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# NON DEGENERACY OF AFFINELIKE LIE ALGEBRA

2

MICHEL NGUIFO BOYOM<sup>†</sup> AND STEPHANE PUECHMOREL<sup>††</sup>

**ABSTRACT.** The aim of this note is to prove some cohomological vanishing theorems for non solvable affinelike Lie algebras, say ALLA. There are some relevant consequences of our vanishing theorems:

- (1) Every real non solvable affinelike Lie algebra is formally nondegenerate in the sense of A. Weinstein, (Theorem 1).
  - (2) Let  $G$  be a non solvable Lie group whose Lie algebra is an affinelike Lie algebra  $(\mathfrak{g}, \mathfrak{e})$ . If the radical of  $[\ker(\text{ad}(\mathfrak{e})), \ker(\text{ad}(\mathfrak{e}))]$  is commutative, then  $G$  admits a left invariant symplectic structure if and only if it has an open coadjoint orbit, (Theorem 2).
- (a) In Section 6, we use our vanishing theorems to supply an algebraic proof of the normal form theorem for Lie non solvable a-algebroids.

3

## 1. INTRODUCTION

4

Given a smooth Riemannian manifold  $(M, g)$ , a gauge structure on  $M$  is a couple  $(M, \nabla)$  where  $\nabla$  is a Koszul connection on  $TM$ . Using the metric  $g$ , one defines the dual connection  $\nabla^*$  of  $\nabla$  by the relation:

5

$$(1) \quad X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

7

with  $X, Y, Z$  arbitrary vector fields. In the field of information geometry, if both  $\nabla$  and  $\nabla^*$  are torsionless, the structure  $(M, g, \nabla, \nabla^*)$  is called a statistical manifold.

8

**Definition 1.** A section  $\theta \in \Gamma(TM^* \otimes TM)$  is said to be a solution of the gauge equation if, for any vector fields  $X, Y$ :

9

$$(2) \quad \nabla_X \theta Y = \theta \nabla_X^* Y.$$

If  $\theta$  is an invertible, antisymmetric (with respect to  $g$ ) solution of the gauge equation, then:

12

$$\omega: (X, Y) \in TM^2 \mapsto g(\theta X, Y).$$

is a  $\nabla$ -parallel symplectic form. In this paper,  $M$  is a smooth manifold of finite dimension,  $TM$  (resp.  $T^*M$ ) its tangent (resp. cotangent) bundle. A Poisson tensor on  $M$ , or a Poisson bivector, is a smooth section  $\Pi \in \Gamma(\wedge^2 TM)$  of the vector bundle  $\wedge^2 TM$  that satisfies:

15

$$(3) \quad [\Pi, \Pi] = 0$$

with  $[\cdot, \cdot]$  the Schouten bracket [21]. It is, in some sense, a dual notion to a 2-form.

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18 An example of such a tensor, that in fact has partly motivated this work, arises from the  
19 2-form  $\omega$  defined above. It is given by the relation:

$$\Pi_\omega(df, dg) = dg(X_f), \quad f, g \in C^\infty(M),$$

20 where  $X_f$  is the Hamiltonian vector field associated to  $f$ . The couple  $(M, \Pi)$  is called a  
21 Poisson manifold. It is coherent with the more common definition using the Poisson bracket  
22  $\{\cdot, \cdot\}$  on  $C^\infty(M)$ -function by the identification:

$$\{f, g\} = \Pi(df, dg).$$

23 *Remark 1.* A Poisson tensor can be expressed in local coordinates as:

$$\Pi = \pi^{ij} X_i X_j$$

24 where  $\pi^{ij} = -\phi^{ji}$  and  $X_i$  is the  $i$ -th coordinate field.

25 A point  $x \in M$  is said to be regular for  $\Pi$  if the linear mapping:

$$(4) \quad \omega \in T^*M \mapsto \Pi(x; \eta, \omega)$$

26 vanishes identically only when  $\eta = 0$ . The next definition is dual to the  $\sharp$  morphism in the  
27 context of Riemannian manifolds.

28 **Definition 2.** For any  $\omega \in T^*M$ , the mapping:

$$(5) \quad \omega^\natural: \alpha \in T^*M \rightarrow \Pi(\omega, \alpha)$$

29 is an element of  $TM$ .

30 *Remark 2.* The morphism  $\natural$  is commonly represented by  $\sharp$ . In this paper, we want to  
31 distinguish it from its metric version, hence the notation.

32 **Definition 3.** A Lie algebroid is a triple  $(A, M, \rho)$  where  $M$  is a smooth manifold,  $A$  is  
33 a vector bundle on  $M$  with a Lie bracket  $[\cdot, \cdot]$  defined on its sections and  $\natural: A \rightarrow TM$  a  
34 vector bundle morphism such that, for any  $f \in C^\infty(M)$  and any vector fields  $X, Y \in \Gamma(A)$ :

$$(6) \quad [X, fY] = f[X, Y] + \rho(X)(f)Y.$$

35 The next proposition was obtained in [11][Theorem 8.1]

36 **Proposition 1.** A Poisson manifold  $(M, \Pi)$  gives rise to a Lie algebroid  $(T^*M, M, \natural)$ ,  
37 where:

$$\natural(\alpha) = i(\alpha)\Pi.$$

38 and the Lie bracket on sections of  $T^*M$  is given by the relation:

$$(7) \quad [\alpha, \beta] = i_{\natural\alpha}d\beta - i_{\natural\beta}d\alpha + d(i_{\Pi}\alpha \wedge \beta)$$

