) \tag{7.16}$$

Then the key point to get the expression of the Lagrange multipliers $\underline{\lambda}(t)$ is to set them as follows where $\mathbf{X}$ is a constant and symmetric matrix:

$$\begin{cases} \underline{\lambda}(t) = \mathbf{X}\underline{x}(t) \\ \mathbf{X} = \mathbf{X}^T \end{cases} \tag{7.17}$$

Thus using *constraint* (7.12) in equation (7.16) we get:

$$\begin{array}{rl} \mathbf{C}_1^T\mathbf{C}_1\underline{x}(t) + \mathbf{A}^T\mathbf{X}\underline{x}(t) & = -\mathbf{X}\underline{\dot{x}}(t) \\ \Leftrightarrow \left(\mathbf{C}_1^T\mathbf{C}_1 + \mathbf{A}^T\mathbf{X}\right)\underline{x}(t) & = -\mathbf{X}\left(\mathbf{A}\underline{x}(t) + \mathbf{B}_1\underline{w}(t) + \mathbf{B}_2\underline{u}(t)\right) \end{array} \tag{7.18}$$

In a last step we use the expression of the *optimal* control signals $\underline{u}(t)$ and $\underline{w}(t)$ to obtain the following equation:

$$\begin{array}{l} \left(\mathbf{C}_1^T\mathbf{C}_1 + \mathbf{A}^T\mathbf{X}\right)\underline{x}(t) = -\mathbf{X}\left(\mathbf{A}\underline{x}(t) + \mathbf{B}_1\gamma^{-2}\mathbf{B}_1^T\underline{\lambda}(t) - \mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\underline{\lambda}(t)\right) \\ \Leftrightarrow \left(\mathbf{C}_1^T\mathbf{C}_1 + \mathbf{A}^T\mathbf{X}\right)\underline{x}(t) = -\mathbf{X}\left(\mathbf{A} + \gamma^{-2}\mathbf{B}_1\mathbf{B}_1^T\mathbf{X} - \mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\mathbf{X}\right)\underline{x}(t) \\ \Leftrightarrow \left(\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{X}\left(\gamma^{-2}\mathbf{B}_1\mathbf{B}_1^T - \mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\right)\mathbf{X} + \mathbf{C}_1^T\mathbf{C}_1\right)\underline{x}(t) = \underline{0} \end{array} \tag{7.19}$$

The preceding equation is verified whatever the value of the state vector $\underline{x}(t)$ as soon as symmetric matrix $\mathbf{X}$ is the solution of the following Algebraic Riccati Equation (ARE):

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{X}\left(\gamma^{-2}\mathbf{B}_1\mathbf{B}_1^T - \mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\right)\mathbf{X} + \mathbf{C}_1^T\mathbf{C}_1 = \mathbf{0} \tag{7.20}$$

The following Hamiltonian matrix $\mathbf{H}_X$ is associated to equation (7.20) :

$$\mathbf{H}_X = \begin{bmatrix} \mathbf{A} & \gamma^{-2}\mathbf{B}_1\mathbf{B}_1^T - \mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T \\ -\mathbf{C}_1^T\mathbf{C}_1 & -\mathbf{A}^T \end{bmatrix} \tag{7.21}$$

In order to get a stabilizing solution the symmetric matrix $\mathbf{X}$ shall be chosen such that it is the positive semi-definite ($\mathbf{X} \geq 0$) solution of the Algebraic Riccati Equation (ARE) (7.20).

Thus the control $\underline{u}(t)$ has the following expression:

$$\underline{u}(t) = -\mathbf{K}\underline{x}(t) \tag{7.22}$$

Where:

$$\mathbf{K} = \left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\mathbf{X} \tag{7.23}$$

Furthermore the worth case disturbance $\underline{w}(t)$ is given by:

$$\underline{w}(t) = \mathbf{K}_w\underline{x}(t) \tag{7.24}$$

Where:

$$\mathbf{K}_w = \gamma^{-2}\mathbf{B}_1^T\mathbf{X} \tag{7.25}$$

Example 7.1. *Consider the double integrator in Figure 7.2 where the input control $u(t)$ is affected by a disturbance $d(t)$.*

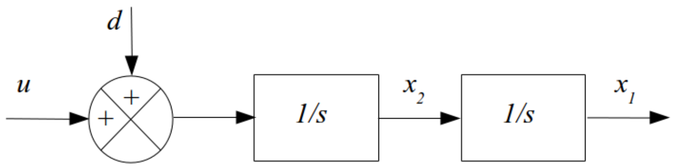

Figure 7.2: Double integrator affected by noise and disturbance

The state space form is the following:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) + d(t) \end{cases} \tag{7.26}$$

The virtual *output* $\underline{z}(t)$ *used only for design that we wish to maintain as small as possible is the actual position* $x_1(t)$ *and the control* $u(t)$ *(we include control signal* $u(t)$ *in order to bound it):*

$$\underline{z}(t) = \begin{bmatrix} x_1(t) \\ u(t) \end{bmatrix} \tag{7.27}$$

The exogenous input vector is:

$$w(t) = d(t) \tag{7.28}$$

Furthermore we will assume that the state vector $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ *is available for control.*

Using the standard state space equations (7.6) *for state feedback* H_∞ *control we get:*

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbb{I} & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{7.29}$$

Where:

$$\begin{cases} \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathbf{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \mathbf{D}_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \tag{7.30}$$

It is clear that the following assumptions are satisfied:

– $(\mathbf{A}, \mathbf{C}_1)$ *as no unobservable modes on the imaginary axis*

– $(\mathbf{A}, \mathbf{B}_2)$ *is stabilizable*

– $\mathbf{D}_{12}^T \mathbf{D}_{12} > 0$ *(here* $\mathbf{D}_{12}^T \mathbf{D}_{12} = 1$*)*

– $\mathbf{D}_{12}^T \mathbf{C}_1 = \mathbf{0}$

Let the symmetric matrix $\mathbf{X}$ *be written as follows:*

$$\mathbf{X} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{7.31}$$

Then expanding (7.20) *leads to the following equations:*

$$\begin{array}{l} \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{X}\left(\gamma^{-2}\mathbf{B}_1\mathbf{B}_1^T - \mathbf{B}_2\left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\right)\mathbf{X} + \mathbf{C}_1^T\mathbf{C}_1 = \mathbf{0} \\ \Leftrightarrow \begin{bmatrix} 1 + \left(\frac{1}{\gamma^2} - 1\right)p_{12}^2 & p_{11} + \left(\frac{1}{\gamma^2} - 1\right)p_{12}p_{22} \\ p_{11} + \left(\frac{1}{\gamma^2} - 1\right)p_{12}p_{22} & 2p_{12} + \left(\frac{1}{\gamma^2} - 1\right)p_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \tag{7.32}$$

From which we get:

$$\begin{cases} p_{12} = +\sqrt{\frac{1}{1-\frac{1}{\gamma^2}}} \; (\ + \textit{ sign because of } (p_{22}, p_{11}) \textit{ expressions}) \\ p_{22} = \pm\sqrt{\frac{2p_{12}}{1-\frac{1}{\gamma^2}}} \\ p_{11} = \pm\sqrt{2p_{12}} \end{cases} \tag{7.33}$$

Finally the positive semi-definite solution $\mathbf{X}$ *of the Algebraic Riccati Equation (ARE) is obtained by taking the positive values of* p_{ij}*:*

$$\mathbf{X} = \begin{bmatrix} \sqrt{2p_{12}} & p_{12} \\ p_{12} & \sqrt{\frac{2p_{12}}{1-\frac{1}{\gamma^2}}} \end{bmatrix} \textit{ where } p_{12} = \sqrt{\frac{1}{1 - \frac{1}{\gamma^2}}} \tag{7.34}$$

■

7.3 State estimation

7.3.1 Problem to be solved

The H_∞ state estimation problem consists in asymptotically reconstruct the system's state $\underline{x}(t)$. More precisely, the state estimation problem consists in finding the matrix gain $\mathbf{L}$ of a observer (also called estimator) which estimates the full state of the system $\underline{\hat{x}}(t)$ thanks to the measurement vector $\underline{y}(t)$ while minimizing the influence of the exogenous disturbance $\underline{w}(t)$.

Let's consider the following state-space realization of the generalized plant:

$$\begin{cases} \underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}_1\underline{w}(t) + \mathbf{B}_2\underline{u}(t) \\ \underline{y}(t) = \mathbf{C}_2\underline{x}(t) + \mathbf{D}_{21}\underline{w}(t) \end{cases} \tag{7.35}$$

The observer for this system is defined as follows where gain $\mathbf{L}$ has to be set:

$$\begin{cases} \dot{\underline{\hat{x}}}(t) = \mathbf{A}\underline{\hat{x}}(t) + \mathbf{B}_2\underline{u}(t) + \mathbf{L}\left(\underline{y}(t) - \underline{\hat{y}}(t)\right) \\ \underline{\hat{y}}(t) = \mathbf{C}_2\underline{\hat{x}}(t) \end{cases} \tag{7.36}$$

As in section 6.3.1, the H_∞ state estimation problem consists in finding the gain $\mathbf{L}$ which renders $\mathbf{A} - \mathbf{L}\mathbf{C}_2$ Hurwitz while minimizing the H_∞-norm of the

State feedback	State estimation
$\mathbf{A}$	$\mathbf{A}^T$
$\mathbf{B}_1$	$\mathbf{C}_1^T$
$\mathbf{B}_2$	$\mathbf{C}_2^T$
$\mathbf{C}_1$	$\mathbf{B}_1^T$
$\mathbf{D}_{12}$	$\mathbf{D}_{21}^T$
$\mathbf{K}$	$\mathbf{L}^T$

Table 7.1: Duality principle for state feedback and state estimation matrices

transfer matrix $\mathbf{T}_{zw}(s)$ from the exogenous disturbance $\underline{w}(t)$ to the estimation error $\underline{z}(t) = \mathbf{C}_1\,\underline{e}(t)$ where the state error $\underline{e}(t)$ reads:

$$\underline{e}(t) = \underline{x}(t) - \underline{\hat{x}}(t) \tag{7.37}$$

In addition the following assumptions are made:

- $(\mathbf{A}, \mathbf{B}_1)$ has no uncontrollable modes on the imaginary axis
- $(\mathbf{A}, \mathbf{C}_2)$ is detectable
- $\mathbf{D}_{21}\mathbf{D}_{21}^T > 0$ (thus the product is invertible)
- $\mathbf{D}_{21}\mathbf{B}_1^T = \mathbf{0}$

7.3.2 Observer design

Thanks to the duality principle shown in Table 7.1, we can reuse the results of the section dealing with the H_∞ state feedback control problem. This leads to the following conclusion: the H_∞ optimal observer (or estimator) which estimates the full state of the system $\underline{\hat{x}}(t)$ thanks to the measurement vector $\underline{y}(t)$ while minimizing the influence of the exogenous disturbance $\underline{w}(t)$ has the following the realization:

$$\left\{ \begin{array}{l} \dot{\underline{\hat{x}}}(t) = \mathbf{A}\underline{\hat{x}}(t) + \mathbf{B}_2\underline{u}(t) + \mathbf{L}\left(\underline{y}(t) - \underline{\hat{y}}(t)\right) \\ \underline{\hat{y}}(t) = \mathbf{C}_2\underline{\hat{x}}(t) \end{array} \right. \tag{7.38}$$

where gain matrix $\mathbf{L}$ is obtained thanks to identification with the matrices of the dual problem and depends on the symmetric positive definite matrix $\mathbf{Y} = \mathbf{Y}^T > \mathbf{0}$:

$$\mathbf{K} = \left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1}\mathbf{B}_2^T\mathbf{X} \Rightarrow \mathbf{L}^T = \left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2\mathbf{Y} \tag{7.39}$$

That is:

$$\mathbf{L} = \mathbf{Y}\mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1} \tag{7.40}$$

Again applying the identification process with the matrices of the dual problem we conclude that positive semi-definite matrix $\mathbf{Y}$ is the solution of the following Algebraic Riccati Equation (ARE) which comes from (6.16):

$$\mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^T + \mathbf{Y}\left(\gamma^{-2}\mathbf{C}_1^T\mathbf{C}_1 - \mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2\right)\mathbf{Y} + \mathbf{B}_1\mathbf{B}_1^T = \mathbf{0} \tag{7.41}$$

The Algebraic Riccati Equation (ARE) (7.41) is associated with the following Hamiltonian matrix $\mathbf{H}_Y$:

$$\mathbf{H}_Y = \begin{bmatrix} \mathbf{A}^T & \gamma^{-2}\mathbf{C}_1^T\mathbf{C}_1 - \mathbf{C}_2^T\left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}\mathbf{C}_2 \\ -\mathbf{B}_1\mathbf{B}_1^T & -\mathbf{A} \end{bmatrix} \tag{7.42}$$

In order to get a stabilizing solution the symmetric matrix $\mathbf{Y}$ which solves the Algebraic Riccati Equation (ARE) (7.41) is chosen to be positive semi-definite. Such a choice leads to a stable closed loop matrix $\mathbf{A} - \mathbf{L}\mathbf{C}_2$, i.e. all its eigenvalues have negative real parts.

7.4 Observer-based controller

In this section, the H_∞ observer-based controller is considered. We will combine state-feedback and estimator results. The proof of the results can be found in the lecture of Professor Hannu T. Toivonen [1].

We recall the general form of the realization of the Linear Time Invariant (LTI) plant:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{7.43}$$

Assume that:

- $(\mathbf{A}, \mathbf{B}_1)$ has no uncontrollable modes on the imaginary axis
- $(\mathbf{A}, \mathbf{C}_1)$ has no unobservable modes on the imaginary axis
- $(\mathbf{A}, \mathbf{B}_2, \mathbf{C}_2)$ is stabilizable and detectable
- $\mathbf{D}_{12}^T\mathbf{D}_{12} > 0$ and $\mathbf{D}_{21}\mathbf{D}_{21}^T > 0$ (thus the product is invertible)
- $\mathbf{D}_{12}^T\mathbf{C}_1 \neq \mathbf{0}$ and $\mathbf{D}_{21}\mathbf{B}_1^T \neq \mathbf{0}$

The realization of the controller $\mathbf{C}(s)$ which achieves the bound $\|\mathbf{T}_{zw}(s)\|_\infty \leq \gamma$ may be split between a state estimator and a state feedback control law of the estimated state as follows:

$$\begin{cases} \dot{\hat{\underline{x}}}(t) = \mathbf{A}\hat{\underline{x}}(t) + \mathbf{B}_2\underline{u}(t) + \mathbf{B}_1\hat{\underline{w}}(t) + \mathbf{Z}_\infty\mathbf{L}\left(\underline{y}(t) - \hat{\underline{y}}(t)\right) \\ \underline{u}(t) = -\mathbf{K}\hat{\underline{x}}(t) \end{cases} \tag{7.44}$$

Where:

$$\begin{cases} \hat{\underline{w}}(t) = \gamma^{-2}\mathbf{B}_1^T\mathbf{X}\hat{\underline{x}}(t) \\ \hat{\underline{y}}(t) = \left(\mathbf{C}_2 + \gamma^{-2}\mathbf{D}_{21}\mathbf{B}_1^T\mathbf{X}\right)\hat{\underline{x}}(t) \end{cases} \tag{7.45}$$

In other words the realization of the controller $\mathbf{C}(s)$ is the following:

$$\begin{cases} \underline{u}(s) = \mathbf{C}(s)\underline{y}(s) \\ \mathbf{C}(s) = \mathbf{C}_K\left(s\mathbb{I} - \mathbf{A}_K\right)^{-1}\mathbf{B}_K \end{cases} \Leftrightarrow \begin{bmatrix} \dot{\hat{\underline{x}}}(t) \\ \underline{u}(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A}_K & \mathbf{B}_K \\ \hline \mathbf{C}_K & \mathbf{0} \end{array}\right] \begin{bmatrix} \hat{\underline{x}}(t) \\ \underline{y}(t) \end{bmatrix} \tag{7.46}$$

Where[2]:

$$\begin{cases} \mathbf{A}_K = \mathbf{A} - \mathbf{B}_2\mathbf{K} - \mathbf{Z}_\infty\mathbf{L}\mathbf{C}_2 + \gamma^{-2}\left(\mathbf{B}_1\mathbf{B}_1^T - \mathbf{Z}_\infty\mathbf{L}\mathbf{D}_{21}\mathbf{B}_1^T\right)\mathbf{X} \\ \mathbf{B}_K = \mathbf{Z}_\infty\mathbf{L} \\ \mathbf{C}_K = -\mathbf{K} \\ \mathbf{Z}_\infty = \left(\mathbb{I} - \gamma^{-2}\mathbf{Y}\mathbf{X}\right)^{-1} \end{cases} \tag{7.47}$$

Doyle and al.[3] have proven that the gains $\mathbf{K}$ and $\mathbf{L}$ of the stabilizing controller $\mathbf{C}(s)$ which guarantees $\|\mathbf{T}_{zw}(s)\|_\infty \leq \gamma \leq \gamma_{opt}$ have the following expression:

$$\begin{cases} \mathbf{K} = \widetilde{\mathbf{D}}_{12}\left(\mathbf{B}_2^T\mathbf{X} + \mathbf{D}_{12}^T\mathbf{C}_1\right) \\ \mathbf{L} = \left(\mathbf{Y}\mathbf{C}_2^T + \mathbf{B}_1\mathbf{D}_{21}^T\right)\widetilde{\mathbf{D}}_{21} \end{cases} \tag{7.48}$$

The symmetric and positive semi-definite matrices $\mathbf{X} = \mathbf{X}^T \geq \mathbf{0}$ and $\mathbf{Y} = \mathbf{Y}^T \geq \mathbf{0}$ are solutions of two Algebraic Riccati Equations (AREs) expressed through the following Hamiltonian matrices:

$$\begin{cases} \mathbf{H}_X = \begin{bmatrix} \mathbf{A} - \mathbf{B}_2\widetilde{\mathbf{D}}_{12}\mathbf{D}_{12}^T\mathbf{C}_1 & \gamma^{-2}\mathbf{B}_1\mathbf{B}_1^T - \mathbf{B}_2\widetilde{\mathbf{D}}_{12}\mathbf{B}_2^T \\ -\widetilde{\mathbf{C}}_1^T\widetilde{\mathbf{C}}_1 & -\left(\mathbf{A} - \mathbf{B}_2\widetilde{\mathbf{D}}_{12}\mathbf{D}_{12}^T\mathbf{C}_1\right)^T \end{bmatrix} \\ \mathbf{H}_Y = \begin{bmatrix} \left(\mathbf{A} - \mathbf{B}_1\mathbf{D}_{21}^T\widetilde{\mathbf{D}}_{21}\mathbf{C}_2\right)^T & \gamma^{-2}\mathbf{C}_1^T\mathbf{C}_1 - \mathbf{C}_2^T\widetilde{\mathbf{D}}_{21}\mathbf{C}_2 \\ -\widetilde{\mathbf{B}}_1\widetilde{\mathbf{B}}_1^T & -\left(\mathbf{A} - \mathbf{B}_1\mathbf{D}_{21}^T\widetilde{\mathbf{D}}_{21}\mathbf{C}_2\right) \end{bmatrix} \end{cases} \tag{7.49}$$

With:

$$\begin{cases} \widetilde{\mathbf{D}}_{12} = \left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1} \\ \widetilde{\mathbf{D}}_{21} = \left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1} \end{cases} \tag{7.50}$$

And:

$$\begin{cases} \widetilde{\mathbf{B}}_1 = \mathbf{B}_1\left(\mathbb{I} - \mathbf{D}_{21}^T\widetilde{\mathbf{D}}_{21}\mathbf{D}_{21}\right) \\ \widetilde{\mathbf{C}}_1 = \left(\mathbb{I} - \mathbf{D}_{12}\widetilde{\mathbf{D}}_{12}\mathbf{D}_{12}^T\right)\mathbf{C}_1 \end{cases} \tag{7.51}$$

Moreover the two positive semi-definite solutions $\mathbf{X}$ and $\mathbf{Y}$ satisfy:

$$\rho\left(\mathbf{X}\mathbf{Y}\right) \leq \gamma^2 \tag{7.52}$$

where $\rho\left(\mathbf{A}\right) := \lambda_{max}\left(\mathbf{A}\right)$ denotes the spectral radius of matrix $\mathbf{A}$.