39 Let  $\Pi$  be a smooth Poisson tensor on  $M$  and let:

$$A_\Pi = (T^*M, \natural)$$

40 be its associated Lie algebroid as in proposition 1. Let  $x_0$  be a singular point of  $\Pi$ , that is  
41 a point where the Poisson tensor vanishes. In a local coordinate system  $x^i, i = 1 \dots n$ , one  
42 has the Taylor expansion for each component  $\pi^{ij}$  of  $\Pi$ :

$$(8) \quad \pi^{ij} = \sum_{k \geq 1} \sum_{a_1, \dots, a_k=1}^2 \frac{1}{k!} (X_{a_1} \dots X_{a_k} \pi^{ij}) x^{a_1} \dots x^{a_k}.$$

43 Gathering terms, one can write:

$$(9) \quad \Pi = \Pi^1 + \Pi^2 + \dots$$

44 where  $\Pi^k$  stands for the bivector corresponding the homogeneous expansion coefficients  
45 of order  $k$ . The same expansion can be conducted on  $\mathfrak{b}$ . The Taylor expansion of  $\Pi$  (resp.  
46  $\mathfrak{b}$ ) at  $x_0$  will be designated in the sequel by  $\tilde{\Pi}$  (resp.  $\tilde{A}_\Pi$ ). Both  $\tilde{\Pi}$  and  $\tilde{A}_\Pi$  have the same  
47 linear part, namely  $\Pi^1$ , whose expression in local coordinates is:

$$(10) \quad \Pi^1 = X_k(\pi^{ij})(x_0)x^k X_i X_j.$$

48 This linear part defines a Lie algebra structure in the vector space  $T_{x_0}^*M$  with structure  
49 constants  $c_{ij}^k = X_k(\pi^{ij})$ . We denote this Lie algebra by  $(T_{x_0}^*M, \Pi^1)$  and call it the  
50 linearization of the Poisson structure.

51 **Definition 4.** A Lie algebra  $\mathfrak{g}$  is said to be non-degenerate in the sense of A. Weinstein  
52 if, for any Poisson structure  $\Pi$  admitting  $A$  as a linearization at a singular point  $x_0$ , there  
53 exists a local coordinate system  $x^i, i = 1 \dots n$  such that each  $\pi^{ij}$  is a linear function:  
54  $\pi^{ij}(x) = a_k^{ij} x^k$ .

55 An important and largely open question in the Poisson geometry is to determinate those  
56 Lie algebras which are non degenerate in the sense of A. Weinstein. If a Lie algebra  $\mathfrak{g}$  is  
57 formally ( resp. analytically or smoothly ) non degenerate in the sense of A. Weinstein,  
58 then, for every formal (resp. analytic or smooth ) Poisson tensor  $\Pi$  the equality:

$$(T_{x_0}^*M, \Pi^1) = \mathfrak{g}$$

59 implies the formal (resp. analytic or smooth ) linearization (near  $x_0$ ) of the Poisson tensor  
60  $\Pi$ . Jean-Paul Dufour and Nguyen Tien Zung have proved that for every positive integer  $m$ ,  
61 the affine Lie algebra

$$\text{Aff}(m) = \mathfrak{gl}(m) \ltimes \mathbb{R}^m$$

62 is analytically non degenerate in the sense of Weinstein, [13]. The list of known types of  
63 non degenerate Lie algebras is short, (see [15], [13]). The linearization of  $\Pi$  implies that  
64  $A_\Pi$ , is linearizable as well. The inverse generally fails to hold. However, Jean-Paul Dufour  
65 conjectures that the formal linearization would imply the analytic linearization of Poisson  
66 structures whose ranks are  $> 2$ , [14]. In the analytic category, the formal linearization of  $\Pi$   
67 implies its analytic linearization whenever the linear part  $(T_{x_0}^*M, \Pi^1)$  is a semi-simple Lie  
68 algebra, [9, 10]. The formal linearization problem may be considered from two points of  
69 view. The first one is the splitting problem of extensions of Lie algebras, [27]. The second  
70 viewpoint is that of formal deformations of Lie algebra structures. Both viewpoints are  
71 largely governed by the Chevalley-Eilenberg cohomology of Lie algebras, [15, 25].

72 The concern of this work is the study of the formal nondegeneracy of a new class of Lie  
73 algebras, say affinelike Lie algebras, (or ALLA). We recall that in a Lie algebra, the adjoint  
74  $\text{ad } \mathfrak{g}: \text{Der}(\mathfrak{g})$  maps  $a \in \mathfrak{g}$  to the derivation  $\text{ad}(a): x \rightarrow [a, x]$ . A affinelike Lie algebra is a  
75 Lie algebra that contains contains an element  $e$  such that  $\text{ad}(e)$  satisfies:

$$\text{ad}^2(e) = \text{ad}(e)$$

76 Let us represent an affinelike Lie algebra by the couple  $(\mathfrak{g}, e)$ .

77 From the Poisson geometry viewpoint, central results of this work yield the next two  
78 theorems:

79 **Theorem 1.** *Every non solvable affinlike Lie algebra is formally non degenerate in the*  
 80 *sense of A. Weinstein. A connected Lie group  $G$  whose Lie algebra  $\mathfrak{g}$  is an affinlike Lie*  
 81 *algebra, namely  $(\mathfrak{g}, e)$ , is labelled affinlike Lie group.*

82 **Theorem 2.** *Let:  $(\mathfrak{g}, e)$  be the Lie algebra of a connected non solvable affinlike Lie*  
 83 *group  $G$ . Let us set  $\mathfrak{g}_e = \ker(\text{ad}(e))$  and,  $\Delta\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$ . Suppose the radical of  $\Delta\mathfrak{g}_e$*   
 84 *to be commutative. If  $G$  carries a left invariant symplectic structure, then it has an open*  
 85 *coadjoint orbit in the dual vector space of its Lie algebra. Moreover, if the ground field*  
 86 *is the field of complex numbers, then the open coadjoint orbit is unique and everywhere*  
 87 *dense.*

88 The paper is organized as follow: In section 2, The notion of affinlike Lie algebra is  
 89 introduced, then in section 3, the cohomology of polynomial maps is defined, with a view  
 90 towards linearization problems. Section 4 is a short one on weak degeneracy. Finally, section  
 91 5 and 6 give the contributions of the paper, the last one being an application of theorem  
 92 4.