The case where $\mathbf{D}_{22} \neq \mathbf{0}$ does not pose any problem since it is easy to form an equivalent problem with $\mathbf{D}_{22} = \mathbf{0}$ by a linear fractional transformation on the controller $\mathbf{C}(s)$ (cf. Doyle & al.[4] p. 261):

$$\mathbf{C}_D(s) = \mathbf{C}(s)\left(\mathbb{I} + \mathbf{D}_{22}\mathbf{C}(s)\right)^{-1} \tag{7.53}$$

[2] Luigi Fortuna and Mattia Frasca, Optimal and robust control: advanced topics with Matlab, CRC Press, 2012

[3] Doyle J.C., Glover K., Khargonekar P.P., Francis B. A., State-space solutions to standard H_2 and H_∞ control problems, IEEE Transactions on Automatic Control (Volume 34 , Issue 8), Aug 1989

[4] Zhou K., Doyle J., Essentials of Robust Control, Prentice Hall, 1997, ISBN 0-13-525833-2

It is worth noticing that the optimal H_∞ controller has the order of the plant plus the orders of all the weighting filters.

Now assume that $\mathbf{D}_{11} \neq \mathbf{0}$, which will complicates the formulas substantially. However Doyle & al.[4] (p. 289) have shown that symmetric and positive semi-definite matrix $\mathbf{X} = \mathbf{X}^T \geq \mathbf{0}$ is now solution of the following Algebraic Riccati Equations (AREs) expressed through the following Hamiltonian matrix:

$$\begin{aligned}\mathbf{H}_X = &\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C}_1^T\mathbf{C}_1 & -\mathbf{A}^T \end{bmatrix} \\ &- \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ -\mathbf{C}_1^T\mathbf{D}_{11} & -\mathbf{C}_1^T\mathbf{D}_{12} \end{bmatrix} \widetilde{\mathbf{R}}_X^{-1} \begin{bmatrix} \mathbf{D}_{11}^T\mathbf{C}_1 & \mathbf{B}_1^T \\ \mathbf{D}_{12}^T\mathbf{C}_1 & \mathbf{B}_2^T \end{bmatrix}\end{aligned} \tag{7.54}$$

where

$$\widetilde{\mathbf{R}}_X = \begin{bmatrix} \mathbf{D}_{11}^T \\ \mathbf{D}_{12}^T \end{bmatrix} \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{7.55}$$

Furthermore state feedback gain matrix $\mathbf{K}$ now reads as follows:

$$\mathbf{K} = \widetilde{\mathbf{R}}_X^{-1} \left(\begin{bmatrix} \mathbf{B}_1^T \\ \mathbf{B}_2^T \end{bmatrix} \mathbf{X} + \begin{bmatrix} \mathbf{D}_{11}^T \\ \mathbf{D}_{12}^T \end{bmatrix} \mathbf{C}_1 \right) \tag{7.56}$$

Similarly when $\mathbf{D}_{11} \neq \mathbf{0}$ Doyle & al.[4] have shown that symmetric and positive semi-definite matrix $\mathbf{Y} = \mathbf{Y}^T \geq \mathbf{0}$ is now solution of the following Algebraic Riccati Equations (AREs) expressed through the following Hamiltonian matrix:

$$\begin{aligned}\mathbf{H}_Y = &\begin{bmatrix} \mathbf{A}^T & \mathbf{0} \\ -\mathbf{B}_1\mathbf{B}_1^T & -\mathbf{A} \end{bmatrix} \\ &- \begin{bmatrix} \mathbf{C}_1^T & \mathbf{C}_2^T \\ -\mathbf{B}_1\mathbf{D}_{11}^T & -\mathbf{B}_1\mathbf{D}_{21}^T \end{bmatrix} \widetilde{\mathbf{R}}_Y^{-1} \begin{bmatrix} \mathbf{D}_{11}\mathbf{B}_1^T & \mathbf{C}_1 \\ \mathbf{D}_{21}\mathbf{B}_1^T & \mathbf{C}_2 \end{bmatrix}\end{aligned} \tag{7.57}$$

where

$$\widetilde{\mathbf{R}}_Y = \begin{bmatrix} \mathbf{D}_{11} \\ \mathbf{D}_{21} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{11}^T & \mathbf{D}_{21}^T \end{bmatrix} - \begin{bmatrix} \gamma^2\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{7.58}$$

Furthermore observer gain matrix $\mathbf{L}$ now reads as follows:

$$\mathbf{L} = \left(\mathbf{Y} \begin{bmatrix} \mathbf{C}_1^T & \mathbf{C}_2^T \end{bmatrix} + \mathbf{B}_1 \begin{bmatrix} \mathbf{D}_{11}^T & \mathbf{D}_{21}^T \end{bmatrix} \right) \widetilde{\mathbf{R}}_Y^{-1} \tag{7.59}$$

7.5 Practical considerations

7.5.1 Optimal bound approximation: γ-iteration algorithm

The optimal bound γ_{opt} can be approximated arbitrarily by a bisection method, also call γ-iteration, which consists to iteratively find an upper bound $\overline{\gamma}$ and a lower bound $\underline{\gamma}$ for γ_{opt}; at each step those bounds come closer and closer to the optimal bound. The algorithm is the following:

1. Set $\underline{\gamma} = 0$ and $\overline{\gamma}$ to the value of $\|\mathbf{T}_{zw}(s)\|_\infty$ obtained with the controller where $\gamma = \infty$ in the Algebraic Riccati Equation (this is the solution of the so called H_2 robust control problem). Alternatively, given the generalized *stable* plant $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, its H_∞-norm can be found by considering the following Hamiltonian matrix:
$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T \\ -\mathbf{C}^T\mathbf{C} & -\mathbf{A}^T \end{bmatrix} \tag{7.60}$$
Thus we may set a value of $\overline{\gamma} := \gamma$ and check that $\mathbf{H}$ has no eigenvalues on the imaginary axis. If it does, we have to choose a larger value of $\overline{\gamma}$ and run the test again.

2. Let $\gamma = \frac{\overline{\gamma}+\underline{\gamma}}{2}$ and check whether a controller $\mathbf{C}(s)$ exists such that $\|\mathbf{T}_{zw}(s)\|_\infty < \gamma$; if yes set $\overline{\gamma} = \gamma$, otherwise set $\underline{\gamma} = \gamma$

3. Repeat step 2 until $\overline{\gamma} - \underline{\gamma}$ is lower than a prescribed small value ϵ

7.5.2 Procedure for solving the H_∞ control problem

The procedure for solving the general H_∞ control problem is the following [2]:

1. Get the state-space realization of the generalized plant $\mathbf{G}(s)$

2. Verify that $\mathbf{D}_{12}^T\mathbf{D}_{12} > 0$ and $\mathbf{D}_{21}\mathbf{D}_{21}^T > 0$ (thus the product is invertible)

3. Set a positive value of γ large enough to solve the two Algebraic Riccati Equations (7.49)

4. Compute the solutions $\mathbf{X}$ and $\mathbf{Y}$ of the two Algebraic Riccati Equations (7.49)

5. Verify that the condition $\rho(\mathbf{XY}) < \gamma^2$ is met

6. If steps 4 and 5 are verified, it is possible to repeat the procedure by lowering γ at step 3 using for example the γ-iteration algorithm presented in section 7.5.1.

It is worth noticing that when $\gamma \to \infty$ we retrieve the results of the H_2 control problem.

7.6 Mixed sensitivity design

7.6.1 Design specifications

The mixed-sensitivity control design approach was proposed by G. Zames in 1981[5]. It is a powerful design tool for One-Degree-Of-Freedom feedback control loop which relies on shaping the closed-loop response characteristics through SISO weighting filters $\mathbf{W}_e(s)$ and $\mathbf{W}_u(s)$.

[5] Zames G., Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses, IEEE Transactions on Automatic Control, Volume 26, Issue 2, April 1981

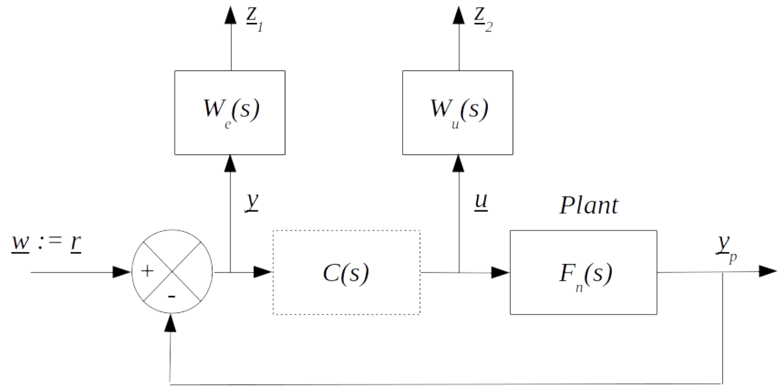

Figure 7.3: One-Degree-Of-Freedom feedback control loop with weighting filters

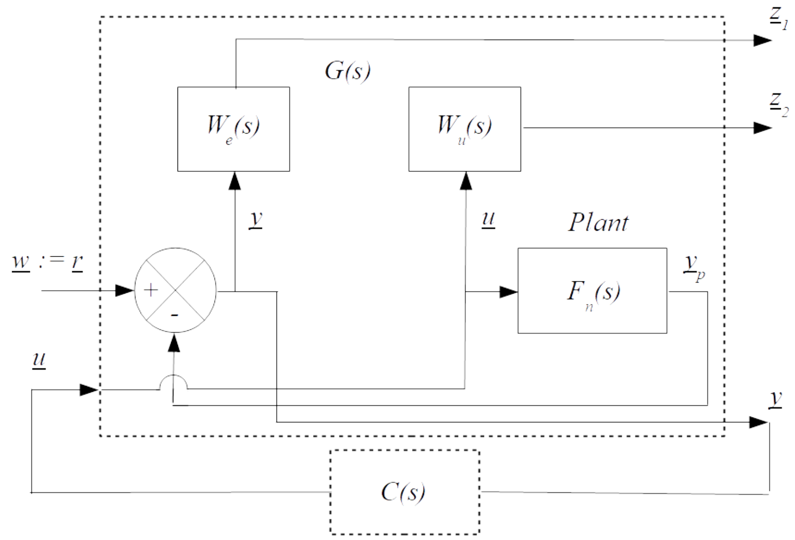

Figure 7.4: Rearrangement to generalized plant $\mathbf{G}(s)$ with controller $\mathbf{C}(s)$

More specifically let's consider Figure 7.3 and define the performance output vector $\underline{z}(s)$ as follows:

$$\underline{z}(s) := \begin{bmatrix} \underline{z}_1(s) \\ \underline{z}_2(s) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_e(s)\left(\underline{r}(s) - \underline{y}_p(s)\right) \\ \mathbf{W}_u(s)\underline{u}(s) \end{bmatrix} \tag{7.61}$$

The input vector $\underline{w}$ formed by *exogenous inputs* is defined as follows:

$$\underline{w} := \underline{r} \tag{7.62}$$

Figure 7.3 can be shaped to the general framework presented in Figure 7.1 by some rearrangement as shown in Figure 7.4

Let's compute the expression of the performance error $\underline{r}(s) - \underline{y}_p(s)$. From Figure 7.3 let's compute $\underline{y}_p(s)$ on the *nominal* plant $\mathbf{F}_n(s)$ (that is assuming no uncertainty: $\mathbf{\Delta}(s) = 0$):

$$\begin{array}{l} \mathbf{\Delta}(s) = 0 \Rightarrow \underline{y}_p(s) = \mathbf{F}_n(s)\mathbf{C}(s)\left(\underline{r}(s) - \underline{y}_p(s)\right) \\ \Leftrightarrow \underline{y}_p(s) = \left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1}\mathbf{F}_n(s)\mathbf{C}(s)\underline{r}(s) \end{array} \tag{7.63}$$

We recognize in $\left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1}\mathbf{F}_n(s)\mathbf{C}(s)$ the complementary (output) sensitivity matrix $\mathbf{T}(s)$:

$$\mathbf{T}(s) = \left(\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s)\right)^{-1}\mathbf{F}_n(s)\mathbf{C}(s) \tag{7.64}$$

Thus:

$$\underline{y}_p(s) = \mathbf{T}(s)\underline{r}(s) \tag{7.65}$$

Using relation $\mathbf{S}(s) + \mathbf{T}(s) = \mathbb{I}$ between the (output) sensitivity matrix $\mathbf{S}(s)$ and the complementary (output) sensitivity matrix $\mathbf{T}(s)$ we finally get for the first component $\underline{r}(s) - \underline{y}_p(s)$ of the performance output vector $\underline{z}$:

$$\mathbf{S}(s) + \mathbf{T}(s) = \mathbb{I} \Rightarrow \underline{r}(s) - \underline{y}_p(s) = \left(\mathbb{I} - \mathbf{T}(s)\right)\underline{r}(s) = \mathbf{S}(s)\underline{r}(s) \tag{7.66}$$

Finally the performance error $\underline{r}(s) - \underline{y}_p(s)$ reads:

$$\underline{r}(s) - \underline{y}_p(s) = \mathbf{S}(s)\,\underline{r}(s) \tag{7.67}$$

Furthermore from relations (7.65) and $\mathbf{S}(s) + \mathbf{T}(s) = \mathbb{I}$ we have:

$$\begin{array}{ll} \underline{u}(s) & = \mathbf{C}(s)\left(\underline{r}(s) - \underline{y}_p(s)\right) \\ & = \mathbf{C}(s)\left(\underline{r}(s) - \mathbf{T}(s)\underline{r}(s)\right) \\ & = \mathbf{C}(s)\mathbf{S}(s)\,\underline{r}(s) \end{array} \tag{7.68}$$

Thus relation (7.61) reads:

$$\underline{z}(s) := \begin{bmatrix} \underline{z}_1(s) \\ \underline{z}_2(s) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_e(s)\left(\underline{r}(s) - \underline{y}_p(s)\right) \\ \mathbf{W}_u(s)\underline{u}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_e(s)\mathbf{S}(s) \\ \mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s) \end{bmatrix}\underline{r}(s) \tag{7.69}$$

The preceding relation is rewritten as follows where matrix $\mathbf{T}_{zw}(s)$ is the performance transfer matrix between the performance output vector $\underline{z}$ and the input vector $\underline{w} := \underline{r}$ on the *nominal* plant $\mathbf{F}_n(s)$.

$$\underline{z}(s) = \mathbf{T}_{zw}(s)\,\underline{r}(s) \text{ where } \mathbf{T}_{zw}(s) := \begin{bmatrix} \mathbf{W}_e(s)\mathbf{S}(s) \\ \mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s) \end{bmatrix} \tag{7.70}$$

As a conclusion design specifications assuming *additive uncertainty* imply that the norm of matrices $\mathbf{S}(s)$ and $\mathbf{C}(s)\mathbf{S}(s)$ (and possibly $\mathbf{T}(s)$) should be kept *small*. Thus the basic robust control problem assuming *additive uncertainty* consists in minimizing the norm (either the H_2 or the H_∞-norm) of the following performance transfer function $\mathbf{T}_{zw}(s)$:

$$\mathbf{F}(s) = \mathbf{F}_n(s) + \mathbf{W}_u(s)\mathbf{\Delta}(s) \Rightarrow \mathbf{T}_{zw}(s) = \begin{bmatrix} \mathbf{W}_e(s)\mathbf{S}(s) \\ \mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s) \end{bmatrix} \tag{7.71}$$

In other words to ensure a robust controller design the following statements shall hold:

– For the stability of the closed-loop uncertain plant, the norm of the product $\mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s)$ is required to be *small* assuming *additive uncertainty*. We have seen that this result comes from the small gain theorem;

– For good command following and disturbance rejection, the norm of the matrix $\mathbf{W}_e(s)\mathbf{S}(s)$ is required to be *small*.

Others performance specifications such as disturbance rejection and sensor noise attenuation may be added. It can be shown that to remain insensitive to sensor noise the norm of the complementary sensitivity matrix $\mathbf{T}(s)$ shall be *small*.

The SISO weighting filters $\mathbf{W}_e(s)$ and $\mathbf{W}_u(s)$ are tuning parameters which help to shape the closed-loop response characteristics. They are selected to reflect the frequency dependence of the signals and performance specifications. They are defined as rational, stable, minimum-phase transfer functions (i.e., no poles or zeros in the right half plane). Usually $\mathbf{W}_e(s)$ is chosen as a low pass filter, enabling low frequency reference inputs to be considered for tracking, whereas $\mathbf{W}_u(s)$ is chosen as a high pass filter, enabling high frequency unmodeled dynamics to be taken into account.

A good starting point is to choose[6]

$$\begin{cases} \mathbf{W}_e(s) = \frac{s/M_1+\omega_1}{s+\epsilon_1\omega_1} \\ \mathbf{W}_u(s) = \frac{1}{M_2} \text{ or } \frac{s+\omega_2/M_2}{\epsilon_2 s+\omega_2} \end{cases} \text{ where } \begin{cases} M_1 = M_2 := M \\ \omega_1 = \omega_2 := \omega_0 \\ \epsilon_1 = \epsilon_2 := \epsilon \\ 0 < \epsilon < 1 \end{cases} \tag{7.72}$$

Parameter ϵ is the maximum allowed steady state offset, ω_0 is the desired bandwidth and M is the sensitivity peak.

$\mathbf{W}_e^{-1}(s)$ is an upper bound of the desired sensitivity function $\mathbf{S}(s)$ whereas $\mathbf{W}_u^{-1}(s)$ will limit the magnitude of the control vector $\underline{u}$.

– For the numerator-denominator perturbations problem of SISO plant that we have studied in section 4.4.3, we have seen that a sufficient condition for robust stability is the following (see (4.35)):

$$\left\| \begin{bmatrix} \mathbf{W}_n(s)\frac{\mathbf{W}_p(s)}{D_n(s)}\mathbf{C}(s)\mathbf{S}(s) \\ \mathbf{W}_d(s)\frac{\mathbf{W}_p(s)}{D_n(s)}\mathbf{S}(s) \end{bmatrix} \right\|_\infty < 1 \tag{7.73}$$

Comparing the preceding relation with (7.71), it can be seen that this problem turns to be equivalent to the basic mixed sensitivity robust control problem assuming *additive uncertainty* through the following choices of weighting filters $\mathbf{W}_e(s)$ and $\mathbf{W}_u(s)$:

$$\begin{cases} \mathbf{W}_e(s) := \mathbf{W}_d(s)\frac{\mathbf{W}_p(s)}{D_n(s)} \\ \mathbf{W}_u(s) := \mathbf{W}_n(s)\frac{\mathbf{W}_p(s)}{D_n(s)} \end{cases} \tag{7.74}$$

[6] Bérard C., Biannic J.M., Saussié D., La commande multivariable - Application au pilotage d'un avion, Dunod, 2012

Note that we have the following relation:

$$\left\| \begin{bmatrix} \mathbf{F}_1(s) \\ \mathbf{F}_2(s) \end{bmatrix} \right\|_\infty < 1 \Rightarrow \left\{ \begin{array}{l} \|\mathbf{F}_1(s)\|_\infty < 1 \\ \|\mathbf{F}_2(s)\|_\infty < 1 \end{array} \right. \tag{7.75}$$

– Finally, the basic robust control problem assuming *output multiplicative uncertainty* consists in minimizing the H_∞-norm of the following performance transfer function $\mathbf{T}_{zw}(s)$:

$$\mathbf{F}(s) = (\mathbb{I} + \mathbf{\Delta}(s)\mathbf{W}_u(s))\, \mathbf{F}_n(s) \Rightarrow \mathbf{T}_{zw}(s) = \begin{bmatrix} \mathbf{W}_e(s)\mathbf{S}(s) \\ \mathbf{W}_u(s)\mathbf{T}(s) \end{bmatrix} \tag{7.76}$$

Where $\mathbf{T}(s)$ is the complementary sensitivity function:

$$\mathbf{T}(s) = \mathbf{S}(s)\mathbf{F}_n(s)\mathbf{C}(s) \tag{7.77}$$

7.6.2 Frequency response shaping

The mixed sensitivity problem may be used for performance design by shaping the sensitivity function $\mathbf{S}(s)$ and the transfer function $\mathbf{C}(s)\mathbf{S}(s)$.