## 93 2. AFFINELIKE LIE ALGEBRAS

94 Let  $\mathbb{K}$  be either the field  $\mathbb{R}$  of real numbers or the field  $\mathbb{C}$  of complex numbers. Let  $\mathfrak{g}$  be a  
 95 Lie algebra over  $\mathbb{K}$ . Given  $e \in \mathfrak{g}$  we shall set:

$$\begin{cases} \Delta\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \\ \Delta_e\mathfrak{g} = [e, \mathfrak{g}] \\ \mathfrak{g}_e = \ker(\text{ad}(e)) \end{cases}$$

96 Let  $\mathfrak{S}$  be a Levi subalgebra of  $\Delta\mathfrak{g}_e$ . In the sequel,  $R(\mathfrak{g})$  will be the radical of the Lie algebra  
 97  $\mathfrak{g}$ .

98 **Definition 5.** A Lie algebra  $\mathfrak{g}$  is called affinlike Lie algebra, 'ALLA', if it contains an  
 99 element  $e$  such that:

100 (alla 1)  $\text{ad}^2(e) = \text{ad}(e)$ .

101 (alla 2)  $\mathfrak{g} = \mathbb{K}e \oplus \Delta\mathfrak{g}$ .

102 (alla 3) If  $\mathfrak{g}$  is non solvable, then  $H^0(\mathfrak{S}, R(\Delta\mathfrak{g})) = 0$ .

103 According to property (alla 2) above,  $\mathfrak{g}$  is a semi-direct product of  $\Delta\mathfrak{g}$  and  $\mathbb{K}e$ . Here are  
 104 two elementary properties of affinlike Lie algebras.

105 **Proposition 2.** *The subspace  $\Delta_e\mathfrak{g}$  is a commutative ideal of  $\mathfrak{g}$ .*

106 *Proof.* Let  $a, b \in \Delta_e\mathfrak{g}$ . Then:

$$[e, [a, b]] = 2[a, b].$$

107 On the other hand one has:

$$[e, [e, [a, b]]] = [e, [a, b]].$$

108 Combining the two equalities above one easily sees that:

$$[a, b] = 0,$$

109 proving that  $\Delta_e \mathfrak{g}$  is commutative. To see that it is an ideal of  $\mathfrak{g}$ , we remark that as a vector  
 110 space,  $\mathfrak{g}$  admits the following decomposition:

$$\mathfrak{g} = \Delta_e \mathfrak{g} \oplus \mathfrak{g}_e$$

111 Now given  $x \in \mathfrak{g}$  let us write it as:

$$x = u + v, u \in \Delta_e \mathfrak{g}, v \in \mathfrak{g}_e$$

112 Given  $a \in \Delta_e \mathfrak{g}$ , one has:

$$[e, [x, a]] = [e, [u, a]] = [u, [e, a]] = [u, a] = [x, a].$$

113 proving that  $\Delta_e \mathfrak{g}$  is an ideal of  $\mathfrak{g}$ . □

114 **Proposition 3.** *Since  $\text{ad}(e)$  is a derivation of the Lie algebra  $\mathfrak{g}$ , the subspace  $\mathfrak{g}_e$  is a*  
 115 *subalgebra of  $\mathfrak{g}$ .*

116 As a Lie algebra,  $\mathfrak{g}$  may be presented as the semi-direct product:

$$\mathfrak{g} = \mathfrak{g}_e \ltimes \Delta_e \mathfrak{g}.$$

117 Putting:

$$\tilde{\mathfrak{g}}_e = \mathfrak{g} / \Delta_e \mathfrak{g}$$

118 we have an exact split sequence of Lie algebras:

$$(11) \quad 0 \longrightarrow \Delta_e \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}_e \longrightarrow 0$$

119 *Remark 3.* Under the adjoint representation of  $\mathfrak{g}$  in itself, 11 is an exact sequence of  
 120  $\mathfrak{g}$ -modules which is generally not split. However, it is split as a sequence of  $\mathfrak{g}_e$ -modules.

121 Now, let  $W$  be a finite dimensional  $\mathfrak{g}$ -module. From 11 we deduce the following exact  
 122 sequence of  $\mathfrak{g}$ -modules:

$$(12) \quad 0 \longrightarrow \text{Hom}(W, \Delta_e \mathfrak{g}) \longrightarrow \text{Hom}(W, \mathfrak{g}) \longrightarrow \text{Hom}(W, \tilde{\mathfrak{g}}_e) \longrightarrow 0$$

123 As an exact sequence of  $\mathfrak{g}_e$ -modules, 12 is split. Let us denote by  $C^*(\mathfrak{g}, \text{Hom}(W, \mathfrak{g}))$  the  
 124 Chevalley-Eilenberg complex of  $\mathfrak{g}$  with coefficients in  $\text{Hom}(W, \mathfrak{g})$ , [8]. Then, for every  
 125 non negative integer  $k$ , we denote by  $H^k(\mathfrak{g}, \text{Hom}(W, \mathfrak{g}))$  the  $k$ -th cohomology space of  
 126  $H(\mathfrak{g}, \text{Hom}(W, \mathfrak{g}))$ . From Remark 3, one has the following identity:

$$H^k(\mathfrak{g}_e, \text{Hom}(W, \mathfrak{g})) = H^k(\mathfrak{g}_e, \text{Hom}(W, \Delta_e \mathfrak{g})) \oplus H^k(\mathfrak{g}_e, \text{Hom}(W, \tilde{\mathfrak{g}}_e))$$

127 This digression might be combined with the Hochschild-Serre spectral sequence to be useful  
 128 in calculating the cohomology spaces  $H^*(\mathfrak{g}, \text{Hom}(W, \mathfrak{g}))$ . However, a direct calculation  
 129 of  $H^*(\mathfrak{g}, \text{Hom}(W, \mathfrak{g}))$  is the purpose of the next sections.