Following Kwakernaak[7], first it is worth noticing that the square of the H_∞-norm of transfer matrix $\mathbf{T}_{zw}(s)$ is given as follows:

$$\|\mathbf{T}_{zw}(s)\|_\infty^2 = sup_{\omega\in\mathbb{R}} \left(|\mathbf{W}_e(j\omega)\mathbf{S}(j\omega)|^2 + |\mathbf{W}_u(j\omega)\mathbf{C}(j\omega)\mathbf{S}(j\omega)|^2 \right) \tag{7.78}$$

The solution of the mixed sensitivity problem has the property that the frequency dependent function $|\mathbf{W}_e(j\omega)\mathbf{S}(j\omega)|^2 + |\mathbf{W}_u(j\omega)\mathbf{C}(j\omega)\mathbf{S}(j\omega)|^2$ is actually a constant when optimal control is achieved[7]:

$$|\mathbf{W}_e(j\omega)\mathbf{S}(j\omega)|^2 + |\mathbf{W}_u(j\omega)\mathbf{C}(j\omega)\mathbf{S}(j\omega)|^2 = \gamma^2 \quad \forall\, \omega \in \mathbb{R} \tag{7.79}$$

This is known as the equalizing property

Thus for the optimal controller $\mathbf{C}(s)$ we have:

$$\left\{ \begin{array}{l} |\mathbf{W}_e(j\omega)\mathbf{S}(j\omega)|^2 \leq \gamma^2 \quad \forall\, \omega \in \mathbb{R} \\ |\mathbf{W}_u(j\omega)\mathbf{C}(j\omega)\mathbf{S}(j\omega)|^2 \leq \gamma^2 \quad \forall\, \omega \in \mathbb{R} \end{array} \right. \tag{7.80}$$

Hence:

$$\left\{ \begin{array}{l} |\mathbf{S}(j\omega)| \leq \frac{\gamma}{|\mathbf{W}_e(j\omega)|} \quad \forall\, \omega \in \mathbb{R} \\ |\mathbf{C}(j\omega)\mathbf{S}(j\omega)| \leq \frac{\gamma}{|\mathbf{W}_u(j\omega)|} \quad \forall\, \omega \in \mathbb{R} \end{array} \right. \tag{7.81}$$

By choosing the weighting filters $\mathbf{W}_e(s)$ and $\mathbf{W}_e(s)$ suitably, the sensitivity function $\mathbf{S}(s)$ and the transfer function $\mathbf{C}(s)\mathbf{S}(s)$ may be made small in appropriate frequency regions.

If the weighting filters are appropriately chosen, in particular, with $\mathbf{W}_e(j\omega)$ large at low frequencies and $\mathbf{W}_u(j\omega)$ large at high frequencies, often the solution

[7]Huibert Kwakernaak, Minimax Frequency Domain Performance and Robustness Optimization of Linear Feedback Systems, IEEE Transactions on Automatic Control AC-30(10), November 1985, Pages 994-1004

of the mixed sensitivity problem has the property that the first term of the criterion dominates at low frequencies and the second at high frequencies:

$$\underbrace{|\mathbf{W}_e(j\omega)\mathbf{S}(j\omega)|^2}_{dominates\, at\, low\, frequencies} + \underbrace{|\mathbf{W}_u(j\omega)\mathbf{C}(j\omega)\mathbf{S}(j\omega)|^2}_{dominates\, at\, high\, frequencies} = \gamma^2 \quad \forall\, \omega \in \mathbb{R} \tag{7.82}$$

As a result:

$$\begin{cases} |\mathbf{S}(j\omega)| \approx \frac{\gamma}{|\mathbf{W}_e(j\omega)|} & \text{for } \omega \text{ small} \\ |\mathbf{C}(j\omega)\mathbf{S}(j\omega)| \approx \frac{\gamma}{|\mathbf{W}_u(j\omega)|} & \text{for } \omega \text{ large} \end{cases} \tag{7.83}$$

This allows quite effective control over the shape of sensitivity function $\mathbf{S}(s)$ and the transfer function $\mathbf{C}(s)\mathbf{S}(s)$ and hence over the performance of the closed-loop.

7.6.3 Generalized plant

Let's consider the one degree of freedom control loop in Figure 7.3. We wish to design a controller that minimizes the H_∞-norm of the following transfer matrix:

$$\mathbf{T}_{zw}(s) = \begin{bmatrix} \mathbf{W}_e(s)\mathbf{S}(s) \\ \mathbf{W}_u s)\mathbf{C}(s)\mathbf{S}(s) \end{bmatrix} \tag{7.84}$$

Where $\mathbf{S}(s)$ is the sensitivity function:

$$\mathbf{S}(s) = (\mathbb{I} + \mathbf{F}_n(s)\mathbf{C}(s))^{-1} \tag{7.85}$$

We will elaborate the state space representation of the generalized open-loop plant $\mathbf{G}(s)$ in Figure 7.3 to match with the formalism of Figure 7.1:

$$\begin{array}{rl} \begin{bmatrix} \underline{z}(s) \\ \underline{y}(s) \end{bmatrix} & = \mathbf{G}(s) \begin{bmatrix} \underline{w}(s) \\ \underline{u}(s) \end{bmatrix} \\ \Leftrightarrow \begin{bmatrix} z_1(s) \\ z_2(s) \\ y(s) \end{bmatrix} & = \begin{bmatrix} \mathbf{W}_e(s) & -\mathbf{W}_e(s)\mathbf{F}_n(s) \\ \mathbf{0} & \mathbf{W}_u(s) \\ \mathbb{I} & -\mathbf{F}_n(s) \end{bmatrix} \begin{bmatrix} r(s) \\ u(s) \end{bmatrix} \end{array} \tag{7.86}$$

The realization of the open-loop plant transfer function $\mathbf{G}(s)$ can be obtained as follows:

– Firstly the state space representation of the plant is the following:

$$\begin{cases} \underline{\dot{x}}_p(t) = \mathbf{A}_p \underline{x}_p(t) + \mathbf{B}_p \underline{u}(t) \\ y_p(t) = \mathbf{C}_p \underline{x}_p(t) \end{cases} \tag{7.87}$$

– Secondly, using the relation $\underline{y}(t) = \underline{r}(t) - \mathbf{C}_p \underline{x}_p(t)$, the state space representation of the error filter $\mathbf{W}_e(s)$ is the following:

$$\begin{cases} \underline{\dot{x}}_e(t) = \mathbf{A}_e \underline{x}_e(t) + \mathbf{B}_e \left(\underline{r}(t) - \mathbf{C}_p \underline{x}_p(t)\right) \\ z_1(t) = \mathbf{C}_e \underline{x}_e(t) + \mathbf{D}_e \left(\underline{r}(t) - \mathbf{C}_p \underline{x}_p(t)\right) \end{cases} \tag{7.88}$$

– Thirdly the state space representation of the controller filter $\mathbf{W}_u(s)$ is the following:

$$\begin{cases} \dot{\underline{x}}_u(t) = \mathbf{A}_u \underline{x}_u(t) + \mathbf{B}_u \underline{u}(t) \\ z_2(t) = \mathbf{C}_u \underline{x}_u(t) + \mathbf{D}_u \underline{u}(t) \end{cases} \tag{7.89}$$

Thus the state vector of the generalized plant is the following:

$$\underline{x}(t) = \begin{bmatrix} \underline{x}_p(t) \\ \underline{x}_e(t) \\ \underline{x}_u(t) \end{bmatrix} \tag{7.90}$$

The exogenous input vector $\underline{w}(t)$ is defined as follows:

$$\underline{w}(t) := \underline{r}(t) \tag{7.91}$$

The vector of output $\underline{y}(t)$ available for the controller is the following:

$$\begin{array}{ll} \underline{y}(t) & = \underline{r}(t) - \mathbf{C}_p \underline{x}_p(t) \\ & = \underline{r}(t) + \begin{bmatrix} -\mathbf{C}_p & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x}_p(t) \\ \underline{x}_e(t) \\ \underline{x}_u(t) \end{bmatrix} \\ & = \underline{w}(t) + \begin{bmatrix} -\mathbf{C}_p & \mathbf{0} & \mathbf{0} \end{bmatrix} \underline{x}(t) \end{array} \tag{7.92}$$

The performance output vector $\underline{z}(t)$ is the following:

$$\begin{array}{ll} \underline{z}(t) & = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \\ & = \begin{bmatrix} -\mathbf{D}_e \mathbf{C}_p & \mathbf{C}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_u \end{bmatrix} \begin{bmatrix} \underline{x}_p(t) \\ \underline{x}_e(t) \\ \underline{x}_u(t) \end{bmatrix} + \begin{bmatrix} \mathbf{D}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_u \end{bmatrix} \begin{bmatrix} \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \\ & = \begin{bmatrix} -\mathbf{D}_e \mathbf{C}_p & \mathbf{C}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_u \end{bmatrix} \underline{x}(t) + \begin{bmatrix} \mathbf{D}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_u \end{bmatrix} \begin{bmatrix} \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \end{array} \tag{7.93}$$

We finally match with the formalism of Equation (5.2):

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array} \right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{7.94}$$

Where the matrices used in the first row have the following expression:

$$\begin{cases} \mathbf{A} = \begin{bmatrix} \mathbf{A}_p & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}_e \mathbf{C}_p & \mathbf{A}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_u \end{bmatrix} \\ \mathbf{B}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_e \\ \mathbf{0} \end{bmatrix} \\ \mathbf{B}_2 = \begin{bmatrix} \mathbf{B}_p \\ \mathbf{0} \\ \mathbf{B}_u \end{bmatrix} \end{cases} \tag{7.95}$$

The matrices used in the second row have the following expression:

$$\begin{cases} \mathbf{C}_1 = \begin{bmatrix} -\mathbf{D}_e\mathbf{C}_p & \mathbf{C}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_u \end{bmatrix} \\ \mathbf{D}_{11} = \begin{bmatrix} \mathbf{D}_e \\ \mathbf{0} \end{bmatrix} \\ \mathbf{D}_{12} = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_u \end{bmatrix} \end{cases} \tag{7.96}$$

And the matrices used in the third line have the following expression:

$$\begin{cases} \mathbf{C}_2 = \begin{bmatrix} -\mathbf{C}_p & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \mathbf{D}_{21} = \mathbb{I} \end{cases} \tag{7.97}$$

Example 7.2. *Let's consider the one degree of freedom control loop in Figure 7.3. The transfer functions of the plant $\mathbf{F}_n(s)$ and weighting filters $\mathbf{W}_e(s)$ and $\mathbf{W}_u(s)$ are the following:*

$$\begin{cases} \mathbf{F}_n(s) = \frac{s-2}{(s+5)(s+1)} = \frac{s-2}{s^2+6s+5} \\ \mathbf{W}_e(s) = \frac{1}{s+0.01} \\ \mathbf{W}_u(s) = \frac{s+2}{s+10} = 1 - \frac{8}{s+10} \end{cases} \tag{7.98}$$

We are looking for the time domain representation of the generalized open-loop plant $\mathbf{G}(s)$ to match with the formalism of Equation (5.2):

– *Firstly the state space representation of the plant is the following:*

$$\begin{cases} \underline{\dot{x}}_p(t) = \mathbf{A}_p\underline{x}_p(t) + \mathbf{B}_p\underline{u}(t) \\ y_p(t) = \mathbf{C}_p\underline{x}_p(t) \end{cases} \tag{7.99}$$

Where:

$$\begin{cases} \mathbf{A}_p = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \\ \mathbf{B}_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathbf{C}_p = \begin{bmatrix} -2 & 1 \end{bmatrix} \end{cases} \tag{7.100}$$

– *Secondly the state space representation of the error filter $\mathbf{W}_e(s)$ is the following:*

$$\begin{cases} \underline{\dot{x}}_e(t) = \mathbf{A}_e\underline{x}_e(t) + \mathbf{B}_e\left(\underline{r}(t) - \mathbf{C}_p\underline{x}_p(t)\right) \\ z_1(t) = \mathbf{C}_e\underline{x}_e(t) \end{cases} \tag{7.101}$$

Where:

$$\begin{cases} \mathbf{A}_e = -0.01 \\ \mathbf{B}_e = 1 \\ \mathbf{C}_e = 1 \\ \mathbf{D}_e = 0 \end{cases} \tag{7.102}$$

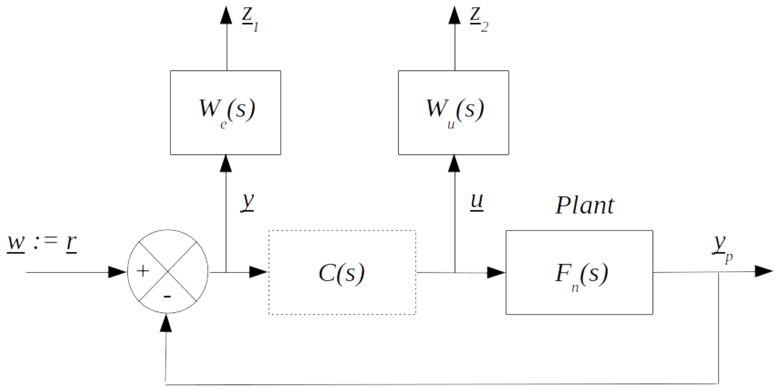

Figure 7.5: One-Degree-Of-Freedom feedback control loop with weighting filters

– *Thirdly the state space representation of the controller filter* $\mathbf{W}_u(s)$ *is the following:*

$$\begin{cases} \underline{\dot{x}}_u(t) = \mathbf{A}_u \underline{x}_u(t) + \mathbf{B}_u \underline{u}(t) \\ z_2(t) = \mathbf{C}_u \underline{x}_u(t) + \mathbf{D}_u \underline{u}(t) \end{cases} \tag{7.103}$$

Where:

$$\begin{cases} \mathbf{A}_u = -10 \\ \mathbf{B}_u = 1 \\ \mathbf{C}_u = -8 \\ \mathbf{D}_u = 1 \end{cases} \tag{7.104}$$

■

7.7 Worked example

We present hereafter the illustrative example proposed by Herzog & Keller[8]. We consider transfer function $F_a(s)$ to be controlled, where uncertain parameter θ varies between 0.02 and 0.05:

$$F_a(s) = \frac{5}{(s+1)(s\theta+1)^2} \text{ where } 0.02 \leq \theta \leq 0.05 \tag{7.105}$$

The robust control design problem consists in designing the controller $\mathbf{C}(s)$ which achieves the following performance criteria:

– Stability of the closed-loop plant, despite the uncertainty in transfer function (7.105) ;

– Good command following and disturbance rejection.

For this worked example, we will design the controller $\mathbf{C}(s)$ through the mixed-sensitivity H_∞ robust control framework. We will use the standard form shown in Figure 7.5, including the weighting functions $W_e(s)$ and $W_u(s)$. The observed-based controller to be designed is denoted $\mathbf{C}(s)$.

[8]Raoul Herzog, Jürg Keller, An Overview on Robust Control, MSE, 2010

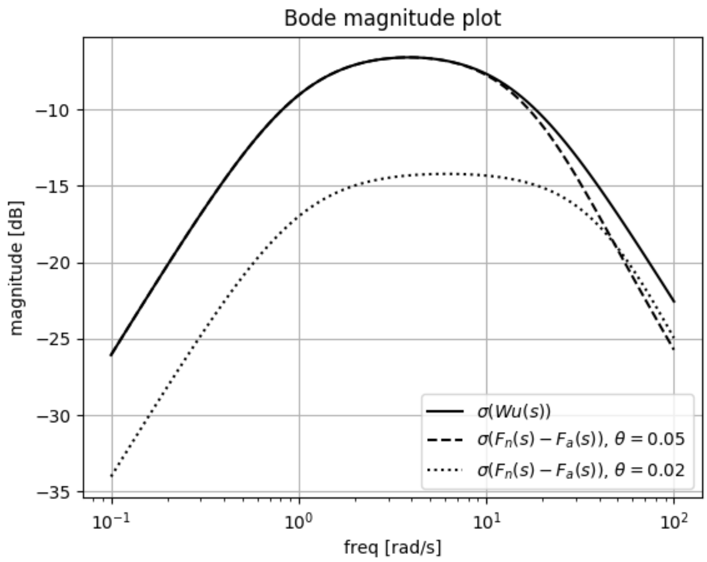

Figure 7.6: Uncertainty bounds

7.7.1 Modelling of uncertainty

This uncertain transfer function $F_a(s)$ is split as the sum of a nominal transfer function, $F_n(s)$, and an additive uncertainty whose bound is denoted $W_u(s)$:

$$F_a(s) = F_n(s) + \Delta(s) W_u(s) \text{ where } \begin{cases} F_n(s) = \frac{5}{s+1} \\ W_u(s) = \frac{0.0075\, s\,(s+1000)}{(s+1)(s+15)} \end{cases} \tag{7.106}$$

For technical reason, it is worth noticing that the degree of the denominator of $W_u(s)$ shall be the same than the degree of its numerator.

Figure 7.6 shows the Bode magnitude plot of $F_a(s) - F_n(s)$, for $\theta = 0.02$ and 0.05, as well as the Bode magnitude plot of $W_u(s)$. It appears clearly that additive uncertainty $W_u(s)$ is an upper bound for the uncertainty.

A state-space realization of $\mathbf{W}_u(s)$ is thus:

$$\begin{array}{ll} \mathbf{W}_u(s) & = 0.0075 - \frac{0.0075(984s-15)}{s^2+16s+15} \\ & = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -15 & -16 & 1 \\ \hline -0.1125 & 7.38 & 0.0075 \end{array} \right) := \left(\begin{array}{c|c} \mathbf{A}_u & \mathbf{B}_u \\ \hline \mathbf{C}_u & \mathbf{D}_u \end{array} \right) \end{array} \tag{7.107}$$

7.7.2 Selection of weighting filter $\mathbf{W}_e(s)$

For good command following and disturbance rejection, the norm of the matrix $\mathbf{W}_e(s)\mathbf{S}(s)$ is required to be *small*. $\mathbf{W}_e^{-1}(s)$ is an upper bound of the desired sensitivity function $\mathbf{S}(s)$. Nevertheless the sensitivity function has to meet the Bode integral theorem presented in section 2.3. An unsuitable selection of $\mathbf{S}(s)$, and thus of $\mathbf{W}_e(s)$, may lead to an unsolvable H_∞ controller design problem. Zero steady-state error is achieved if $\mathbf{W}_e(s)$ has a pole at $\omega = 0$. Selecting the bandwidth of the closed-loop to be about 1 rad/sec (which is approximatively the bandwidth of the uncertainty filter $\mathbf{W}_u(s)$), we may choose:

$$\mathbf{W}_e(s) = \frac{s+1}{s} \tag{7.108}$$

Weighting filter defined in (7.108) has the drawback that there is no limit to possible resonance peaks of the sensitivity function. With an additional pole at 0.001, this drawback is eliminated:

$$\mathbf{W}_e(s) = \frac{s+1}{s+0.001} \tag{7.109}$$

A state-space realization of $\mathbf{W}_e(s)$ is thus:

$$\mathbf{W}_e(s) = 1 + \frac{0.999}{s+0.001} = \left(\begin{array}{c|c} -0.001 & 1 \\ \hline 0.999 & 1 \end{array}\right) := \left(\begin{array}{c|c} \mathbf{A}_e & \mathbf{B}_e \\ \hline \mathbf{C}_e & \mathbf{D}_e \end{array}\right) \tag{7.110}$$

7.7.3 Generalized plant

We will elaborate the state-space realization of the generalized open-loop plant $\mathbf{G}(s)$ in Figure 7.5 (that is the plant without controller $\mathbf{C}(s)$) to match with the following formalism:

$$\begin{bmatrix} \underline{\dot{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{7.111}$$

– Firstly the state-space realization of the plant is the following:

$$\begin{cases} \underline{\dot{x}}_p(t) = \mathbf{A}_p \underline{x}_p(t) + \mathbf{B}_p u(t) \\ y_p(t) = \mathbf{C}_p \underline{x}_p(t) \end{cases} \tag{7.112}$$

A state-space realization of $\mathbf{F}_n(s)$ is the following:

$$\mathbf{F}_n(s) = \frac{5}{s+1} = \left(\begin{array}{c|c} -1 & 1 \\ \hline 5 & 0 \end{array}\right) := \left(\begin{array}{c|c} \mathbf{A}_p & \mathbf{B}_p \\ \hline \mathbf{C}_p & \mathbf{0} \end{array}\right) \tag{7.113}$$

– Secondly the state-space realization of the error filter $\mathbf{W}_e(s)$ is the following:

$$\begin{cases} \underline{\dot{x}}_e(t) = \mathbf{A}_e \underline{x}_e(t) + \mathbf{B}_e \left(r(t) - \mathbf{C}_p \underline{x}_p(t)\right) \\ z_1(t) = \mathbf{C}_e \underline{x}_e(t) + \mathbf{D}_e \left(\underline{r}(t) - \mathbf{C}_p \underline{x}_p(t)\right) \end{cases} \tag{7.114}$$

– Thirdly the state-space realization of the controller filter $\mathbf{W}_u(s)$ is the following:

$$\begin{cases} \underline{\dot{x}}_u(t) = \mathbf{A}_u \underline{x}_u(t) + \mathbf{B}_u u(t) \\ z_2(t) = \mathbf{C}_u \underline{x}_u(t) + \mathbf{D}_u u(t) \end{cases} \tag{7.115}$$

As shown in section 7.6.3, and setting $\underline{w} := \underline{r}$, we finally match with the formalism of Equation (7.111) assuming $\underline{x} = \begin{bmatrix} \underline{x}_p^T & \underline{x}_e^T & \underline{x}_u^T \end{bmatrix}^T$ by making the following settings:

$$\underline{x} = \begin{bmatrix} \underline{x}_p \\ \underline{x}_e \\ \underline{x}_u \end{bmatrix} \Rightarrow \begin{cases} \mathbf{A} = \begin{bmatrix} \mathbf{A}_p & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}_e\mathbf{C}_p & \mathbf{A}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_u \end{bmatrix} = \left[\begin{array}{c|c|cc} -1 & 0 & 0 & 0 \\ \hline -5 & -0.001 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -15 & -16 \end{array}\right] \\ \mathbf{B}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_e \\ \mathbf{0} \end{bmatrix} = \left[\begin{array}{c} 0 \\ \hline 1 \\ \hline 0 \\ 0 \end{array}\right] \\ \mathbf{B}_2 = \begin{bmatrix} \mathbf{B}_p \\ \mathbf{0} \\ \mathbf{B}_u \end{bmatrix} = \left[\begin{array}{c} 1 \\ \hline 0 \\ \hline 0 \\ 1 \end{array}\right] \end{cases} \tag{7.116}$$

The matrices used in the second row have the following expression:

$$\underline{z} = \begin{bmatrix} \underline{z}_1 \\ \underline{z}_2 \end{bmatrix} \Rightarrow \begin{cases} \mathbf{C}_1 = \begin{bmatrix} -\mathbf{D}_e\mathbf{C}_p & \mathbf{C}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_u \end{bmatrix} = \left[\begin{array}{c|c|cc} -5 & 0.999 & 0 & 0 \\ \hline 0 & 0 & -0.1125 & 7.38 \end{array}\right] \\ \mathbf{D}_{11} = \begin{bmatrix} \mathbf{D}_e \\ \mathbf{0} \end{bmatrix} = \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right] \\ \mathbf{D}_{12} = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_u \end{bmatrix} = \left[\begin{array}{c} 0 \\ \hline 0.0075 \end{array}\right] \end{cases} \tag{7.117}$$

And the matrices used in the third line have the following expression:

$$\begin{cases} \mathbf{C}_2 = \begin{bmatrix} -\mathbf{C}_p & \mathbf{0} & \mathbf{0} \end{bmatrix} = \left[\begin{array}{c|c|cc} -5 & 0 & 0 & 0 \end{array}\right] \\ \mathbf{D}_{21} = 1 \end{cases} \tag{7.118}$$

7.7.4 Observer-based controller

The H_∞ controller $\mathbf{C}(s)$ can be obtained by using either the functions *gamitg* and *ccontrg* (or *hinf*) with Scilab or the function *hinfsyn* with Matlab or with the Python Control Systems Library.

Using Python Control Systems Library we obtain the following results:

$$\begin{cases} \gamma \approx 1 \\ \mathbf{C}(s) = \frac{8.981e07s^3+1.524e09s^2+2.738e09s+1.304e09}{s^4+4.178e08s^3+7.16e09s^2+6.749e09s+6.742e06} \end{cases} \tag{7.119}$$

It is worth noticing that the controller is strictly proper with the same order than the generalized plant. Nevertheless when the two positive semi-definite solutions $\mathbf{X}$ and $\mathbf{Y}$ satisfy $\rho(\mathbf{XY}) = \gamma^2$ then the order of the controller drops. This order reduction is due to the simplification of poles at infinity[9]. In this example, factorizing by the term $8.981e07s^3$ the numerator and the denominator

[9] Robustesse et commande optimale, Daniel Alazard, Pierre Apkarian, Christelle Cumer, Gilles Ferreres, Michel Gauvrit, Cepadues, 1999, page 106

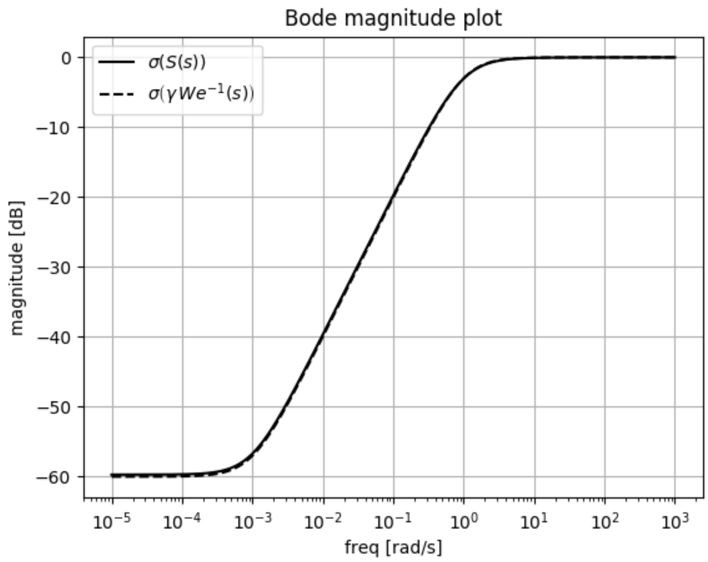

Figure 7.7: Bode magnitude plot of sensitivity function $\mathbf{S}(s)$

of $\mathbf{C}(s)$ and omitting the very small term in s^4 in the denominator leads to the following reduced order controller $\mathbf{C}_r(s)$:

$$\mathbf{C}_r(s) = \frac{s^3 + 17.14s^2 + 16.15s + 0.01614}{4.653s^3 + 79.73s^2 + 75.15s + 0.07507} \tag{7.120}$$

In Figure 7.7 it can be seen that sensitivity function $\mathbf{S}(s)$ and $\mathbf{W}_e^{-1}(s)$ match pretty well for *all* frequencies. Usually $\mathbf{S}(s)$ and $\mathbf{W}_e^{-1}(s)$ match pretty well only for *low* frequencies.

In Figure 7.8 it can be seen that $\|\mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s)\|_\infty \approx 0.1\,(-20\,dB) < 1$. From the small gain theorem, we conclude that the closed-loop system remains stable despite the uncertainty (we recall that this is a conservative stability condition).

In Figure 7.9 the step response of the closed-loop system is shown. It can bee seen that the settling time is approximately $3\,sec$, that is three times the inverse of the bandwidth of the sensitivity function.

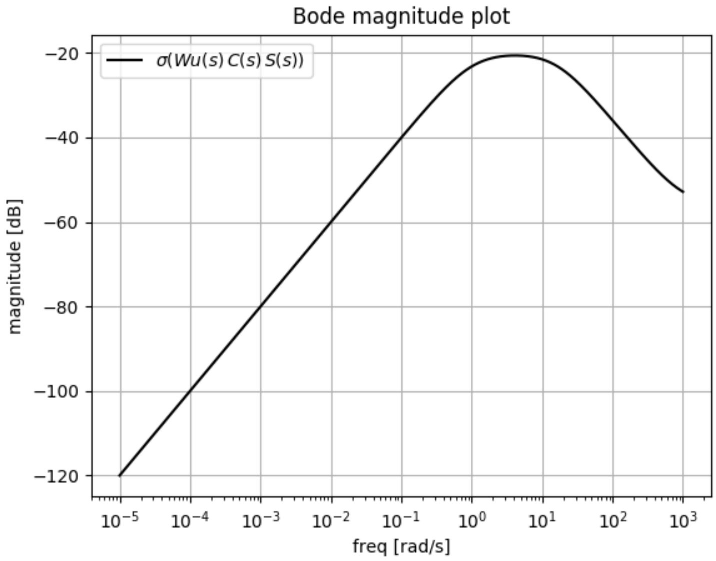

Figure 7.8: Bode magnitude plot of $\mathbf{W}_u(s)\mathbf{C}(s)\mathbf{S}(s)$

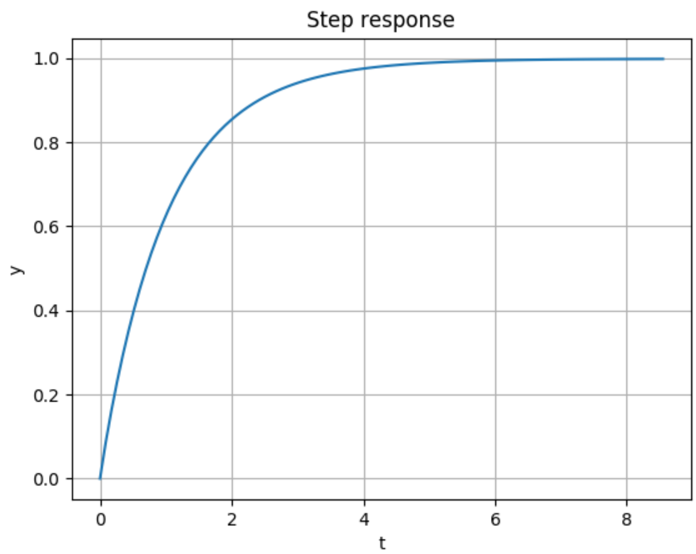

Figure 7.9: Step response of the closed-loop system

Chapter 8

Introduction to Linear Matrix Inequalities (LMI)

8.1 Definitions

8.1.1 Positive definite and positive semi-definite matrix

A positive definite matrix $\mathbf{M}$ is denoted $\mathbf{M} > \mathbf{0}$ where $\mathbf{0}$ denotes here the zero matrix. We remind that a real $n \times n$ symmetric matrix $\mathbf{M} = \mathbf{M}^T$ is called positive definite if and only if we have either:

– $\underline{x}^T \mathbf{M} \underline{x} > 0$ for all $\underline{x} \neq 0$;

– All eigenvalues of $\mathbf{M}$ are strictly positive;

– All of the leading principal minors are strictly positive (the leading principal minor of order k is the minor of order k obtained by deleting the last $n - k$ rows and columns);

– $\mathbf{M}$ can be written as $\mathbf{M}_s^T \mathbf{M}_s$ where matrix $\mathbf{M}_s$ is square and invertible.

Similarly a semi-definite positive matrix $\mathbf{M}$ is denoted $\mathbf{M} \geq 0$ where 0 denotes here the zero matrix. We remind that a $n \times n$ real symmetric matrix $\mathbf{M} = \mathbf{M}^T$ is called positive semi-definite if and only if we have either:

– $\underline{x}^T \mathbf{M} \underline{x} \geq 0$ for all $\underline{x} \neq 0$;

– All eigenvalues of $\mathbf{M}$ are non-negative;

– All of the principal (not only leading) minors are non-negative (the principal minor of order k is the minor of order k obtained by deleting $n - k$ rows and the $n - k$ columns with the same position than the rows. For instance, in a principal minor where you have deleted rows 1 and 3, you should also delete columns 1 and 3);

– $\mathbf{M}$ can be written as $\mathbf{M}_s^T \mathbf{M}_s$ where matrix $\mathbf{M}_s$ is full row rank.

Furthermore a real symmetric matrix $\mathbf{M}$ is called negative (semi-)definite if $-\mathbf{M}$ is positive (semi-)definite.

Example 8.1. *Check that* $\mathbf{M}_1 = \mathbf{M}_1^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ *is not positive definite and that* $\mathbf{M}_2 = \mathbf{M}_2^T = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ *is positive definite.* ■

8.1.2 Linear Matrix Inequalities (LMI)

A Linear Matrix Inequality (LMI) is a matrix inequality of the form:

$$\mathbf{M}(\underline{p}) = \mathbf{M}_0 + \sum_{i=1}^{n} p_i \mathbf{M}_i > 0 \tag{8.1}$$

where $\mathbf{M}_i = \mathbf{M}_i^T$, $i = 1, \cdots, n$, are given symmetric matrices, $\underline{p} = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}^T$ is a vector of real scalar variables. Matrix inequality $\mathbf{M}(\underline{p}) > 0$ means that symmetric matrix $\mathbf{M}(\underline{p}) = \mathbf{M}(\underline{p})^T$ is positive definite.

It can be shown that the set of all solutions $\underline{p}$ of (8.1) is a convex set.

8.1.3 Schur complement lemma

Schur complement lemma converts convex nonlinear inequalities to an LMI[1]. The convex nonlinear inequalities are the sets of the following equivalent relations where $\mathbf{M}_{11} = \mathbf{M}_{11}^T$ and $\mathbf{M}_{22} = \mathbf{M}_{22}^T$:

$$\begin{cases} \mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{M}_{22}^{-1}\mathbf{M}_{12}^T > 0 \\ \mathbf{M}_{22} > 0 \end{cases} \Leftrightarrow \begin{cases} \mathbf{M}_{22} - \mathbf{M}_{12}^T\mathbf{M}_{11}^{-1}\mathbf{M}_{12} > 0 \\ \mathbf{M}_{11} > 0 \end{cases} \tag{8.2}$$

The Schur complement lemma converts the previous sets of inequalities into the following equivalent linear matrix inequality (LMI):

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{12}^T & \mathbf{M}_{22} \end{bmatrix} > 0 \tag{8.3}$$

The following matrix inversion lemma may also be useful:

$$(\mathbf{W} + \mathbf{XYZ})^{-1} = \mathbf{W}^{-1} - \mathbf{W}^{-1}\mathbf{X}\left(\mathbf{ZW}^{-1}\mathbf{X} + \mathbf{Y}^{-1}\right)^{-1}\mathbf{ZW}^{-1} \tag{8.4}$$

8.2 Stability analysis

8.2.1 Lyapunov's second method

Let's consider the following homogeneous nonlinear system with state vector $\underline{x}$:

$$\dot{\underline{x}}(t) = f(\underline{x}(t)) \tag{8.5}$$

We said that $\underline{x}_e$ is an equilibrium point if the following relation holds:

$$\underline{0} = f(\underline{x}_e) \tag{8.6}$$

[1]VanAntwerp J., Braatz R., A tutorial on linear and bilinear matrix inequalities, Journal of Process Control, vol. 10, pp. 363-385, 2000

Furthermore the equilibrium point $\underline{x}_e$ is said to be asymptotically stable if the response of the system to any arbitrary initial condition $\underline{x}(0)$ tends asymptotically towards the equilibrium point $\underline{x}_e$.

According to Lyapunov's second method for stability the equilibrium point $\underline{x}_e$ is asymptotically stable if there exists a scalar function $V(\underline{x}) : \mathbb{R}^n \to \mathbb{R}$ such that the three inequalities hold:

$$\begin{cases} V(\underline{x}_e) = 0 \\ V(\underline{x}) > 0 \ \forall \, \underline{x} \neq \underline{x}_e \\ \dot{V}(\underline{x}) < 0 \ \forall \, \underline{x} \neq \underline{x}_e \end{cases} \tag{8.7}$$

In addition if the scalar function $V(\underline{x})$ is radially unbounded, meaning that $\lim_{\|x\| \to \infty} V(\underline{x}) = \infty$, then the equilibrium point $\underline{x}_e$ is said to be globally asymptotically stable.

Moreover, Bath[2] has shown that if there exists real numbers $c > 0$ and $\alpha \in [0, 1[$ such that:

$$\dot{V}(\underline{x}) + c\,(V(\underline{x}))^{\alpha} \leq 0 \ \forall \, \underline{x} \neq \underline{x}_e \tag{8.8}$$

then the origin is a finite-time-stable equilibrium point and the settling time function $t_s(\underline{x})$ is bounded as follows:

$$t_s(\underline{x}) \leq \frac{(V(\underline{x}))^{1-\alpha}}{c\,(1-\alpha)} \tag{8.9}$$

Homogeneous linear systems are defined by the following relation where $\mathbf{A}_{cl}$ stands for the closed loop state matrix:

$$f(\underline{x}(t)) = \mathbf{A}_{cl}\underline{x}(t) \Rightarrow \underline{\dot{x}}(t) = \mathbf{A}_{cl}\underline{x}(t) \tag{8.10}$$

For linear system there is a unique equilibrium point (assuming that state matrix $\mathbf{A}_{cl}$ is non singular) which is $\underline{x}_e = \underline{0}$. An usual choice for the Lyapunov's function $V(\underline{x})$ is the following where $\mathbf{P} = \mathbf{P}^T > 0$ is a symmetric positive definite matrix:

$$V(\underline{x}) = \underline{x}^T \mathbf{P} \underline{x} \ \text{ where } \mathbf{P} = \mathbf{P}^T > 0 \tag{8.11}$$

It is clear that $V(\underline{x})$ is radially unbounded and that the first two conditions of (8.7) are fulfilled. If $V(\underline{x})$ is such $\dot{V}(\underline{x}) \leq -\underline{x}^T \mathbf{Q} \underline{x} < 0 \ \forall \, \underline{x} \neq \underline{0}$ where $\mathbf{Q} = \mathbf{Q}^T > 0$, then equilibrium point $\underline{x}_e$ is globally asymptotically stable ($\underline{x}_e$ is said to be globally exponentially stable) and $\|\underline{x}(t)\|$ satisfies:

$$\begin{cases} V(\underline{x}) = \underline{x}^T \mathbf{P} \underline{x} \ \text{ where } \mathbf{P} = \mathbf{P}^T > 0 \\ \dot{V}(\underline{x}) \leq -\underline{x}^T \mathbf{Q} \underline{x} < 0 \ \forall \, \underline{x} \neq \underline{0} \ \text{ where } \mathbf{Q} = \mathbf{Q}^T > 0 \end{cases}$$
$$\Rightarrow \|\underline{x}(t)\| \leq \sqrt{\frac{\lambda_{max}(\mathbf{P})}{\lambda_{min}(\mathbf{P})} \exp\left(-\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}\, t\right)} \, \|\underline{x}(0)\| \tag{8.12}$$

[2] S.P. Bhat & D.S. Bernstein, Finite-time stability of continuous autonomous systems, SIAM J. Control Optim. 38 (3) (2000) 751–766

Indeed since $\mathbf{P} = \mathbf{P}^T > 0$ we get:

$$\begin{array}{l} 0 < \lambda_{min}(\mathbf{P})\, \underline{x}^T \underline{x} \leq V(\underline{x}) = \underline{x}^T \mathbf{P} \underline{x} \leq \lambda_{max}(\mathbf{P})\, \underline{x}^T \underline{x} \quad \forall\, \underline{x} \neq \underline{0} \\ \Rightarrow -\underline{x}^T \underline{x} \leq -\frac{V(\underline{x})}{\lambda_{max}(\mathbf{P})} \end{array} \tag{8.13}$$

Similarly, from $\dot{V}(\underline{x}) \leq -\underline{x}^T \mathbf{Q} \underline{x}$ and since $\mathbf{Q} = \mathbf{Q}^T > 0$ we get, using the previous result:

$$\dot{V}(\underline{x}) \leq -\underline{x}^T \mathbf{Q} \underline{x} \leq -\lambda_{min}(\mathbf{Q})\, \underline{x}^T \underline{x} \leq -\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} V(\underline{x}) \tag{8.14}$$

Integration of this inequality yields:

$$V(\underline{x}(t)) \leq \exp\left(-\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}\, t \right) V(\underline{x}(0)) \tag{8.15}$$

Using this result in the inequalities (8.13) implies:

$$\begin{array}{l} \left\{ \begin{array}{l} \lambda_{min}(\mathbf{P})\, \underline{x}(t)^T \underline{x}(t) \leq V(\underline{x}(t)) \\ V(\underline{x}(0)) \leq \lambda_{max}(\mathbf{P})\, \underline{x}(0)^T \underline{x}(0) \end{array} \right. \\ \quad \Rightarrow \underline{x}(t)^T \underline{x}(t) \leq \frac{V(\underline{x}(t))}{\lambda_{min}(\mathbf{P})} \quad \leq \frac{1}{\lambda_{min}(\mathbf{P})} \exp\left(-\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}\, t \right) V(\underline{x}(0)) \\ \qquad\qquad\qquad\qquad\qquad\quad \leq \frac{\lambda_{max}(\mathbf{P})}{\lambda_{min}(\mathbf{P})} \exp\left(-\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}\, t \right) \underline{x}(0)^T \underline{x}(0) \\ \Rightarrow \|\underline{x}(t)\| := \sqrt{\underline{x}(t)^T \underline{x}(t)} \quad \leq \sqrt{\frac{\lambda_{max}(\mathbf{P})}{\lambda_{min}(\mathbf{P})} \exp\left(-\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}\, t \right)} \|\underline{x}(0)\| \end{array} \tag{8.16}$$

This completes the proof. ■

Moreover the third condition of (8.7) reads:

$$\dot{V}(\underline{x}) = \dot{\underline{x}}^T \mathbf{P} \underline{x} + \underline{x}^T \mathbf{P} \dot{\underline{x}} = \underline{x}^T \left(\mathbf{A}_{cl}^T \mathbf{P} + \mathbf{P} \mathbf{A}_{cl} \right) \underline{x} < 0 \ \forall\, \underline{x} \neq \underline{0} \tag{8.17}$$

Assume that there exists $\alpha > 0$ such that:

$$\mathbf{A}_{cl}^T \mathbf{P} + \mathbf{P} \mathbf{A}_{cl} + 2\alpha \mathbf{P} < 0 \tag{8.18}$$

Then:

$$\dot{V}(\underline{x}) = \underline{x}^T \left(\mathbf{A}_{cl}^T \mathbf{P} + \mathbf{P} \mathbf{A}_{cl} \right) \underline{x} < \underline{x}^T \left(-2\alpha \mathbf{P} \right) \underline{x} \tag{8.19}$$

Then using similar relations than those in the previous proof, it can be shown that:

$$\|\underline{x}(t)\| := \sqrt{\underline{x}(t)^T \underline{x}(t)} \leq \sqrt{\frac{\lambda_{max}(\mathbf{P})}{\lambda_{min}(\mathbf{P})}} \exp\left(-\alpha\, t \right) \|\underline{x}(0)\| \tag{8.20}$$

Therefore for linear system the equilibrium point $\underline{x}_e = \underline{0}$ is globally asymptotically stable if there exits a symmetric positive definite matrix $\mathbf{P} = \mathbf{P}^T$ satisfying the following Linear Matrix Inequalities (LMI):

$$\left\{ \begin{array}{l} \mathbf{P} = \mathbf{P}^T > 0 \\ \mathbf{A}_{cl}^T \mathbf{P} + \mathbf{P} \mathbf{A}_{cl} < 0 \end{array} \right. \tag{8.21}$$

Linear Matrix Inequalities (LMI) (8.21) represents a necessary and sufficient condition for the matrix $\mathbf{A}_{cl}$ to have all its eigenvalues in the left half plane.