130

## 3. COHOMOLOGY OF POLYNOMIAL MAPS

131 Let  $S(\mathfrak{g})$  be the symmetric algebra of the vector space  $\mathfrak{g}$ . Our purpose is to calculate  
 132 the cohomology space  $H^2(\mathfrak{g}, \text{Hom}(W, \mathfrak{g}))$  where  $W$  is a homogeneous submodule of the  
 133 graded  $\mathfrak{g}$ -module:

$$S^+(\mathfrak{g}) = \bigoplus_{i \geq 0} S^i(\mathfrak{g})$$

134 In the sequel, we shall identify  $\text{Hom}(W, F)$  with  $W^* \otimes F$ , where  $W^*$  is the dual vector space  
 135 of  $W$  and  $F \in \{\Delta_e \mathfrak{g}, \mathfrak{g}, \tilde{\mathfrak{g}}_e\}$ . According to the exact sequence 12, we have the classical long  
 136 exact sequence:

$$(13) \quad \begin{array}{ccccc} H^i(\mathfrak{g}, W^* \otimes \Delta_e \mathfrak{g}) & \longrightarrow & H^i(\mathfrak{g}, W^* \otimes \mathfrak{g}) & \longrightarrow & H^i(\mathfrak{g}, W^* \otimes \tilde{\mathfrak{g}}_e) \\ & & & \swarrow & \\ H^{i+1}(\mathfrak{g}, W^* \otimes \Delta_e \mathfrak{g}) & \longrightarrow & \dots & & \end{array}$$

137 Before pursuing, let us give some examples of affinelike Lie algebras.

138 *Example 1.* Let  $V$  be a vector space over  $\mathbb{K}$ , then the Lie algebra:

$$\text{Aff}(V) = \mathfrak{gl}(V) \ltimes V$$

139 is an ALLA where  $e$  is the identity endomorphism of  $V$ .

140 *Example 2.* Let  $(V, \langle \cdot, \cdot \rangle)$  be a real euclidean space. Its Lie algebra of affine endomorphisms  
 141 with conformal linear part is an ALLA.

142 *Example 3.* Let  $\mathfrak{g} = \mathbb{K} \text{Id}_{\mathbb{K}^m} \ltimes K^m$ . Then  $\mathfrak{g}$  is an ALLA.

143 *Example 4.* Let  $M(m, n)$  be the vector space of  $m \times n$  matrices with entries in  $\mathbb{K}$ . Let:

$$\mathfrak{g}_{m,n} = \mathfrak{gl}(\mathbb{K}^m) \ltimes M(m, n)$$

144 with bracket:

$$[(a, b), (a', b')] = ([a, a'], ab' - a'b)$$

145 Let  $e$  be the unit matrix in  $\mathfrak{gl}(\mathbb{K}^m)$ . Then  $(\mathfrak{g}, e)$  is an ALLA. The algebras  $\mathfrak{g}_{m,n}$  are sometimes  
 146 called affine algebras. Their coadjoint representations are deeply studied in [24].

147 **Proposition 4.** *Let  $\mathfrak{g}$  be an ALLA. Then:*

$$H^2(\mathfrak{g}, \mathbb{K}) = H^2([\mathfrak{g}_e, \mathfrak{g}_e], \mathbb{K})$$

148 *Proof.* Let  $e \in \mathfrak{g}$  be an element satisfying properties (alla 1) and (alla 2). Given a scalar  
 149 2-cocycle  $\theta \in C^2(\mathfrak{g}, \mathbb{K})$ , the cartan formula:

$$L(e)\theta = i(e)\partial + \partial i(e)$$

150 implies that:

$$L(e)\theta = \partial i(e)\theta.$$

151 Now, let  $x = s + u, x' = s' + u' \in \mathfrak{g}_e \oplus \Delta_e \mathfrak{g}$ . One easily verifies that:

$$\theta(u, u') = \theta(e, [s, s']) = 0$$

152 The restriction map from  $H^2(\mathfrak{g}, \mathbb{K})$  to  $H^2(\mathfrak{g}_e, \mathbb{K})$  is thus one-to-one. To see that it is onto,  
153 we remark that (alla 2) implies that:

$$\mathfrak{g}_e = \mathbb{K}e \oplus [\mathfrak{g}_e, \mathfrak{g}_e]$$

154 Moreover, the same (alla 2) shows that the Lie algebra  $[\mathfrak{g}_e, \mathfrak{g}_e]$  is perfect. Thus, we have:

$$H^1(\mathfrak{g}_e, \mathbb{K}) = 0$$

155 and

$$(H^1(\mathbb{K}e, \mathbb{K}))^{\mathfrak{g}_e} = 0$$

156 Owing to the Hochschild-Serres exact sequence [18], one sees that the inflation map from  
157  $H^2([\mathfrak{g}_e, \mathfrak{g}_e], \mathbb{K})$  to  $H^2(\mathfrak{g}_e, \mathbb{K})$  is injective, thus proving the claim.  $\square$

158 In the particular case where  $\mathfrak{g} = \text{Aff}(m)$  and  $e = Id_{\mathbb{K}^m}$ , we have  $\mathfrak{g}_e = \mathfrak{gl}(m, \mathbb{K})$  and:

$$H^2(\mathfrak{gl}(m, \mathbb{K}), \mathbb{K}) = \sum_{p+q=2} H^p(\mathbb{K}e, \mathbb{K}) \otimes H^q(\mathfrak{sl}(m, \mathbb{K}), \mathbb{K}) = 0$$