Defining by $\mathbf{1}_{ij}$ the matrix whose elements are all zero except elements at position (i,j) and (j,i) which are equal to one, and by $p_{i,j}$ the elements of the $n \times n$ symmetric positive definite matrix $\mathbf{P}$ we can write $\mathbf{P} = \sum_{i=1}^{n}\sum_{j=1}^{i} p_{i,j}\mathbf{1}_{ij}$. The two previous inequalities become:

$$\mathbf{M}(\underline{p}) = \sum_{i=1}^{n}\left(\sum_{j=1}^{i} p_{i,j}\begin{bmatrix} \mathbf{1}_{ij} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_{cl}^T\mathbf{1}_{ij} - \mathbf{1}_{ij}\mathbf{A}_{cl} \end{bmatrix}\right) > 0 \tag{8.22}$$

This yields a Linear Matrix Inequalities (LMI) in the form of (8.1) with vector $\underline{p}$ including all the coefficients $p_{i,j}$ of the symmetric positive definite matrix $\mathbf{P}$. Nevertheless the writing (8.21) in terms of Linear Matrix Inequalities (LMI) in the matrix variable $\mathbf{P}$ is more concise. Furthermore available software operate directly on matrix variables so that it is usually not necessary to carry out transformation (8.22).

A result that can be used to complement Lyapunov analysis is the following theorem, attributed to Franklin (1969):

$$\mathbf{A}^T\mathbf{B} + \mathbf{B}^T\mathbf{A} \leq \gamma\mathbf{A}^T\mathbf{A} + \frac{1}{\gamma}\mathbf{B}^T\mathbf{B} \quad \forall \gamma > 0 \tag{8.23}$$

8.2.2 Application to controller and observer design

In the state feedback case the closed loop state matrix $\mathbf{A}_{cl}$ reads as follows where $\mathbf{K}$ is the controller gain to be set:

$$\mathbf{A}_{cl} = \mathbf{A} - \mathbf{BK} \tag{8.24}$$

Thus Linear Matrix Inequalities (LMI) (8.21) reads:

$$\begin{cases} \mathbf{P} = \mathbf{P}^T > 0 \\ (\mathbf{A} - \mathbf{BK})^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BK}) < 0 \end{cases} \tag{8.25}$$

Because $\mathbf{P} = \mathbf{P}^T > 0$ we have $\mathbf{P}^{-1} = \left(\mathbf{P}^{-1}\right)^T > 0$. The preceding relation can be rewritten by pre and post multiplying by $\mathbf{P}^{-1}$ and by renaming $\mathbf{P}^{-1}$ as $\mathbf{P}$:

$$\begin{cases} \mathbf{P} = \mathbf{P}^T > 0 \\ (\mathbf{A} - \mathbf{BK})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BK})^T < 0 \end{cases} \tag{8.26}$$

In order to exhibit Linear Matrix Inequalities (LMI) we define matrix $\mathbf{Y}$ as follows:

$$\mathbf{Y} = -\mathbf{KP} \Leftrightarrow \mathbf{K} = -\mathbf{YP}^{-1} \tag{8.27}$$

Thus we conclude that the closed loop state matrix $\mathbf{A}_{cl} = \mathbf{A} - \mathbf{BK}$ has all its eigenvalues in the left half plane if and only if there exists a symmetric positive definite matrix $\mathbf{P} = \mathbf{P}^T > 0$ and a matrix $\mathbf{Y}$ such that the following Linear Matrix Inequalities (LMI) hold:

$$\begin{cases} \mathbf{P} = \mathbf{P}^T > 0 \\ \mathbf{AP} + \mathbf{BY} + (\mathbf{AP} + \mathbf{BY})^T < 0 \end{cases} \tag{8.28}$$

Similarly in the observer design case it can be shown that the closed loop state matrix $\mathbf{A}_{cl} = \mathbf{A} - \mathbf{LC}$ has all its eigenvalues in the left half plane if and only if there exists a symmetric positive definite matrix $\mathbf{Q} = \mathbf{Q}^T > 0$ and a matrix $\mathbf{Z}$ such that the following Linear Matrix Inequalities (LMI) hold:

$$\begin{cases} \mathbf{Q} = \mathbf{Q}^T > 0 \\ \mathbf{QA} + \mathbf{ZC} + (\mathbf{QA} + \mathbf{ZC})^T < 0 \end{cases} \tag{8.29}$$

where

$$\mathbf{Z} = -\mathbf{QL} \Leftrightarrow \mathbf{L} = -\mathbf{Q}^{-1}\mathbf{Z} \tag{8.30}$$

8.2.3 Bounded-real lemma

We consider the following realization of a system:

$$\begin{cases} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{w}(t) \\ \underline{z}(t) = \mathbf{C}\underline{x}(t) \\ \underline{x}(0) = \underline{0} \end{cases} \tag{8.31}$$

and the following performance index where $\mathbf{S} = \mathbf{S}^T \geq 0$:

$$J(\underline{w}) = \underline{x}^T(t_f)\mathbf{S}\underline{x}(t_f) + \int_0^{t_f} \underline{z}^T(t)\underline{z}(t)\, dt - \gamma^2 \int_0^{t_f} \underline{w}^T(t)\underline{w}(t)\, dt \tag{8.32}$$

The bounded-real lemma states that a necessary and sufficient condition for $J(\underline{w}) < 0$ for all nonzero $\underline{w}(t)$ is the existence of a solution $\mathbf{P}(t) = \mathbf{P}(t)^T \geq 0$ to the following differential Riccati equation[3]:

$$\begin{cases} -\dot{\mathbf{P}}(t) = \mathbf{A}^T\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A} + \frac{1}{\gamma^2}\mathbf{P}(t)\mathbf{B}\mathbf{B}^T\mathbf{P}(t) + \mathbf{C}^T\mathbf{C} \\ \mathbf{P}(t_f) = \mathbf{S} \end{cases} \tag{8.33}$$

The solution to the above differential Riccati equation can be obtained by solving the following differential linear matrix inequality:

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{P}}(t) + \mathbf{A}^T\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{C}^T\mathbf{C} & \mathbf{P}(t)\mathbf{B} \\ \mathbf{B}^T\mathbf{P}(t) & -\gamma^2\mathbb{I} \end{bmatrix} \leq 0 \\ \mathbf{P}(t_f) = \mathbf{S} \end{cases} \tag{8.34}$$

This result can be obtained through optimal control theory. Indeed consider the problem of *minimizing* the following quadratic performance index under the dynamical constraint (8.31):

$$J(\underline{w}) = \underline{x}^T(t_f)\mathbf{S}\underline{x}(t_f) + \int_0^{t_f} \underline{z}^T(t)\underline{z}(t) + \underline{w}^T(t)\mathbf{R}\underline{w}(t)\, dt \tag{8.35}$$

where:

$$\begin{cases} \mathbf{S} = \mathbf{S}^T \geq 0 \\ \mathbf{R} = \mathbf{R}^T > 0 \end{cases} \tag{8.36}$$

[3] U. Shaked and V. Suplin, A New Bounded Real Lemma Representation for the Continuous-Time Case, IEEE Transactions On Automatic Control, Vol. 46, No. 9, September 2001

Then the optimal control $\underline{w}^*(t)$ reads as follows:

$$\left\{\begin{array}{l} \underline{w}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) \\ -\dot{\mathbf{P}}(t) = \mathbf{A}^T\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A} - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) + \mathbf{C}^T\mathbf{C} \\ \mathbf{P}(t_f) = \mathbf{S} \end{array}\right. \tag{8.37}$$

Furthermore, assuming that $\underline{x}(t_f) = \underline{0}$, the following inequality for the minimum performance index $J^* := J(\underline{w}^*(t))$, $\forall\, \underline{z}(t)$, $\underline{w}(t)$ holds:

$$\underline{x}(t_f) = \underline{0} \Rightarrow J^* = \underline{x}^T(0)\mathbf{P}(0)\underline{x}(0) \leq \int_0^{t_f} \underline{z}^T(t)\underline{z}(t) + \underline{w}^T(t)\mathbf{R}\underline{w}(t)\, dt \tag{8.38}$$

Now, we consider to problem of *maximizing* the following quadratic performance index under the dynamical constraint (8.31):

$$J(\underline{w}) = \underline{x}^T(t_f)\mathbf{S}\underline{x}(t_f) + \int_0^{t_f} \underline{z}^T(t)\underline{z}(t) - \underline{w}^T(t)\mathbf{R}\underline{w}(t)\, dt \tag{8.39}$$

Then the optimal control $\underline{w}^*(t)$ reads as follows:

$$\left\{\begin{array}{l} \underline{w}^*(t) = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) \\ -\dot{\mathbf{P}}(t) = \mathbf{A}^T\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) + \mathbf{C}^T\mathbf{C} \\ \mathbf{P}(t_f) = \mathbf{S} \end{array}\right. \tag{8.40}$$

Assuming that $\underline{x}(t_f) = \underline{0}$, this leads to the following inequality for the maximum performance index $J^* := J(\underline{w}^*(t)) = max\,(J(\underline{w}))\,, \forall\, \underline{z}(t)$, $\underline{w}(t)$, where we use the fact that $max\,(J(\underline{w})) = -min\,(J(\underline{w}))$:

$$J^* := J(\underline{w}^*(t)) = -\underline{x}^T(0)\mathbf{P}(0)\underline{x}(0) \geq \int_0^{t_f} \underline{z}^T(t)\underline{z}(t) - \underline{w}^T(t)\mathbf{R}\underline{w}(t)\, dt \tag{8.41}$$

Noticing that $\underline{x}^T(0)\mathbf{P}(0)\underline{x}(0) > 0\ \forall\, \underline{x}(0) \neq \underline{0}$ and setting $\mathbf{R} = \gamma^2\mathbb{I} > 0$ we get:

$$\begin{array}{c} \underline{x}(t_f) = \underline{0} \Rightarrow \int_0^{t_f} \underline{z}^T(t)\underline{z}(t) - \gamma^2\underline{w}^T(t)\underline{w}(t)\, dt \leq -\underline{x}^T(0)\mathbf{P}(0)\underline{x}(0) < 0 \\ \Rightarrow \int_0^{t_f} \underline{z}^T(t)\underline{z}(t) < \gamma^2 \int_0^{t_f} \underline{w}^T(t)\underline{w}(t)\, dt \end{array} \tag{8.42}$$

Conversely, assume that $\underline{w}(t)$ is an exogenous disturbance. We are seeking for the lowest value of $\gamma > 0$ such that the following inequality holds:

$$\begin{array}{c} \int_0^t \underline{w}(\tau)^T\underline{w}(\tau)d\tau \leq \frac{1}{\gamma^2}\int_0^t \underline{z}(\tau)^T\underline{z}(\tau)d\tau \quad \forall t \geq 0 \\ \Leftrightarrow \int_0^t \left(\underline{z}(\tau)^T\underline{z}(\tau) - \gamma^2\underline{w}(\tau)^T\underline{w}(\tau)\right) d\tau \geq 0 \quad \forall t \geq 0 \end{array} \tag{8.43}$$

To get γ we consider the following candidate Lyapunov function where matrix $\mathbf{P}(t) = \mathbf{P}(t)^T > 0$:

$$V(\underline{x}, \underline{z}) = \underline{x}^T\mathbf{P}(t)\underline{x} + \int_0^t \left(\underline{z}^T(\tau)\underline{z}(\tau) - \gamma^2\underline{w}^T(\tau)\underline{w}(\tau)\right)\, d\tau \geq 0 \tag{8.44}$$

Evaluating the time derivative of $V(\underline{x}, \underline{z})$ yields:

$$\begin{array}{rl} \dot{V}(\underline{x}, \underline{z}) & = \underline{\dot{x}}^T \mathbf{P} \underline{x} + \underline{x}^T \mathbf{P} \underline{\dot{x}} + \underline{z}^T \underline{z} - \gamma^2 \underline{w}^T \underline{w} \\ & = (\mathbf{A}\underline{x} + \mathbf{B}\underline{w})^T \mathbf{P}\underline{x} + \underline{x}^T \mathbf{P} (\mathbf{A}\underline{x} + \mathbf{B}\underline{w}) + (\mathbf{C}\underline{x})^T \mathbf{C}\underline{x} - \gamma^2 \underline{w}^T \underline{w} \\ & = \begin{bmatrix} \underline{x}^T & \underline{w}^T \end{bmatrix} \begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{C}^T\mathbf{C} & \mathbf{P}\mathbf{B} \\ \mathbf{B}^T\mathbf{P} & -\gamma^2 \mathbb{I} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{w} \end{bmatrix} \end{array} \tag{8.45}$$

Thus $\dot{V}(\underline{x}, \underline{z}) \leq 0$, meaning that assuming that matrix $\mathbf{A}$ is stable the system remains stable despite disturbance $\underline{w}(t)$, as soon as the following LMI holds:

$$\begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{C}^T\mathbf{C} & \mathbf{P}\mathbf{B} \\ \mathbf{B}^T\mathbf{P} & -\gamma^2 \mathbb{I} \end{bmatrix} \leq 0 \tag{8.46}$$

The level of allowed uncertainty can be maximized by minimizing γ.

More generally let $\mathbf{G}(s)$ be the following transfer function:

$$\mathbf{G}(s) = \mathbf{C}\left(s\mathbb{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D} := \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right) \tag{8.47}$$

Let $\mathbf{R}_\gamma = \gamma^2\mathbb{I} - \mathbf{D}^T\mathbf{D}$ and assume that $\mathbf{A}$ has no eigenvalue on the imaginary axis. Then the following assertions are equivalent[4]:

- $J(\underline{w}) = \int_0^\infty \underline{z}^T(t)\underline{z}(t)\,dt - \gamma^2 \int_0^\infty \underline{w}^T(t)\underline{w}(t)\,dt \leq 0$
- $\|\mathbf{G}(s)\|_\infty \leq \gamma$
- $\bar{\sigma}(\mathbf{D}) \leq \gamma$ and the following Hamiltonian matrix $\mathbf{H}$ has no eigenvalue on the imaginary axis:

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} + \mathbf{B}\mathbf{R}_\gamma^{-1}\mathbf{D}^T\mathbf{C} & \mathbf{B}\mathbf{R}_\gamma^{-1}\mathbf{B}^T \\ -\mathbf{C}^T\left(\mathbb{I} + \mathbf{D}\mathbf{R}_\gamma^{-1}\mathbf{D}^T\right)\mathbf{C} & -\left(\mathbf{A} + \mathbf{B}\mathbf{R}_\gamma^{-1}\mathbf{D}^T\mathbf{C}\right)^T \end{bmatrix} \tag{8.48}$$

Furthermore, the assertion that $\mathbf{A}$ is stable and that $\|\mathbf{G}(s)\|_\infty < \gamma$ is equivalent to the fact that there exists a matrix $\mathbf{P} = \mathbf{P}^T \geq 0$ solving the following LMI:

$$\begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{C}^T\mathbf{C} & \mathbf{P}\mathbf{B} + \mathbf{C}^T\mathbf{D} \\ \mathbf{D}^T\mathbf{C} + \mathbf{B}^T\mathbf{P} & -\mathbf{R}_\gamma \end{bmatrix} \leq 0 \tag{8.49}$$

8.2.4 Inescapable set

Let's consider the following *minimal* realization of a system where state matrix $\mathbf{A}$ is assumed to be *stable*:

$$\begin{cases} \underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{w}(t) \\ \underline{z}(t) = \mathbf{C}\underline{x}(t) \end{cases} \tag{8.50}$$

[4] Uwe Mackenroth, Robust Control Systems: Theory and Case Studies, Springer-Verlag Berlin Heidelberg 2004

A set $\mathcal{X}$ is said to be inescapable if it contains the origin and if $\underline{x}(0) \in \mathcal{X}$ and $\|\underline{w}(t)\|_\infty \leq 1$ implies that $\underline{x}(t) \in \mathcal{X}$ for all future time $t > 0$:

$$\left\{ \begin{array}{l} \underline{x}(0) \in \mathcal{X} \\ \|\underline{w}(t)\|_\infty \leq 1 \end{array} \right. \Rightarrow \underline{x}(t) \in \mathcal{X}\, \forall\, t \geq 0 \tag{8.51}$$

Applying Lyapunov's second method, it can be seen that the ellipsoid $\underline{x}^T \mathbf{P} \underline{x} \leq 1$, where $\underline{x} \in \mathbb{R}^n$ and $\mathbf{P} = \mathbf{P}^T > 0$, is inescapable if and only if the following LMI holds:

$$\left\{ \begin{array}{l} \begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} & \mathbf{P}\mathbf{B} \\ \mathbf{B}^T\mathbf{P} & \mathbf{0} \end{bmatrix} \leq 0 \\ \mathbf{P} = \mathbf{P}^T > 0 \end{array} \right. \tag{8.52}$$

Using the S-procedure[5], it can be shown that the preceding LMI is equivalent to the following one[6] for some scalar α:

$$\left\{ \begin{array}{l} \begin{bmatrix} \mathbf{A}^T\mathbf{P}_\alpha + \mathbf{P}_\alpha\mathbf{A} + \alpha\mathbf{P}_\alpha & \mathbf{P}_\alpha\mathbf{B} \\ \mathbf{B}^T\mathbf{P}_\alpha & -\alpha\mathbb{I} \end{bmatrix} \leq 0 \\ \mathbf{P}_\alpha = \mathbf{P}_\alpha^T > 0 \end{array} \right. \tag{8.53}$$

Scalar α is such that:

$$0 < \alpha < -2\, max\,(real\,(spec\,(\mathbf{A}))) \tag{8.54}$$

With the new matrix $\mathbf{Q}_\alpha = \mathbf{P}_\alpha^{-1} > 0$ the preceding LMI is equivalent to the following one:

$$\left\{ \begin{array}{l} \begin{bmatrix} \mathbf{A}\mathbf{Q}_\alpha + \mathbf{Q}_\alpha\mathbf{A}^T + \alpha\,\mathbf{Q}_\alpha & \mathbf{B} \\ \mathbf{B}^T & -\alpha\mathbb{I} \end{bmatrix} \leq 0 \\ \mathbf{Q}_\alpha = \mathbf{Q}_\alpha^T > 0 \end{array} \right. \tag{8.55}$$

It is worth noticing that when $0 < \alpha < -2\, max\,(real\,(spec\,(\mathbf{A})))$ the matrix $\mathbf{A} + \frac{1}{2}\alpha\mathbb{I}$ is stable. Thus there exists a unique solution $\mathbf{Q}_\alpha = \mathbf{Q}_\alpha^T > 0$ to the following Lyapunov equation.

$$\left(\mathbf{A} + \frac{1}{2}\alpha\mathbb{I}\right)\mathbf{Q}_\alpha + \mathbf{Q}_\alpha\left(\mathbf{A} + \frac{1}{2}\alpha\mathbb{I}\right)^T + \frac{1}{\alpha}\mathbf{B}\mathbf{B}^T = \mathbf{0} \tag{8.56}$$

Indeed:

$$\begin{array}{l} \mathbf{A}\mathbf{Q}_\alpha + \mathbf{Q}_\alpha\mathbf{A}^T + \alpha\,\mathbf{Q}_\alpha + \frac{1}{\alpha}\mathbf{B}\mathbf{B}^T = \mathbf{0} \\ \Leftrightarrow \left(\mathbf{A} + \frac{1}{2}\alpha\mathbb{I}\right)\mathbf{Q}_\alpha + \mathbf{Q}_\alpha\left(\mathbf{A} + \frac{1}{2}\alpha\mathbb{I}\right)^T + \frac{1}{\alpha}\mathbf{B}\mathbf{B}^T = \mathbf{0} \end{array} \tag{8.57}$$

Furthermore the following inequality holds[6]:

$$\sup_{\|\underline{w}(t)\|_\infty \leq 1} \underline{z}(t) = \sup_{\underline{x}^T\mathbf{Q}_\alpha^{-1}\underline{x} \leq 1} \|\mathbf{C}\underline{x}(t)\|_2 = \sqrt{\|\mathbf{C}\mathbf{Q}_\alpha\mathbf{C}^T\|_2} \tag{8.58}$$

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[5]Derinkuyu, K., Pınar, M.Ç. On the S-procedure and Some Variants. Math Meth Oper Res 64, 55–77 (2006).