159 The reader may consult [1] for another proof of the latter vanishing theorem, namely:

$$H^2(\text{Aff}(\mathbb{K}^m), \mathbb{K}) = 0$$

160 The proof given here is simpler. Another remark to be made is that every affine group:

$$\text{Aff}(\mathbb{K}^m) = \text{GL}(\mathbb{K}^m) \ltimes \mathbb{K}^m$$

161 has open coadjoint orbits. In other words, every affine group  $\text{Aff}(V)$  carries a left invariant  
162 symplectic structure, [24]. Indeed, it is sufficient to identify the dual vector space of the  
163 Lie algebra  $\mathfrak{aff}(m)$  with itself under the transposition map. In other words, considering  
164  $\mathfrak{aff}(m)$  as a subspace of  $\mathfrak{gl}(m+1, \mathbb{K})$ , one deals with the inner product:

$$\langle A, B \rangle = \text{Tr}(A^t B)$$

165 Let:

$$(A, v) \in \mathfrak{gl}(m) \ltimes \mathbb{K}$$

166 be a  $m$ -cyclic couple in the following sense:  $\{v, Av, \dots, A^{m-1}v\}$  is a basis of the vector  
167 space  $\mathbb{K}^m$ . The couple  $(A, v)$  being regarded as an element of  $\mathfrak{gl}(m+1, \mathbb{K})$ , its transpose,  
168 say  $(A, v)^t$ , is an element of the dual space of  $\mathfrak{aff}(m)$ . The property to be  $m$ -cyclic implies  
169 that the coadjoint orbit of  $(A, v)^t$  is an open subset of the dual vector space of  $\mathfrak{aff}(m)$ . For  
170 instance, take the square  $m \times m$  matrix  $A = (a_{ij})$  whose nonzero entries are:

$$\begin{cases} a_{i+1, i} = 1 \\ a_{1, m} = 1 \end{cases}$$

171 Thus,  $A$  is an  $m$ -cyclic matrix. Take the vector  $v = (1, 0, \dots, 0) \in \mathbb{K}^m$ . Then, the couple  
172  $(A, v)$  being regarded as an element of the dual space of  $\mathfrak{aff}(m)$  has an open coadjoint  
173 orbit.



## 4. WEAK NONDEGENERACY

174

175 Let  $\Pi$  be a smooth Poisson tensor on the smooth manifold  $\mathbb{K}^m$ . Let us suppose  $0$  to be a  
 176 singularity of  $\Pi$  and let  $\hat{\Pi}$  be the Taylor expansion at  $0$  of  $\Pi$ . The linear part of  $\Pi$ , say  $\Pi^1$ ,  
 177 defines a Lie algebra structure in the vector space of linear functions. Let us denote this  
 178 Lie algebra structure by  $(\mathbb{K}^{*m}, \Pi^1)$ . Following Weinstein, a Lie algebra  $\mathfrak{g}$  is called formally  
 179 (resp. analytically or smoothly) non degenerate if every formal (resp. analytic or smooth)  
 180 Poisson tensor  $\Pi$  satisfying both conditions:

$$\Pi(0) = 0$$

181 and

$$(\mathbb{K}^{*m}, \Pi^1) = \mathfrak{g}$$

182 is formally (resp. analytically or smoothly) linearizable at  $0$ . In other words,  $\Pi$  is formally  
 183 (resp. analytically or smoothly) linearizable at  $0$  if (near  $0$ ) there exist a system of formally  
 184 (resp. analytic or smooth) coordinate functions:

$$F := (f_1, \dots, f_m)$$

185 such that:

$$\Pi(df_i, df_j) = \sum_k c_{ij}^k f_k$$

186 with  $c_{ij}^k \in \mathbb{K}$ . Now, to every smooth Poisson tensor  $\Pi$  on a smooth manifold  $M$  is assigned  
 187 the Lie algebroid structure  $(T^*M, \mathfrak{h})$ . The anchor map  $\mathfrak{h}$  is the vector bundle homomor-  
 188 phism from  $T^*M$  to  $TM$  defined by:

$$\mathfrak{h}(\alpha) = i(\alpha)\Pi.$$

189 The anchor map  $\mathfrak{h}$  and the Poisson tensor  $\Pi$  have the same singularities. In particular  
 190  $\mathfrak{h}(0) = 0$ . The linear part of the Taylor expansion at  $0$  of  $\mathfrak{h}$  defines a Lie algebra structure  
 191 in the cotangent space  $T_0^*M$ . That Lie algebra structure is the same as the one defined by  
 192  $\Pi^1$ . So, it is easy to see that the Taylor expansion at  $0$  of  $\mathfrak{h}$  defines a formal deformation of  
 193 the linear Lie algebroid:

$$\mathfrak{g} \times T_0M$$

194 where:

$$\mathfrak{g} = (T_0^*M, \Pi^1).$$

195 **Definition 6.** A Lie algebra  $\mathfrak{g}$  is called weakly formally (resp. weakly analytically or weakly  
 196 smoothly) non degenerate in the sense of Weinstein if the following conditions hold: let  $\Pi$   
 197 be a formal (resp. analytical or smooth) Poisson structure such that:

$$\begin{cases} \Pi(0) = 0. \\ \mathfrak{g} = (T_0^*M, \Pi^1). \end{cases}$$

198 then the corresponding formal (resp. analytic or smooth) Lie algebroid structure  $(T^*M, \mathfrak{h})$   
 199 is formally (resp. analytically or smoothly) linearizable at  $0$ .

200 The formal (resp. analytic or smooth) linearization at  $0$  of  $\Pi$  implies the formal (resp.  
 201 analytic or smooth) linearization at  $0$  of its associated Lie algebroid. The converse doesn't  
 202 hold, even formally.

203

## 5. MAIN RESULTS

204 From now on, we shall represent an ALLA by the couple  $(\mathfrak{g}, e)$ . We fix once for all an  
 205 homogeneous submodule  $W$  of the graded  $\mathfrak{g}$ -module  $S^+(\mathfrak{g})$ . Then, one has the following  
 206 exact sequence of  $\mathfrak{g}$ -modules:

$$(14) \quad 0 \longrightarrow \text{Hom}(W, \Delta_e \mathfrak{g}) \longrightarrow \text{Hom}(W, \mathfrak{g}) \longrightarrow \text{Hom}(W, \tilde{\mathfrak{g}}_e) \longrightarrow 0$$

207 We shall denote by  $W^l$  the subspace of  $W$  consisting of homogeneous elements of degree  $l$ .  
 208 Let  $Z(\Delta_e \mathfrak{g})$  be the centralizer of  $\Delta_e \mathfrak{g}$  in the Lie algebra  $\mathfrak{g}$  and let  $R(\Delta_e \mathfrak{g})$  be the radical of  
 209 the Lie algebra  $\Delta_e \mathfrak{g}$ . Keeping these notations, the main concern of this section is to prove  
 210 the next statements:

**Theorem 3.** *Let  $(\mathfrak{g}, e)$  be a non solvable affineline Lie algebra. Then, for every positive integer  $l$ , the following Chevalley-Eilenberg cohomology spaces vanish:*

$$H^2(\mathfrak{g}, \text{Hom}(W^l, \Delta_e \mathfrak{g})),$$

$$H^2(\mathfrak{g}, \text{Hom}(W^l, \tilde{\mathfrak{g}}_e)).$$

211 **Theorem 4.** *Let  $(\mathfrak{g}, e)$  be a non solvable affineline Lie algebra. Then, for every homogeneous*  
 212 *submodule  $V$  of the graded  $\mathfrak{g}$ -module  $S^+ \mathfrak{g}$ :*

$$\text{Ext}_{\mathfrak{g}}^2(V, \mathfrak{g}) = 0$$

213 **Theorem 5.** *Every non solvable affineline Lie algebra is formally (resp. analytically)*  
 214 *weakly non degenerate in the sense of Alan Weinstein.*

215 Before stating the proofs of the theorems which are stated above, it is useful to fix some  
 216 notations. Let  $\mathfrak{g}_e = R(\mathfrak{g}_e) + S$  be a fixed Levi decomposition of  $\mathfrak{g}_e$ . Therefore, the vector  
 217 space  $\mathfrak{g}$  may be decomposed as follows:

$$\mathfrak{g} = \Delta_e \mathfrak{g} + R(\mathfrak{g}_e) + S$$

218 Each subspace  $W^l$  is multi-graded as indicated below:

$$W^l = \bigoplus W^{t,r,s}$$

219 where  $t, r$  and  $s$  are non negative integers such that  $t + r + s = l$  and  $W^{t,r,s}$  is a subspace of  
 220 the vector space:

$$S^t(\Delta_e \mathfrak{g}) \otimes S^r(R(\mathfrak{g}_e)) \otimes S^s(S).$$

221 Every vector space  $C^k(\mathfrak{g}, \text{Hom}(W^l, \Delta_e \mathfrak{g}))$  is a  $\mathfrak{g}$ -module under the prolongation of the  
 222 adjoint representation. Thus, for each  $a \in \mathfrak{g}$ , the action of  $a$  (resp. the inner product by  $a$ )  
 223 is denoted by  $L_{(a)}$  (resp.  $i(a)$ ). The Chevalley-Eilenberg coboundary operator, namely  $\partial$ ,  
 224 is related to  $L_{(a)}$  by the next Cartan formula:

$$L_{(a)} = \partial i(a) + i(a)\partial.$$

225 Let  $\xi \in W^l$ . According to the previous notations, we shall write  $\xi$  as follows:

$$\xi = \sum_{t,r,s} \xi^{t,r,s}$$

226 It is easy to see that  $L_{(e)} \xi^{t,r,s} = t \xi^{t,r,s}$ . Now, let  $\theta \in Z^2(\mathfrak{g}, W^l \otimes \Delta_e \mathfrak{g})$ . Following the  
 227 Cartan formula we get:

$$L_{(e)} \theta = \partial i(e)\theta.$$

228 Before pursuing, let  $\wedge^k \mathfrak{g}$  be the  $k$ -th exterior power of the vector space  $\mathfrak{g}$ . Then, we can  
 229 decompose it as follows:

$$(15) \quad \wedge^k \mathfrak{g} = \bigoplus^m \wedge^m \Delta_e \mathfrak{g} \wedge \wedge^n R(\mathfrak{g}_e) \wedge \wedge^p S.$$

230 with  $m + n + p = k$ . We are now in position to prove theorem 3.

231 **5.1. Proof of theorem 3.** The proof has been divided into two parts.

**Part 1**

*Step 1*

232 Let  $u, v \in \Delta_e \mathfrak{g}$  and  $\xi^{t,r,s} \in W^{t,r,s}$ . A direct calculation of:

$$\left( (L_{(e)}\theta)(u, v) \right) (\xi^{t,r,s})$$

233 yields the next result:

$$\left( (L_{(e)}\theta)(u, v) \right) (\xi^{t,r,s}) = (-1 + t) (\theta(u, v)) (\xi^{t,r,s})$$

234 For each  $\phi \in W^l \oplus \Delta_e \mathfrak{g}$ , the  $W^{t,r,s}$  component of  $\phi$  is denoted by  $\phi^{t,r,s}$ . The calculations  
 235 made above show that for every 2-cocycle:

$$\theta \in C^2 \left( \mathfrak{g}, W^l \oplus \Delta_e \mathfrak{g} \right)$$

236 one has:

$$(-1 + t) \theta^{t,r,s}(u, v) = (\partial i(e)\theta)(u, v).$$

237 From the identity above we deduce that if  $t \neq 1$ , then the map  $(u, v) \mapsto \theta^{t,r,s}(u, v)$  coincides  
 238 with the boundary of a 1-cochain.