[6]Nguyen T., Jabbari F., H_∞ design for systems with input saturation: an LMI approach, Proceedings of the American Control Conference (Cat. No.97CH36041), 1997

To obtain the smallest value of these upper bounds, that is the smallest value of $\sup_{\|\underline{w}(t)\|_\infty \leq 1} \underline{z}(t)$, the quantity $\|\mathbf{C}\mathbf{Q}_\alpha \mathbf{C}^T\|_2$ shall be minimized over $0 < \alpha < -2\, max\, (real\, (spec\, (\mathbf{A})))$.

We recall that the induced matrix 2-norm $\|\mathbf{X}\|_2$ of matrix $\mathbf{X}$ is defined as the largest singular value of $\mathbf{X}$, that is the root square of the largest eigenvalue of $\mathbf{X}^T\mathbf{X}$ (or $\mathbf{X}\mathbf{X}^T$).

8.3 State feedback control

8.3.1 Kronecker product

Notation $\otimes$ will be used to denote the Kronecker product. The Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$ and is a block matrix $\mathbf{C}$ with generic block entry $[\mathbf{C}]_{ij} = [\mathbf{A}]_{ij}\, \mathbf{B}$:

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B} & \cdots & \mathbf{A}_{1n}\mathbf{B} \\ \vdots & & \vdots \\ \mathbf{A}_{n1}\mathbf{B} & \cdots & \mathbf{A}_{nn}\mathbf{B} \end{bmatrix} \tag{8.59}$$

Furthermore the Kronecker product has the following properties:

$$\begin{cases} 1 \otimes \mathbf{A} = \mathbf{A} \otimes 1 = \mathbf{A} \\ (\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C} \\ (\mathbf{A} \otimes \mathbf{B})\, (\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}) \\ (\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T \\ (\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \end{cases} \tag{8.60}$$

8.3.2 LMI regions

Following Chilali & al.[7] an LMI region is any subset $\mathcal{D}$ of the complex plane that can be defined as follows where $\mathbf{L} = \mathbf{L}^T$ and $\mathbf{M}$ are real matrices and $\bar{z}$ is the conjugate of complex number z:

$$\mathcal{D} = \left\{ z \in \mathbb{C} \,:\, \mathbf{L} + \mathbf{M}\, z + \bar{z}\, \mathbf{M}^T < 0 \right\} \tag{8.61}$$

The characteristic function of $\mathcal{D}$ is the matrix-valued function $f_{\mathcal{D}}(z)$ defined as follows:

$$f_{\mathcal{D}}(z) = \mathbf{L} + \mathbf{M}\, z + \bar{z}\, \mathbf{M}^T \tag{8.62}$$

Here are some examples of LMI regions[7]:

– Half-plane $Re(z) < \alpha$:

$$f_{\mathcal{D}}(z) = -2\alpha + z + \bar{z} < 0 \tag{8.63}$$

Thus:

$$\begin{cases} \mathbf{L} = -2\alpha \\ \mathbf{M} = 1 \end{cases} \tag{8.64}$$

[7] Chilali M., Gahinet P. and Apkarian P., Robust Pole Placement in LMI Regions, IEEE Transactions on Automatic Control, vol. 44, no. 12, pp. 2257-2270, Dec 1999

– Half-plane $Re(z) > \alpha$:

$$f_{\mathcal{D}}(z) = 2\alpha - z - \bar{z} < 0 \tag{8.65}$$

Thus:

$$\begin{cases} \mathbf{L} = 2\alpha \\ \mathbf{M} = -1 \end{cases} \tag{8.66}$$

– Disk centered at $(q, 0)$ with radius $r > 0$: Such a region is defined by:

$$\begin{array}{l} z\bar{z} - q(z+\bar{z}) + q - r^2 < 0 \\ \Leftrightarrow (z-q)(\bar{z}-q) - r^2 < 0 \\ \Leftrightarrow r\left(\frac{(z-q)(\bar{z}-q)}{r} - r\right) < 0 \end{array} \tag{8.67}$$

As far as the radius r of the circle is positive (i.e. $r > 0 \Leftrightarrow -r < 0$), the preceding inequality reduces as follows:

$$-r < 0 \Rightarrow \frac{(z-q)(\bar{z}-q)}{r} - r < 0 \tag{8.68}$$

By using Schur complement lemma (8.2) we finally get the following LMI region:

$$f_{\mathcal{D}}(z) = \begin{bmatrix} -r & z-q \\ \bar{z}-q & -r \end{bmatrix} < 0 \tag{8.69}$$

Thus:

$$\begin{cases} \mathbf{L} = \begin{bmatrix} -r & -q \\ -q & -r \end{bmatrix} \\ \mathbf{M} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{cases} \tag{8.70}$$

– Conic sector with apex at the origin and inner angle 2θ (see Figure 8.1 with $\alpha = 0$). Such a region is defined by the intersection of two half-planes:

$$\begin{cases} j(z-\bar{z})\cos(\theta) - (z+\bar{z})\sin(\theta) > 0 \\ j(z-\bar{z})\cos(\theta) + (z+\bar{z})\sin(\theta) < 0 \end{cases} \tag{8.71}$$

The product of those two conditions reads:

$$\begin{array}{l} (j(z-\bar{z})\cos(\theta) - (z+\bar{z})\sin(\theta)) \\ \qquad \times (j(z-\bar{z})\cos(\theta) + (z+\bar{z})\sin(\theta)) < 0 \\ \Leftrightarrow -(z-\bar{z})^2\cos(\theta)^2 - (z+\bar{z})^2\sin(\theta)^2 < 0 \\ \Leftrightarrow (-(z+\bar{z})\sin(\theta))\left((z+\bar{z})\sin(\theta) + \frac{(z-\bar{z})^2\cos(\theta)^2}{(z+\bar{z})\sin(\theta)}\right) < 0 \end{array} \tag{8.72}$$

Assuming that $-(z+\bar{z})\sin(\theta) > 0$, that is $0 < \theta < \frac{\pi}{2}$ and $Re(z) < 0$ (i.e. z is located in the stability region), the use of Schur complement lemma (8.2) leads to the following LMI region:

$$f_{\mathcal{D}}(z) = \begin{bmatrix} (z+\bar{z})\sin(\theta) & (z-\bar{z})\cos(\theta) \\ -(z-\bar{z})\cos(\theta) & (z+\bar{z})\sin(\theta) \end{bmatrix} < 0 \tag{8.73}$$

Thus:

$$\begin{cases} \mathbf{L} = \mathbf{0} \\ \mathbf{M} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{bmatrix} \end{cases} \tag{8.74}$$

Chilali & al. have shown[7] that a real matrix $\mathbf{A}$ is $\mathcal{D}$-stable, i.e., has all the eigenvalues of $\mathbf{A}$ are located inside the LMI region $\mathcal{D}$, if and only if there exists a symmetric matrix $\mathbf{P} > 0$ such that the following relations hold:

$$\begin{cases} \mathbf{P} \otimes \mathbf{L} + (\mathbf{PA}) \otimes \mathbf{M} + \left(\mathbf{A}^T\mathbf{P}\right) \otimes \mathbf{M}^T < 0 \\ \mathbf{P} = \mathbf{P}^T > 0 \end{cases} \tag{8.75}$$

Relation (8.75) can also be used to check the $\mathcal{D}$-stability of polynomials. Indeed polynomial $\chi(s) = a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + 1 \times s^n$ is $\mathcal{D}$-stable, i.e., has all the roots of $\chi(s)$ are located inside the LMI region $\mathcal{D}$, if and only if relation (8.75) holds with:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \tag{8.76}$$

Relation (8.75) can be seen as a generalization of the Lyapunov theorem because for the usual stability region $f_{\mathcal{D}}(z) = z + \bar{z} < 0$ then relation (8.75) reduces as follows:

$$\begin{cases} \mathbf{P} \otimes 0 + (\mathbf{PA}) \otimes 1 + \left(\mathbf{A}^T\mathbf{P}\right) \otimes 1 < 0 \\ \mathbf{P} = \mathbf{P}^T > 0 \end{cases} \Leftrightarrow \begin{cases} \mathbf{PA} + \mathbf{A}^T\mathbf{P} < 0 \\ \mathbf{P} = \mathbf{P}^T > 0 \end{cases} \tag{8.77}$$

More generally[8] an LMI region is defined as follows where $\mathbf{L} = \mathbf{L}^T$, $\mathbf{N} = \mathbf{N}^T$ and $\mathbf{M}$ are real matrices and $\bar{z}$ is the conjugate of complex number z:

$$\mathcal{D} = \left\{ z \in \mathbb{C} : \begin{bmatrix} \mathbb{I} & \bar{z}\mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbf{L} & \mathbf{M} \\ \mathbf{M}^T & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbb{I} \\ z\mathbb{I} \end{bmatrix} < 0 \right\} \tag{8.78}$$

Then all eigenvalues of $\mathbf{A} \in \mathbb{R}^{n \times n}$ are contained in the LMI region $\mathcal{D}$ if and only if there exists a symmetric matrix $\mathbf{P} > 0$ such that the following relations hold:

$$\begin{cases} \begin{bmatrix} \mathbb{I} & \mathbf{A}^T \otimes \mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbf{P} \otimes \mathbf{L} & \mathbf{P} \otimes \mathbf{M} \\ \mathbf{P} \otimes \mathbf{M}^T & \mathbf{P} \otimes \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbb{I} \\ \mathbf{A} \otimes \mathbb{I} \end{bmatrix} < 0 \\ \mathbf{P} = \mathbf{P}^T > 0 \end{cases} \tag{8.79}$$

[8] Linear Matrix Inequalities in Control, Carsten Scherer and Siep Weiland, Lectures Notes

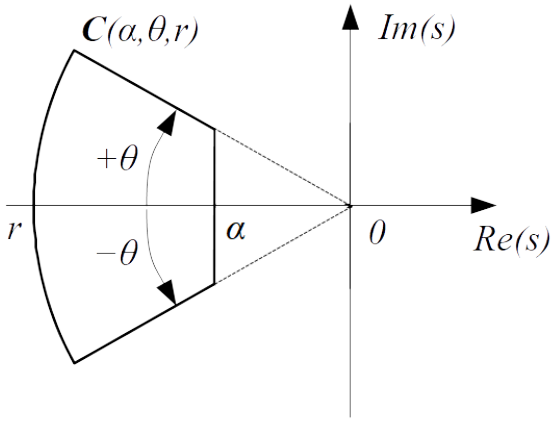

Figure 8.1: Pole placement constraints

8.3.3 LMI regions for pole placement constraints

We consider the state-space realization (8.80) where the state vector $\underline{x}$ is of dimension n (that is the size of state matrix $\mathbf{A}$). In addition $\underline{y}(t)$ denotes the output vector and $\underline{u}(t)$ the input vector. We will assume that the feedforward gain matrix $\mathbf{D}$ is zero ($\mathbf{D} = \mathbf{0}$):

$$\begin{cases} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \\ \underline{y}(t) = \mathbf{C}\underline{x}(t) \end{cases} \tag{8.80}$$

From a designer point of view system specifications are usually limited to a minimum speed of response and a minimum damping ratio. Furthermore the required control effort is larger when the poles of the closed loop system are moved far away from their original locations. All those constraints can be drawn as a region in the complex plane where the closed-loop poles should be located. A typical convex region $\mathcal{C}(\alpha, \theta, r)$ in the complex plane where all the poles of the closed loop system are constrained to be located is shown in Figure 8.1:

- The real part of all the poles shall be lower than α in order to achieve a minimum speed of response;

- All the poles shall be located in a conic sector with inner angle $2\,\theta$ in order to achieve a minimum damping ratio;

- All the poles shall be located inside the circle of radius r to limit the control effort.

Chilali & al.[7] have shown that in the state feedback case the eigenvalues of the closed loop state matrix $\mathbf{A} - \mathbf{BK}$ will be situated in the convex region $\mathcal{C}(\alpha, \theta, r)$ if and only if there exists a symmetric positive definite matrix $\mathbf{P} = \mathbf{P}^T > 0$ and a matrix $\mathbf{Y}$ such that the following Linear Matrix Inequalities

(LMI) hold:

$$\begin{cases} \mathbf{P} = \mathbf{P}^T > 0 \\ \mathbf{AP} + \mathbf{BY} + (\mathbf{AP} + \mathbf{BY})^T - 2\alpha\mathbf{P} < 0 \\ (\mathbf{AP} + \mathbf{BY}) \otimes \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{bmatrix} \\ \qquad\qquad + (\mathbf{AP} + \mathbf{BY})^T \otimes \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{bmatrix}^T < 0 \\ \begin{bmatrix} -r\mathbf{P} & \mathbf{AP} + \mathbf{BY} \\ (\mathbf{AP} + \mathbf{BY})^T & -r\mathbf{P} \end{bmatrix} < 0 \end{cases} \tag{8.81}$$

where

$$\mathbf{Y} = -\mathbf{KP} \Leftrightarrow \mathbf{K} = -\mathbf{YP}^{-1} \tag{8.82}$$

Similarly, in the observer design case, the eigenvalues of the closed loop state matrix $\mathbf{A} - \mathbf{LC}$ will be situated in the convex region $\mathcal{C}(\alpha, \theta, r)$ if and only if there exists a symmetric positive definite matrix $\mathbf{P} = \mathbf{P}^T > 0$ and a matrix $\mathbf{Z}$ such that the following Linear Matrix Inequalities (LMI) hold:

$$\begin{cases} \mathbf{Q} = \mathbf{Q}^T > 0 \\ \mathbf{QA} + \mathbf{ZC} + (\mathbf{QA} + \mathbf{ZC})^T - 2\alpha\mathbf{Q} < 0 \\ (\mathbf{QA} + \mathbf{ZC}) \otimes \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{bmatrix} \\ \qquad\qquad + (\mathbf{QA} + \mathbf{ZC})^T \otimes \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{bmatrix}^T < 0 \\ \begin{bmatrix} -r\mathbf{Q} & \mathbf{QA} + \mathbf{ZC} \\ (\mathbf{QA} + \mathbf{ZC})^T & -r\mathbf{Q} \end{bmatrix} < 0 \end{cases} \tag{8.83}$$

where

$$\mathbf{Z} = -\mathbf{QL} \Leftrightarrow \mathbf{L} = -\mathbf{Q}^{-1}\mathbf{Z} \tag{8.84}$$

8.4 Observer-based feedback control

8.4.1 Linear-Quadratic-Regulator (LQR) problem

Linear-Quadratic-Regulator (LQR) tackle the problem to design a state feedback stabilizing controller which minimizes the following criteria where $\mathbf{R} = \mathbf{R}^T > \mathbf{0}$ and $\mathbf{Q} = \mathbf{Q}^T \geq \mathbf{0}$:

$$J_{LQR} = \frac{1}{2} \int_0^\infty \left(\underline{x}^T(t)\mathbf{Q}\underline{x}(t) + \underline{u}^T(t)\mathbf{R}\underline{u}(t) \right) dt \tag{8.85}$$

under the following constraint:

$$\underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \tag{8.86}$$

The solution of this problem is closely related to matrix $\mathbf{P} = \mathbf{P}^T > \mathbf{0}$ which solves the following Algebraic Riccati Equation (ARE):

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{8.87}$$

The closed loop state matrix $\mathbf{A}_{cl}$ reads as follows where $\mathbf{K}$ is the controller gain to be set:

$$\left\{\begin{array}{l}\mathbf{A}_{cl} = \mathbf{A} - \mathbf{B}\mathbf{K} \\ \mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\end{array}\right. \tag{8.88}$$

Thus the algebraic Riccati equation can be rewritten as follows:

$$\begin{array}{l}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{K}^T\mathbf{R}\mathbf{K} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{K} - (\mathbf{P}\mathbf{B}\mathbf{K})^T = \mathbf{0} \\ \Leftrightarrow (\mathbf{A} - \mathbf{B}\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}) + \mathbf{K}^T\mathbf{R}\mathbf{K} + \mathbf{Q} = \mathbf{0} \\ \Leftrightarrow (\mathbf{A} - \mathbf{B}\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}) = -\mathbf{K}^T\mathbf{R}\mathbf{K} - \mathbf{Q} = \mathbf{0}\end{array} \tag{8.89}$$

From the preceding relation and the Lyapunov's second method seen in section 8.2.1 we conclude that any LQR gain $\mathbf{K}$ leads to a stable closed loop.

Furthermore, according to Boyd & al. [9], algebraic Riccati equation is equivalent to the following LMI problem: find W = WT , P = PT and Y which solves the following *convex* problem

$$\begin{array}{ll} max & \underline{x}_0^T\mathbf{P}\underline{x}_0 \\ s.t. & \left\{\begin{array}{l}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} \geq \mathbf{0} \\ \mathbf{P} = \mathbf{P}^T \geq \mathbf{0}\end{array}\right.\end{array} \tag{8.90}$$

Because $\underline{x}_0$ is unknown we replace it by $\mathbf{B}$. In addition from that fact that $max(J) = -min(J)$, and using the Schur complement lemma (8.2), the LMI corresponding to LQR consists in finding $\mathbf{P}$ which solves the following problem:

$$\begin{array}{ll} min & -\operatorname{tr}\left(\mathbf{B}^T\mathbf{P}\mathbf{B}\right) \\ s.t. & \left\{\begin{array}{l}\begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} & \mathbf{P}\mathbf{B} \\ (\mathbf{P}\mathbf{B})^T & \mathbf{R}\end{bmatrix} \geq \mathbf{0} \\ \mathbf{P} = \mathbf{P}^T \geq \mathbf{0}\end{array}\right.\end{array} \tag{8.91}$$

8.4.2 Minimizing grammian through LMI

Controllability grammian

We have seen that the solution of the $\mathcal{H}_2$ robust control problem consists in minimizing the H_2-norm of some transfer function $\mathbf{T}_{zw}(s)$.

Let $\mathbf{T}_{zw}(s)$ be defined as follows:

$$\mathbf{T}_{zw}(s) = \mathbf{C}\left(s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K}\right)^{-1}\mathbf{B}_1 \tag{8.92}$$

From the results of section 1.7.2, minimizing the H_2-norm of $\mathbf{T}_{zw}(s)$ consists in solving the following semi-definite program where $\mathbf{P}$ stands for the controllability grammian of $\mathbf{T}_{zw}(s)$:

$$\begin{array}{ll} min & \operatorname{tr}\left(\mathbf{C}\mathbf{P}\mathbf{C}^T\right) \\ s.t. & \left\{\begin{array}{l}(\mathbf{A} - \mathbf{B}_2\mathbf{K})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}_2\mathbf{K})^T + \mathbf{B}_1\mathbf{B}_1^T \leq \mathbf{0} \\ \mathbf{P} = \mathbf{P}^T > \mathbf{0}\end{array}\right.\end{array} \tag{8.93}$$

[9] S. Boyd, L. El Ghaoui, E. Feron,V. Balakrishnan, Linear matrix inequalities in system and control theory, SIAM Studies in Applied Mathematics, vol. 15, Philadelphia, PA, 1994

Introducing $\mathbf{W} \geq \mathbf{C}\mathbf{P}\mathbf{C}^T$ and using $\mathbf{Y} = -\mathbf{K}\mathbf{P}$ we can rewrite the preceding semi-definite program as follows:

$$\begin{array}{ll} min & \operatorname{tr}(\mathbf{W}) \\ s.t. & \left\{ \begin{array}{l} \mathbf{A}\mathbf{P} + \mathbf{B}_2\mathbf{Y} + \mathbf{P}\mathbf{A}^T + (\mathbf{B}_2\mathbf{Y})^T + \mathbf{B}_1\mathbf{B}_1^T \leq \mathbf{0} \\ \mathbf{P} = \mathbf{P}^T > \mathbf{0} \end{array} \right. \end{array} \tag{8.94}$$

Using the inequality $\mathbf{W} \geq \mathbf{C}\mathbf{P}\mathbf{C}^T$ it is then possible to rewrite the problem above as an LMI problem. Indeed:

$$\begin{array}{rl} \mathbf{C}\mathbf{P}\mathbf{C}^T & = \mathbf{C}\mathbf{P}\mathbf{P}^{-1}\mathbf{P}\mathbf{C}^T \\ & = (\mathbf{C}\mathbf{P})\,\mathbf{P}^{-1}\,(\mathbf{C}\mathbf{P})^T \\ \mathbf{W} \geq \mathbf{C}\mathbf{P}\mathbf{C}^T & \Leftrightarrow \mathbf{W} - (\mathbf{C}\mathbf{P})\,\mathbf{P}^{-1}\,(\mathbf{C}\mathbf{P})^T \geq \mathbf{0} \end{array} \tag{8.95}$$

Using Schur's complement, the LMI problem finally reads as follows: find $\mathbf{W} = \mathbf{W}^T$, $\mathbf{P} = \mathbf{P}^T$ and $\mathbf{Y}$ which solves the following problem:

$$\begin{array}{ll} min & \operatorname{tr}(\mathbf{W}) \\ s.t. & \left\{ \begin{array}{l} \mathbf{A}\mathbf{P} + \mathbf{B}_2\mathbf{Y} + \mathbf{P}\mathbf{A}^T + (\mathbf{B}_2\mathbf{Y})^T + \mathbf{B}_1\mathbf{B}_1^T \leq \mathbf{0} \\ \begin{bmatrix} \mathbf{P} & (\mathbf{C}\mathbf{P})^T \\ \mathbf{C}\mathbf{P} & \mathbf{W} \end{bmatrix} \geq \mathbf{0} \end{array} \right. \end{array} \tag{8.96}$$

or, equivalently, when applying again the Schur's complement on the first LMI:

$$\begin{array}{ll} min & \operatorname{tr}(\mathbf{W}) \\ s.t. & \left\{ \begin{array}{l} \begin{bmatrix} \mathbf{A}\mathbf{P} + \mathbf{B}_2\mathbf{Y} + (\mathbf{A}\mathbf{P} + \mathbf{B}_2\mathbf{Y})^T & \mathbf{B}_1 \\ \mathbf{B}_1^T & -\mathbb{I} \end{bmatrix} \leq \mathbf{0} \\ \begin{bmatrix} \mathbf{P} & (\mathbf{C}\mathbf{P})^T \\ \mathbf{C}\mathbf{P} & \mathbf{W} \end{bmatrix} \geq \mathbf{0} \end{array} \right. \end{array} \tag{8.97}$$

The required state feedback gain matrix can be computed as $\mathbf{K} = -\mathbf{Y}\mathbf{P}^{-1}$.

Observability grammian

A similar semi-definite program can be obtained by using the observability grammian.

Let $\mathbf{T}_{zw}(s)$ be defined as follows:

$$\mathbf{T}_{zw}(s) = \mathbf{C}_1\,(s\mathbb{I} - \mathbf{A} + \mathbf{L}\mathbf{C}_2)^{-1}\,\mathbf{B} \tag{8.98}$$

From the results of section 1.7.2, minimizing the H_2-norm of $\mathbf{T}_{zw}(s)$ consists in solving the following semi-definite program where $\mathbf{Q}$ stands for the observability grammian of $\mathbf{T}_{zw}(s)$:

$$\begin{array}{ll} min & \operatorname{tr}\left(\mathbf{B}^T\mathbf{Q}\mathbf{B}\right) \\ s.t. & \left\{ \begin{array}{l} \mathbf{Q}\,(\mathbf{A} - \mathbf{L}\mathbf{C}_2) + (\mathbf{A} - \mathbf{L}\mathbf{C}_2)^T\,\mathbf{Q} + \mathbf{C}_1^T\mathbf{C}_1 \leq \mathbf{0} \\ \mathbf{Q} = \mathbf{Q}^T > \mathbf{0} \end{array} \right. \end{array} \tag{8.99}$$

Introducing $\mathbf{W} \geq \mathbf{B}^T\mathbf{QB}$ and using $\mathbf{Z} = -\mathbf{QL}$ we can rewrite the preceding semi-definite program as follows:

$$\begin{array}{ll} min & \operatorname{tr}(\mathbf{W}) \\ s.t. & \left\{ \begin{array}{l} \mathbf{QA} + \mathbf{ZC}_2 + \mathbf{A}^T\mathbf{Q} + (\mathbf{ZC}_2)^T + \mathbf{C}_1^T\mathbf{C}_1 \leq \mathbf{0} \\ \mathbf{Q} = \mathbf{Q}^T > \mathbf{0} \end{array} \right. \end{array} \tag{8.100}$$

Using the inequality $\mathbf{W} \geq \mathbf{B}^T\mathbf{QB}$, it is then possible to rewrite the problem above as an LMI problem. Indeed:

$$\begin{array}{rl} \mathbf{B}^T\mathbf{QB} & = \mathbf{B}^T\mathbf{QQ}^{-1}\mathbf{QB} \\ & = (\mathbf{QB})^T\mathbf{Q}^{-1}(\mathbf{QB}) \\ \mathbf{W} \geq \mathbf{B}^T\mathbf{QB} & \Leftrightarrow \mathbf{W} - (\mathbf{QB})^T\mathbf{Q}^{-1}(\mathbf{QB}) \geq \mathbf{0} \end{array} \tag{8.101}$$

Using Schur's complement, the LMI problem finally reads as follows: find $\mathbf{W} = \mathbf{W}^T$, $\mathbf{Q} = \mathbf{Q}^T$ and $\mathbf{Z}$ which solves the following problem:

$$\begin{array}{ll} min & \operatorname{tr}(\mathbf{W}) \\ s.t. & \left\{ \begin{array}{l} \mathbf{QA} + \mathbf{ZC}_2 + \mathbf{A}^T\mathbf{Q} + (\mathbf{ZC}_2)^T + \mathbf{C}_1^T\mathbf{C}_1 \leq \mathbf{0} \\ \begin{bmatrix} \mathbf{Q} & \mathbf{QB} \\ (\mathbf{QB})^T & \mathbf{W} \end{bmatrix} \geq \mathbf{0} \end{array} \right. \end{array} \tag{8.102}$$

or, equivalently, when applying again the Schur's complement on the first LMI:

$$\begin{array}{ll} min & \operatorname{tr}(\mathbf{W}) \\ s.t. & \left\{ \begin{array}{l} \begin{bmatrix} \mathbf{QA} + \mathbf{ZC}_2 + (\mathbf{QA} + \mathbf{ZC}_2)^T & \mathbf{C}_1^T \\ \mathbf{C}_1 & -\mathbb{I} \end{bmatrix} \leq \mathbf{0} \\ \begin{bmatrix} \mathbf{Q} & \mathbf{QB} \\ (\mathbf{QB})^T & \mathbf{W} \end{bmatrix} \geq \mathbf{0} \end{array} \right. \end{array} \tag{8.103}$$

The required estimator gain matrix can be computed as $\mathbf{L} = -\mathbf{Q}^{-1}\mathbf{Z}$.

8.4.3 $\mathcal{H}_2$ robust control problem

We have seen in section 6.4.1 that the H_2 robust control problem can be seen as two separate Linear-Quadratic-Regulator (LQR) design problems:

State feedback problem

The H_2 state feedback problem consists finding a gain $\mathbf{K}$ such that the input $\underline{u}(t) = -\mathbf{K}\underline{x}(t)$ stabilizes the closed loop system and minimizes the H_2 norm of transfer function $\mathbf{T}_{zw}(s)$.

For this problem we assume that state-space realization of the generalized plant is the following:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{D}_{12} \\ \mathbb{I} & \mathbf{0} & \mathbf{0} \end{array} \right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{8.104}$$

In addition the following assumptions are made:

- $(\mathbf{A}, \mathbf{C}_1)$ has no unobservable modes on the imaginary axis
- $(\mathbf{A}, \mathbf{B}_2)$ is stabilizable
- $\mathbf{D}_{12}^T \mathbf{D}_{12} > 0$ (thus the product is invertible)
- $\mathbf{D}_{12}^T \mathbf{C}_1 = \mathbf{0}$

The two last assumptions are not very restrictive and can be relaxed but they are convenient because they lead to simplifications in the solution of the problem.

We have seen in section 6.2.1 that the H_2 state feedback problem consists in finding the gain $\mathbf{K}$ which stabilizes the closed loop system and minimizes the H_2 norm of the following transfer function:

$$\mathbf{T}_{zw}(s) = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})\left(s\mathbb{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K}\right)^{-1}\mathbf{B}_1 \tag{8.105}$$

From the results of section 8.4.2, and setting $\mathbf{C} = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})$, it can be seen that minimizing the H_2-norm of $\mathbf{T}_{zw}(s)$ consists in solving the following semi-definite program where $\mathbf{P}$ stands for the controllability grammian of $\mathbf{T}_{zw}(s)$:

$$\begin{array}{ll} min & \text{tr}\left((\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})\,\mathbf{P}\,(\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})^T\right) \\ s.t. & \left\{ \begin{array}{l} (\mathbf{A} - \mathbf{B}_2\mathbf{K})\,\mathbf{P} + \mathbf{P}\,(\mathbf{A} - \mathbf{B}_2\mathbf{K})^T + \mathbf{B}_1\mathbf{B}_1^T \leq \mathbf{0} \\ \mathbf{P} = \mathbf{P}^T > \mathbf{0} \end{array} \right. \end{array} \tag{8.106}$$

Setting $\mathbf{C} = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})$ and using the fact that $\mathbf{Y} = -\mathbf{K}\mathbf{P}$ we get:

$$\mathbf{C}\mathbf{P} = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})\,\mathbf{P} = \mathbf{C}_1\mathbf{P} + \mathbf{D}_{12}\mathbf{Y} \tag{8.107}$$

Thus the results of section 8.4.2 now reads as follows: find $\mathbf{W} = \mathbf{W}^T$, $\mathbf{P} = \mathbf{P}^T$ and $\mathbf{Y}$ which solves the following problem:

$$\begin{array}{ll} min & \text{tr}\,(\mathbf{W}) \\ s.t. & \left\{ \begin{array}{l} \begin{bmatrix} \mathbf{A}\mathbf{P} + \mathbf{B}_2\mathbf{Y} + (\mathbf{A}\mathbf{P} + \mathbf{B}_2\mathbf{Y})^T & \mathbf{B}_1 \\ \mathbf{B}_1^T & -\mathbb{I} \end{bmatrix} \leq \mathbf{0} \\ \begin{bmatrix} \mathbf{P} & (\mathbf{C}_1\mathbf{P} + \mathbf{D}_{12}\mathbf{Y})^T \\ \mathbf{C}_1\mathbf{P} + \mathbf{D}_{12}\mathbf{Y} & \mathbf{W} \end{bmatrix} \geq \mathbf{0} \end{array} \right. \end{array} \tag{8.108}$$

The required state feedback gain matrix can be computed as $\mathbf{K} = -\mathbf{Y}\mathbf{P}^{-1}$.

Now assume $\mathbf{D}_{12}^T\mathbf{C}_1 \neq \mathbf{0}$. Then Doyle & al.[10] have shown that matrix $\mathbf{A}$ shall be replaced by matrix $\widetilde{\mathbf{A}}$ defined as follows:

$$\left\{ \begin{array}{l} \widetilde{\mathbf{A}} = \mathbf{A} - \mathbf{B}_2\widetilde{\mathbf{D}}_{12}\mathbf{D}_{12}^T\mathbf{C}_1 \\ \widetilde{\mathbf{D}}_{12} = \left(\mathbf{D}_{12}^T\mathbf{D}_{12}\right)^{-1} \end{array} \right. \tag{8.109}$$

[10]Doyle J.C., Glover K., Khargonekar P.P., Francis B. A., State-space solutions to standard H_2 and H_∞ control problems, IEEE Transactions on Automatic Control (Volume 34 , Issue 8), Aug 1989

State estimation problem

The H_2 state estimation problem consists finding a gain $\mathbf{L}$ such that the output $\underline{z}$ of the generalized plant properly estimates the actual state vector $\underline{x}(t)$ of the closed loop plant thanks to the measurement of output $\underline{y}(t)$ while minimizing the H_2 norm of transfer function $\mathbf{T}_{zw}(s)$.

For this problem we assume that state-space realization of the generalized plant is the following:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{array}\right] \begin{bmatrix} \underline{x}(t) \\ \underline{w}(t) \\ \underline{u}(t) \end{bmatrix} \tag{8.110}$$

while the state-space realization of the estimator is the following:

$$\begin{cases} \dot{\hat{\underline{x}}}(t) = \mathbf{A}\hat{\underline{x}}(t) + \mathbf{B}_2\underline{u}(t) + \mathbf{L}\left(\underline{y}(t) - \hat{\underline{y}}(t)\right) \\ \hat{\underline{y}}(t) = \mathbf{C}_2\hat{\underline{x}}(t) \end{cases} \tag{8.111}$$

In addition the following assumptions are made:

– $(\mathbf{A}, \mathbf{B}_1)$ has no uncontrollable modes on the imaginary axis

– $(\mathbf{A}, \mathbf{C}_2)$ is detectable

– $\mathbf{D}_{21}\mathbf{D}_{21}^T > 0$ (thus the product is invertible)

– $\mathbf{B}_1\mathbf{D}_{21}^T = \mathbf{0}$

The two last assumptions are not very restrictive and can be relaxed but they are convenient because they lead to simplifications in the solution of the problem.

We have seen in section 6.3.1 that the H_2 state feedback problem consists in finding the gain $\mathbf{K}$ which stabilizes the closed loop system and minimizes the H_2 norm of the following transfer function:

$$\mathbf{T}_{zw}(s) = \mathbf{C}_1\left(s\mathbb{I} - \mathbf{A} + \mathbf{L}\mathbf{C}_2\right)^{-1}\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right) \tag{8.112}$$

From the results of section 8.4.2, and setting $\mathbf{B} = (\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21})$, it can be seen that minimizing the H_2-norm of $\mathbf{T}_{zw}(s)$ consists in solving the following semi-definite program where $\mathbf{Q}$ stands for the observability grammian of $\mathbf{T}_{zw}(s)$:

$$\begin{array}{ll} min & \operatorname{tr}\left(\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)^T \mathbf{Q}\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)\right) \\ s.t. & \begin{cases} \mathbf{Q}\left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right) + \left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)^T\mathbf{Q} + \mathbf{C}_1^T\mathbf{C}_1 \leq \mathbf{0} \\ \mathbf{Q} = \mathbf{Q}^T > \mathbf{0} \end{cases} \end{array} \tag{8.113}$$

Setting $\mathbf{B} = (\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21})$ and using the fact that $\mathbf{Z} = -\mathbf{Q}\mathbf{L}$ we get:

$$\mathbf{Q}\mathbf{B} = \mathbf{Q}\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right) = \mathbf{Q}\mathbf{B}_1 + \mathbf{Z}\mathbf{D}_{21} \tag{8.114}$$

Thus the results of section 8.4.2 now reads as follows: find $\mathbf{W} = \mathbf{W}^T$, $\mathbf{Q} = \mathbf{Q}^T$ and $\mathbf{Z}$ which solves the following problem:

$$
\begin{array}{ll}
min & \operatorname{tr}(\mathbf{W}) \\
s.t. & \left\{ \begin{array}{l}
\begin{bmatrix} \mathbf{QA} + \mathbf{ZC}_2 + (\mathbf{QA} + \mathbf{ZC}_2)^T & \mathbf{C}_1^T \\ \mathbf{C}_1 & -\mathbb{I} \end{bmatrix} \leq \mathbf{0} \\
\begin{bmatrix} \mathbf{Q} & \mathbf{QB}_1 + \mathbf{ZD}_{21} \\ (\mathbf{QB}_1 + \mathbf{ZD}_{21})^T & \mathbf{W} \end{bmatrix} \geq \mathbf{0}
\end{array} \right.
\end{array}
\tag{8.115}
$$

The required estimator gain matrix can be computed as $\mathbf{L} = -\mathbf{Q}^{-1}\mathbf{Z}$.

Now assume $\mathbf{B}_1\mathbf{D}_{21}^T \neq \mathbf{0}$. Then Doyle & al.[10] have shown that matrix $\mathbf{A}$ shall be replaced by matrix $\widetilde{\mathbf{A}}$ defined as follows:

$$
\left\{ \begin{array}{l}
\widetilde{\mathbf{A}} = \mathbf{A} - \mathbf{B}_1\mathbf{D}_{21}^T\widetilde{\mathbf{D}}_{21}\mathbf{C}_2 \\
\widetilde{\mathbf{D}}_{21} = \left(\mathbf{D}_{21}\mathbf{D}_{21}^T\right)^{-1}
\end{array} \right.
\tag{8.116}
$$

8.4.4 $\mathcal{H}_\infty$ norm minimization

Similarly Scherer & al.[7] have shown that, for the state feedback problem, the $\mathcal{H}_\infty$ norm of the transfer function from $\underline{w}(t)$ to $\underline{z}(t)$ is less than γ_∞ if and only if there exists a symmetric positive definite matrix $\mathbf{P} = \mathbf{P}^T > 0$ and a matrix $\mathbf{Y}$ such that the following matrix inequalities hold:

$$
\|\mathbf{T}_{zw}(s)\|_\infty < \gamma_\infty \Leftrightarrow \left\{ \begin{array}{l}
\mathbf{P} = \mathbf{P}^T > 0 \\
\begin{bmatrix} \mathbf{A}_K^T\mathbf{P} + \mathbf{PA}_K & \mathbf{B}_1 & \mathbf{C}_1^T \\ \mathbf{B}_1^T & -\gamma_\infty\mathbb{I} & \mathbf{0} \\ \mathbf{C}_1 & \mathbf{0} & -\gamma_\infty\mathbb{I} \end{bmatrix} < 0
\end{array} \right.
\tag{8.117}
$$

where:

$$
\mathbf{A}_K = \mathbf{AP} + \mathbf{B}_2\mathbf{Y} \tag{8.118}
$$

By applying the duality principle and the Schur complement lemma, the state estimation problem is solved if and only if there exists a symmetric positive definite matrix $\mathbf{Q} = \mathbf{Q}^T > 0$ and a matrix $\mathbf{Z}$ such that the following matrix inequalities hold:

$$
\|\mathbf{T}_{zw}(s)\|_\infty < \gamma_\infty \Leftrightarrow \left\{ \begin{array}{l}
\mathbf{Q} = \mathbf{Q}^T > 0 \\
\begin{bmatrix} \mathbf{A}_L\mathbf{Q} + \mathbf{QA}_L^T & \mathbf{C}_1^T & \mathbf{B}_1 \\ \mathbf{C}_1 & -\gamma_\infty\mathbb{I} & \mathbf{0} \\ \mathbf{B}_1^T & \mathbf{0} & -\gamma_\infty\mathbb{I} \end{bmatrix} < 0
\end{array} \right.
\tag{8.119}
$$

where:

$$
\mathbf{A}_L = \mathbf{QA} + \mathbf{ZC}_2 \tag{8.120}
$$

8.5 Static output feedback control

8.5.1 Related LMIs

In this section we present how to design a static output feedback controller for a continuous time system.