*Step 2*

239 Let  $x, x' \in \mathfrak{g}$ . One has the following identity:

$$\left( (L_{(e)}\theta)(x, x') \right) (\xi^{t,r,s}) = (1 + t) (\theta(x, x')) (\xi^{t,r,s})$$

240 Since  $t$  is a non negative integer, the map  $(x, x') \mapsto \theta(x, x')$  coincides with the coboundary  
 241 of a 1-cochain.

*Step 3*

242 It remains to examine  $\theta(x, u)$  for  $(x, u) \in \mathfrak{g}_e \oplus \Delta_e \mathfrak{g}$ . Let us fix  $\xi^{t,r,s}$  as above. It  
 243 comes:

$$\left( (L_{(e)}\theta)(x, u) \right) (\xi^{t,r,s}) = t (\theta(x, u)) (\xi^{t,r,s}).$$

244 Using arguments similar to those used in *step 2*, we deduce that if  $t$  is a positive integer,  
 245 then there is a 1-cochain  $\phi(t) \in C^1(\mathfrak{g}, \text{Hom}(W^l, \Delta_e \mathfrak{g}))$  such that:

$$\theta(x, u) = \partial \phi(t)(x, u).$$

*Step 4*

Now, we take into account the decomposition 15. Given a cochain  $\theta \in C^{m+n+p}(\mathfrak{g}, W \otimes \mathfrak{g})$ ,  
 its restriction to  $\wedge^{m,n,p} \mathfrak{g}$  is denoted by  $\theta_{m,n,p}$ . Therefore, every  $k$ -cochain:

$$\theta^k \in C^k \left( \mathfrak{g}, \text{Hom} \left( W^l, \Delta_e \mathfrak{g} \right) \right)$$

246 is decomposed as follows:

$$\theta^k = \sum_{m+n+p=k} \sum_{t+r+s=l} \theta_{m,n,p}^{t,r,s}$$

247 In particular, when  $\phi$  is a 2-cocycle, *step 1*, *step 2* and *step 3* tell us that all its components  
 248  $\theta_{m,n,p}^{t,r,s}$  are exact, except when  $t = 0$  or  $t = 1$ . So, every cohomology class  $[\theta^l] \in$   
 249  $H^2(\mathfrak{g}, \text{Hom}(W^l, \Delta_e \mathfrak{g}))$  may be represented by a cocycle of the form:

$$\sum_{r+s=l} \theta_{2,0,0}^{1,r,s} + \sum_{n+p=1} \sum_{r+s=l-1} \theta_{1,n,p}^{0,r,s}$$

250 Now, let's consider the map  $\chi: S \rightarrow \text{Hom}(\Delta_e \mathfrak{g} + R(\mathfrak{g}_e), S^{0,r,s}(\mathfrak{g}) \otimes \Delta_e \mathfrak{g})$  that maps  $x$   
 251 to  $\sum_{m+n=1} \sum_{r+s=l-1} i(x) \theta_{m,n,1}^{0,r,s}$ . By virtue of Cartan formula, this map is a cocycle of  
 252  $S$ . Since  $S$  is semi-simple, there exists an element  $\phi \in \text{Hom}(\Delta_e \mathfrak{g} + R(\mathfrak{g}_e), S^l(\mathfrak{g}) \otimes \Delta_e \mathfrak{g})$   
 253 whose coboundary is  $\chi$ . Now, we are in position to prove that the cohomology  $[\theta^k]$  may  
 254 be represented by a cocycle of the form:

$$\sum_{r+s=l-1} \theta_{2,0,0}^{1,r,s} + \sum_{r+s=l} \theta_{1,1,0}^{0,r,s}$$

255 Indeed, let us consider the cochain  $\mu \in C^1(\mathfrak{g}, S^l(\mathfrak{g}) \otimes \Delta_e \mathfrak{g})$  which is defined by:

$$\mu(x, y) = \phi(y).$$

256 where  $x \in S$  and  $y \in \Delta_e \mathfrak{g} + R(\mathfrak{g}_e)$ . A direct calculation shows that the component:

$$\sum_{m+n=l} \sum_{r+s=l} \theta_{m,n,1}^{0,r,s}$$

257 is the coboundary of  $\mu$ . Therefore, the cohomology class of  $[\theta^l]$  may be represented by a  
 258 cocycle of the form:

$$(16) \quad \sum_{r+s=l-1} \theta_{2,0,0}^{1,r,s} + \sum_{r+s=l} \theta_{1,1,0}^{0,r,s}$$

259 For convenience and without loss of generality, we may assume that the ground field  $\mathbb{K}$  is  
 260 the field of complex numbers  $\mathbb{C}$ . Fix a Cartan subalgebra  $\mathfrak{h} \subset S$ . To the couple  $(\mathfrak{h}, S)$  we  
 261 assign the roots system  $\mathcal{R}$  (resp.  $\mathcal{R}'$ ) of the  $S$ -module  $\Delta_e \mathfrak{g} + R(\mathfrak{g}_e)$  (resp.  $S^l \mathfrak{g} \otimes \Delta_e \mathfrak{g}$ ). The  
 262 elements of  $\mathcal{R}$  (resp.  $\mathcal{R}'$ ) are labelled  $\alpha_j$  (resp.  $\gamma_j$ ). It is to be noticed that the multigraduation  
 263 that we are using is preserved by the action of  $\mathfrak{g}_e$ . Thus, given a cochain  $\theta$  and a root  $\gamma \in \mathcal{R}'$ ,  
 264 the  $\gamma$ -component of  $\theta$  is denoted by  $\theta^{(\gamma)}$ . Now, let  $\alpha_i, \alpha_j \in \mathcal{R}$ , and  $\gamma \in \mathcal{R}'$ . Let  $H \in \mathfrak{h}$ .  
 265 Taking into account those notations and the Cartan formula, one easily deduces from the  
 266 expression 16 the following identities;

$$\gamma(H) \theta^{(\gamma)}(x_{\alpha_i}, x_{\alpha_j}) = (\alpha_i + \alpha_j)(H) \theta^{(\gamma)}(\alpha_i, \alpha_j).$$

267 Those identities hold if and only if  $\theta^{(\gamma)} = 0$ ,  $\forall \gamma \in \mathcal{R}'$ . That ends the proof of the vanishing  
 268 of  $H^2(\mathfrak{g}, W \otimes \Delta_e \mathfrak{g})$ .