Let's consider the following linear system:

$$\dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \tag{8.121}$$

With the following static output feedback:

$$\begin{cases} \underline{u}(t) = -\mathbf{K}\underline{y}(t) \\ \underline{y}(t) = \mathbf{C}\underline{x}(t) \end{cases} \Rightarrow \underline{u}(t) = -\mathbf{KC}\underline{x}(t) \tag{8.122}$$

Then the dynamics of the closed loop system reads:

$$\dot{\underline{x}}(t) = (\mathbf{A} - \mathbf{BKC})\underline{x}(t) \tag{8.123}$$

We recall that the linear autonomous system $\dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t)$ is stable if and only if there exists a symmetric and positive definite matrix $\mathbf{P} = \mathbf{P}^T > 0$ satisfying the following Lyapunov inequality:

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} < 0 \tag{8.124}$$

Applying the preceding inequality to the closed loop autonomous system $\dot{\underline{x}}(t) = (\mathbf{A}-\mathbf{BKC})\underline{x}(t)$ it can be stated that the static output feedback controller $\underline{u}(t) = -\mathbf{K}\underline{y}(t)$ leads to a stable closed loop system if and only if there exists a symmetric and positive definite matrix $\mathbf{P} = \mathbf{P}^T > 0$ satisfying the following Lyapunov inequality:

$$(\mathbf{A} - \mathbf{BKC})^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BKC}) < 0 \tag{8.125}$$

Cao & al.[11] has shown that Lyapunov inequality (8.125) is equivalent to the fact that there exists symmetric positive definite matrices $\mathbf{P}$ and a matrix $\mathbf{K}$ satisfying the following matrix inequality:

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PBB}^T\mathbf{P} + \left(\mathbf{B}^T\mathbf{P} + \mathbf{KC}\right)^T\left(\mathbf{B}^T\mathbf{P} + \mathbf{KC}\right) < 0 \tag{8.126}$$

The negative sign of the term $-\mathbf{PBB}^T\mathbf{P}$ in matrix inequality (8.126) renders it difficult to solve. Thus Cao & al.[11] introduces a matrix $\mathbf{\Phi}$ defined as follows where $\mathbf{X} > 0$:

$$\mathbf{\Phi} = \mathbf{X}^T\mathbf{BB}^T\mathbf{P} + \mathbf{PBB}^T\mathbf{X} - \mathbf{X}^T\mathbf{BB}^T\mathbf{X} \leq \mathbf{PBB}^T\mathbf{P} \tag{8.127}$$

It is worth noticing that the equal sign holds, that is $\mathbf{\Phi} = \mathbf{PBB}^T\mathbf{P}$, if and only if $\mathbf{X}^T\mathbf{B} = \mathbf{P}^T\mathbf{B}$.

Thanks to the Schur's complement (8.2), Lyapunov inequality (8.125) is equivalent to the fact that there exists symmetric positive definite matrices $\mathbf{P}$ and a matrix $\mathbf{K}$ satisfying the following linear matrix inequality:

$$\begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{\Phi} & \left(\mathbf{B}^T\mathbf{P} + \mathbf{KC}\right)^T \\ \mathbf{B}^T\mathbf{P} + \mathbf{KC} & -\mathbb{I} \end{bmatrix} < 0 \tag{8.128}$$

Thus, once $\mathbf{X} > 0$ is set, matrix inequality (8.128) reduces to an LMI.

[11] Cao, Y.-Y., Lam, J. & Sun, Y.-X. (1998). Static Output Feedback Stabilization: An ILMI Approach. Automatica 34, 1641- 1645.

8.5.2 Stabilizability by a static output feedback

Following Sadabadi & al.[12], Linear Time invariant (LTI) system (8.80) is stabilizable by a static output feedback $\mathbf{K}$ if and only if there exist matrix $\mathbf{Y}$ and symmetric matrices $\mathbf{P}$, $\mathbf{X}$ and $\mathbf{Z}$ such that the following constraints hold:

$$\left\{ \begin{array}{l} \mathbf{P} > 0 \\ \mathbf{Z} > 0 \\ \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ \mathbf{A} & \mathbf{B} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{P} \\ \mathbf{P} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ \mathbf{A} & \mathbf{B} \end{bmatrix} < \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}^T & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} \\ \mathbf{X} \leq \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}^T \end{array} \right. \tag{8.129}$$

The preceding relations are purely LMI except for the last constraint $\mathbf{X} \leq \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}^T$.

8.5.3 Changing PID controller into static output feedback

We present hereafter some results provided by Zheng & al.[13] which transforms a PID controller to static output feedback.

We consider the following linear time-invariant system:

$$\left\{ \begin{array}{l} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \\ \underline{y}(t) = \mathbf{C}\underline{x}(t) \end{array} \right. \tag{8.130}$$

And the following PID controller where matrices $\mathbf{K}_p$, $\mathbf{K}_i$ and $\mathbf{K}_d$ have to be designed:

$$\underline{u}(t) = -\left(\mathbf{K}_p\,\underline{e}(t) + \mathbf{K}_i \int_0^t \underline{e}(\tau)d\tau + \mathbf{K}_d \frac{d}{dt}\underline{e}(t)\right) \tag{8.131}$$

where:

$$\underline{e}(t) = \underline{y}(t) - \underline{r}(t) \tag{8.132}$$

Let's denote $\underline{x}_a(t)$ the augmented state-space vector defined as follows:

$$\underline{x}_a(t) = \begin{bmatrix} \underline{x}(t) \\ \int_0^t \underline{e}(\tau)d\tau \end{bmatrix} \tag{8.133}$$

Thus:

$$\dot{\underline{x}}_a(t) = \mathbf{A}_a\underline{x}_a(t) + \mathbf{B}_a\underline{u}(t) + \begin{bmatrix} \mathbf{0} \\ -\mathbb{I} \end{bmatrix} \underline{r}(t) \tag{8.134}$$

where:

$$\left\{ \begin{array}{l} \mathbf{A}_a = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \\ \mathbf{B}_a = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \end{array} \right. \tag{8.135}$$

[12] Mahdieh Sadabadi, Dimitri Peaucelle. From Static Output Feedback to Structured Robust Static Output Feedback: A Survey. Annual Reviews in Control, Elsevier, 2016, 42 (11-26), 10.1016/j.arcontrol.2016.09.014.

[13] Zheng, F., Wang, Q.-G. & Lee, T. H. (2002). On the design of multivariable PID controllers via LMI approach. Automatica 38, 517-526

Furthermore, assuming that $\dot{\underline{r}}(t) = \underline{0}$, we have:

$$\dot{\underline{r}}(t) = \underline{0} \Rightarrow \frac{d}{dt}\underline{e}(t) = \mathbf{C}\dot{\underline{x}}(t) = \mathbf{CA}\underline{x}(t) + \mathbf{CB}\underline{u}(t) \tag{8.136}$$

Using the definition of $\underline{x}_a(t)$, the PID controller reads:

$$\begin{array}{rl}
\underline{u}(t) & = -\left(\mathbf{K}_p\,\underline{e}(t) + \mathbf{K}_i \int_0^t \underline{e}(\tau)d\tau + \mathbf{K}_d \frac{d}{dt}\underline{e}(t)\right) \\
& = -\mathbf{K}_p\mathbf{C}\underline{x}(t) + \mathbf{K}_p\mathbf{C}\underline{r}(t) - \mathbf{K}_i\frac{d}{dt}\underline{e}(t) - \mathbf{K}_d\left(\mathbf{CA}\underline{x}(t) + \mathbf{CB}\underline{u}(t)\right) \\
& = -\mathbf{K}_p\left[\begin{array}{cc} \mathbf{C} & \mathbf{0} \end{array}\right]\underline{x}_a(t) - \mathbf{K}_i\left[\begin{array}{cc} \mathbf{0} & \mathbb{I} \end{array}\right]\underline{x}_a(t) - \mathbf{K}_d\left[\begin{array}{cc} \mathbf{CA} & \mathbf{0} \end{array}\right]\underline{x}_a(t) \\
& \qquad -\mathbf{K}_d\mathbf{CB}\underline{u}(t) + \mathbf{K}_p\mathbf{C}\underline{r}(t) \\
& = -\left[\begin{array}{ccc} \mathbf{K}_p & \mathbf{K}_i & \mathbf{K}_d \end{array}\right]\left[\begin{array}{cc} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \\ \mathbf{CA} & \mathbf{0} \end{array}\right]\underline{x}_a(t) - \mathbf{K}_d\mathbf{CB}\underline{u}(t) + \mathbf{K}_p\mathbf{C}\underline{r}(t)
\end{array} \tag{8.137}$$

We will assume that $\mathbb{I}+\mathbf{K}_d\mathbf{CB}$ is invertible and define $\mathbf{C}_a$ and $\mathbf{K}_a$ as follows:

$$\left\{\begin{array}{l}
\mathbf{C}_a = \left[\begin{array}{cc} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \\ \mathbf{CA} & \mathbf{0} \end{array}\right] \\
\mathbf{K}_a = \left(\mathbb{I} + \mathbf{K}_d\mathbf{CB}\right)^{-1}\left[\begin{array}{ccc} \mathbf{K}_p & \mathbf{K}_i & \mathbf{K}_d \end{array}\right]
\end{array}\right. \tag{8.138}$$

Let $\widetilde{\mathbf{K}}_p$, $\widetilde{\mathbf{K}}_i$ and $\widetilde{\mathbf{K}}_d$ be defined as follows:

$$\left\{\begin{array}{l}
\widetilde{\mathbf{K}}_p = \left(\mathbb{I} + \mathbf{K}_d\mathbf{CB}\right)^{-1}\mathbf{K}_p \\
\widetilde{\mathbf{K}}_i = \left(\mathbb{I} + \mathbf{K}_d\mathbf{CB}\right)^{-1}\mathbf{K}_i \\
\widetilde{\mathbf{K}}_d = \left(\mathbb{I} + \mathbf{K}_d\mathbf{CB}\right)^{-1}\mathbf{K}_d
\end{array}\right. \tag{8.139}$$

Assuming that $\widetilde{\mathbf{K}}_p$, $\widetilde{\mathbf{K}}_i$ and $\widetilde{\mathbf{K}}_d$ are known, gains $\mathbf{K}_p$, $\mathbf{K}_i$ and $\mathbf{K}_d$ are obtained as follows where it can be shown[13] that matrix $\mathbb{I} - \mathbf{CB}\widetilde{\mathbf{K}}_d$ is always invertible:

$$\left\{\begin{array}{l}
\mathbf{K}_d = \widetilde{\mathbf{K}}_d\left(\mathbb{I} - \mathbf{CB}\widetilde{\mathbf{K}}_d\right)^{-1} \\
\mathbf{K}_p = \left(\mathbb{I} + \mathbf{K}_d\mathbf{CB}\right)\widetilde{\mathbf{K}}_p \\
\mathbf{K}_i = \left(\mathbb{I} + \mathbf{K}_d\mathbf{CB}\right)\widetilde{\mathbf{K}}_i
\end{array}\right. \tag{8.140}$$

Thus the problem of PID controller design is changed into the following static output feedback problem:

$$\left\{\begin{array}{l}
\dot{\underline{x}}_a(t) = \mathbf{A}_a\underline{x}_a(t) + \mathbf{B}_a\underline{u}(t) \\
\underline{y}_a(t) = \mathbf{C}_a\underline{x}_a(t) \\
\underline{u}(t) = -\mathbf{K}_a\underline{y}_a(t) + \left(\mathbb{I} + \mathbf{K}_d\mathbf{CB}\right)^{-1}\mathbf{K}_p\mathbf{C}\underline{r}(t)
\end{array}\right. \tag{8.141}$$

It is worth noticing that the same results are obtained, but without the assumption that $\dot{\underline{r}}(t) = \underline{0}$, when a PI-D controller is used; for such a controller the term multiplied by $\mathbf{K}_d$ is $\underline{y}(t)$ rather than $\underline{e}(t)$:

$$\underline{u}(t) = -\left(\mathbf{K}_p\,\underline{e}(t) + \mathbf{K}_i\int_0^t \underline{e}(\tau)d\tau + \mathbf{K}_d\frac{d}{dt}\underline{y}(t)\right) \tag{8.142}$$

8.5.4 Optimal control

Vesely [14] has shown that Lyapunov inequality (8.125) is equivalent to the fact that there exists symmetric positive definite matrices $\mathbf{P}$ and $\mathbf{R}$ and a matrix $\mathbf{K}$ satisfying the following matrix inequality:

$$\begin{bmatrix} \mathbf{R} & \mathbf{B}^T\mathbf{P} - \mathbf{RKC} \\ \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right)^T & \mathbf{\Phi} \end{bmatrix} > 0 \tag{8.143}$$

Where $\mathbf{\Phi} = \mathbf{\Phi}^T$ is a symmetric matrix defined as follows:

$$\mathbf{\Phi} = -\left(\mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} - \mathbf{C}^T\mathbf{K}^T\mathbf{RKC}\right) \tag{8.144}$$

Using the Schur's complement (8.2) matrix inequality (8.143) reads:

$$\begin{cases} \mathbf{R} > 0 \\ \mathbf{\Phi} - \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right)^T \mathbf{R}^{-1} \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right) > 0 \end{cases} \tag{8.145}$$

To get Vesely's result [14] let's develop the following relation:

$$\begin{array}{ll} \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right)^T & \mathbf{R}^{-1} \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right) = \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} - \mathbf{PBKC} \\ & \quad -\mathbf{C}^T\mathbf{K}^T\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{K}^T\mathbf{RKC} \\ \Rightarrow \mathbf{\Phi} - & \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right)^T \mathbf{R}^{-1} \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right) = -\mathbf{A}^T\mathbf{P} - \mathbf{PA} \\ & +\cancel{\mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P}} + \cancel{\mathbf{C}^T\mathbf{K}^T\mathbf{RKC}} \\ & -\cancel{\mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P}} + \mathbf{PBKC} + \mathbf{C}^T\mathbf{K}^T\mathbf{B}^T\mathbf{P} - \cancel{\mathbf{C}^T\mathbf{K}^T\mathbf{RKC}} \\ \Leftrightarrow \mathbf{\Phi} - & \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right)^T \mathbf{R}^{-1} \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right) = -\mathbf{A}^T\mathbf{P} - \mathbf{PA} \\ & \quad +\mathbf{PBKC} + \mathbf{C}^T\mathbf{K}^T\mathbf{B}^T\mathbf{P} \end{array} \tag{8.146}$$

We finally get:

$$\begin{array}{l} \mathbf{\Phi} - \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right)^T \mathbf{R}^{-1} \left(\mathbf{B}^T\mathbf{P} - \mathbf{RKC}\right) = \\ \qquad\qquad -\left(\mathbf{A} - \mathbf{BKC}\right)^T\mathbf{P} - \mathbf{P}(\mathbf{A} - \mathbf{BKC}) \end{array} \tag{8.147}$$

Consequently as soon as (8.143) holds the Lyapunov inequality (8.125) also holds.

Using the Schur's complement (8.2) matrix inequality (8.143) can also be written as follows:

$$\begin{cases} \mathbf{\Phi} > 0 \\ \mathbf{R} - (\mathbf{B}^T\mathbf{P} - \mathbf{RKC})\mathbf{\Phi}^{-1}(\mathbf{B}^T\mathbf{P} - \mathbf{RKC})^T > 0 \end{cases} \tag{8.148}$$

From this basis Vesely[14] has demonstrated that the following statements are equivalent:

[14] Vojtech Vesely, Static Output Feedback Controller Design, Kybernetika, Volume 37 (2001), Number 2, Pages 205-221

– The system

$$\begin{cases} \underline{\dot{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \\ \underline{y}(t) = \mathbf{C}\underline{x}(t) \\ \underline{x}(0) = \underline{x}_0 \end{cases} \tag{8.149}$$

is static output feedback stabilizable with the following guaranteed cost J^*:

$$\int_0^\infty \left(\underline{x}^T(t)\mathbf{Q}\underline{x}(t) + u^T(t)\mathbf{R}\underline{u}(t)\right) dt < \underline{x}_0^T\mathbf{P}\underline{x}_0 = J^* \tag{8.150}$$

Where:

$$\begin{cases} \mathbf{P} = \mathbf{P}^T > 0 \\ \mathbf{R} = \mathbf{R}^T > 0 \\ \mathbf{Q} = \mathbf{Q}^T > 0 \end{cases} \tag{8.151}$$

– There exists symmetric positive definite matrices $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and a matrix $\mathbf{K}$ such that the following inequality holds:

$$(\mathbf{A} - \mathbf{BKC})^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BKC}) + \mathbf{Q} + \mathbf{C}^T\mathbf{K}^T\mathbf{RKC} < 0 \tag{8.152}$$

Compared with Lyapunov inequality (8.125), the preceding inequality may be obtained by introducing matrix $\mathbf{Q}$ defined as follows:

$$\mathbf{Q} \geq -\mathbf{C}^T\mathbf{K}^T\mathbf{RKC} \tag{8.153}$$

– There exists symmetric positive definite matrices $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and a matrix $\mathbf{K}$ such that the following inequality holds:

$$\begin{cases} \mathbf{\Phi} > 0 \\ \mathbf{R} - (\mathbf{B}^T\mathbf{P} - \mathbf{RKC})\mathbf{\Phi}^{-1}(\mathbf{B}^T\mathbf{P} - \mathbf{RKC})^T > 0 \end{cases} \tag{8.154}$$

Where $\mathbf{\Phi} = \mathbf{\Phi}^T$ is a symmetric matrix defined as follows:

$$\mathbf{\Phi} = -\left(\mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q}\right) \tag{8.155}$$

Defining $\mathbf{S} = \mathbf{S}^T = \mathbf{P}^{-1} > \gamma\mathbb{I}$ where $\gamma \geq 0$ is some non-negative constant and using Schur's complement the inequality $\mathbf{\Phi} > 0$ is equivalent to the following Linear Matrix Inequality (LMI) in $\mathbf{S}$:

$$\begin{array}{l} \mathbf{\Phi} = -\left(\mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q}\right) > 0 \\ \Leftrightarrow \begin{bmatrix} -\mathbf{SA}^T - \mathbf{AS} + \mathbf{BR}^{-1}\mathbf{B}^T & -\mathbf{S}\sqrt{\mathbf{Q}} \\ -\sqrt{\mathbf{Q}}\mathbf{S} & \mathbb{I} \end{bmatrix} > 0 \end{array} \tag{8.156}$$

Similarly when one knows $\mathbf{P} = \mathbf{S}^{-1}$ inequality $\mathbf{R} - (\mathbf{B}^T\mathbf{P} - \mathbf{RKC})\mathbf{\Phi}^{-1}(\mathbf{B}^T\mathbf{P} - \mathbf{RKC})^T > 0$ turns to be a Linear Matrix Inequality (LMI) in $\mathbf{K}$ thanks to the Schur's complement:

$$\begin{array}{l} \mathbf{R} - (\mathbf{B}^T\mathbf{P} - \mathbf{RKC})\mathbf{\Phi}^{-1}(\mathbf{B}^T\mathbf{P} - \mathbf{RKC})^T > 0 \\ \Leftrightarrow \begin{bmatrix} \mathbf{R} & \mathbf{B}^T\mathbf{P} - \mathbf{RKC} \\ (\mathbf{B}^T\mathbf{P} - \mathbf{RKC})^T & \mathbf{\Phi} \end{bmatrix} > 0 \end{array} \tag{8.157}$$

This leads to the Vesely's[14] algorithm for static output feedback stabilization of system (8.149) with guaranteed cost (8.150):

– Choose two symmetric definite positive matrices $\mathbf{Q}$ and $\mathbf{R}$:

$$\begin{cases} \mathbf{Q} = \mathbf{Q}^T > 0 \\ \mathbf{R} = \mathbf{R}^T > 0 \end{cases} \tag{8.158}$$

– Solve Linear Matrix Inequality (LMI) (8.156) in $\mathbf{S} = \mathbf{S}^T > \gamma\mathbb{I}$ where $\gamma \geq 0$ and set:

$$\mathbf{P} = \mathbf{S}^{-1} \tag{8.159}$$

– Solve Linear Matrix Inequality (LMI) (8.157) in $\mathbf{K}$ and set:

$$\underline{u}(t) = -\mathbf{K}\underline{y}(t) \tag{8.160}$$

– If the solutions of (8.156) and (8.157) are not feasible then system (8.149) is not stabilizable with the predefined values of matrices $\mathbf{Q}$ and $\mathbf{R}$; then change $\mathbf{Q}$ (you may try to decrease $\mathbf{Q}$) or/and $\mathbf{R}$ (you may try to increase $\mathbf{R}$).