## Part 2

269 The concern of Part 2 is to prove that  $\forall l > 1$ , the cohomology space  $H^2(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e)$  vanishes.  
 270 The proof is similar to that in Part 1 and so we shall keep the same notations.

### Step 1

271 Let  $\theta$  be a 2-cocycle in  $C^2(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e)$ . Given  $u, v \in \Delta_e \mathfrak{g}$ , using the Cartan formula, a  
 272 direct calculation of  $((L_{(e)} \theta^{t,r,s}))(u, v)$  yields:

$$(t-2) \theta^{t,r,s}(u, v) = (\partial i(e) \theta)(u, v).$$

273 Therefore, we deduce that the quantity  $\theta(u, v)$  coincides with the coboundary of a 1-cochain  
 274 when  $t \neq 2$ . So, except  $\theta_{2,0,0}^{2,r,s}$ , all of the components  $\theta_{2,0,0}^{t,r,s}$  are exact.

### Step 2.

275 Given  $x, x' \in \tilde{\mathfrak{g}}_e$ , one deduces from the identity:

$$((L_{(e)}\theta)(x, x')) = (\partial i(e)\theta)(x, x')$$

276 the following relation:

$$t\theta^{t,r,s}(x, x') = (\partial i(e)\theta)(x, x').$$

277 So, if  $t \neq 0$ , the component  $\theta_{0,n,p}^{t,r,s}$  coincides with the component of an exact cocycle.  
278 *Step 3*

278 Given  $(x, u) \in \tilde{\mathfrak{g}}_e \oplus \Delta_e \mathfrak{g}$ , we have:

$$(t-1)\theta^{t,r,s}(x, u) = (\partial i(e)\theta)(x, u).$$

279 Thus, except  $\theta_{1,n,p}^{1,r,s}$  all of the components  $\theta_{1,n,p}^{t,r,s}$  coincides with the restriction of exact cocycle.  
280 Now, we can conclude from the three steps above that every cohomology class

$$[\theta] \in H^2(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e)$$

281 may be represented by a cocycle of the following form:

$$\theta = \sum_{r+s=l-2} \theta_{2,0,0}^{2,r,s} + \sum_{n+p=1} \sum_{r+s=l-1} \theta_{1,n,p}^{1,r,s} + \sum_{r+s=l} \sum_{n+p=2} \theta_{0,n,p}^{0,r,s}.$$

282 Now, taking into account this reduced form of  $\theta$ , it is easy to verify that the following  
283 component of  $\theta$ :

$$\phi = \sum_{r+s=l} \theta_{0,0,2}^{0,r,s}$$

284 is a cocycle of  $S$ . Since  $S$  is semi-simple, there exists an element  $\chi \in C^1(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e)$   
285 whose boundary is  $\phi$ . Thus, we can represent the cohomology class of  $\theta^l$  by:

$$(17) \quad \gamma = \sum_{r+s=l-2} \theta_{2,0,0}^{2,r,s} + \sum_{r+s=l-1} (\theta_{1,1,0}^{1,r,s} + \theta_{1,0,1}^{1,r,s}) + \sum_{r+s=l} (\theta_{0,2,0}^{0,r,s} + \theta_{0,1,1}^{0,r,s}).$$

286 Now, we consider the map  $\mu$  (resp.  $\nu$ ) from  $S$  to  $\text{Hom}(\Delta_e \mathfrak{g}, W \otimes \tilde{\mathfrak{g}}_e)$  (resp. from  $S$  to  
287  $\text{Hom}(R\mathfrak{g}_e, W \otimes \tilde{\mathfrak{g}}_e)$ ) which is defined as follows:

$$\mu(x) = \sum_{r+s=l-1} i(x)\theta_{1,0,1}^{1,r,s}$$

288 and:

$$\nu(x) = \sum_{r+s=l} i(x)\theta_{0,1,1}^{0,r,s}.$$

289 Both  $\mu$  and  $\nu$  are cocycles of the semi-simple Lie algebra  $S$ . Thus, it exists  $\xi \in$   
290  $\text{Hom}(\Delta_e \mathfrak{g}, W \otimes \tilde{\mathfrak{g}}_e)$  (resp.  $\zeta \in \text{Hom}(R\mathfrak{g}_e, W \otimes \tilde{\mathfrak{g}}_e)$ ) whose coboundary is  $\mu$  (resp.  $\nu$ ).  
291 Now, let one regard the map  $\eta = \xi + \zeta$  as an element of  $C^1(\mathfrak{g}, W \otimes \tilde{\mathfrak{g}}_e)$  by setting:

$$\eta(u + y + x) = \xi(x) + \zeta(x), \forall (u, y, x) \in \Delta_e \mathfrak{g} \oplus R(\mathfrak{g}_e) \oplus S.$$

292 Now, the cocycle  $\theta' = \gamma - d\eta$ , with  $\gamma$  as in 17, doesn't contain any component of type  
293  $\theta_{0,0,p}^{t,r,s}$  with  $p > 0$ . Thus, any cohomology class in  $H^2(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e)$  may be represented by  
294 a cocycle of the following REDUCED form:

$$\theta' = \sum_{r+s=l-1} \theta_{2,0,0}^{1,r,s} + \sum_{r+s=l} (\theta_{1,1,0}^{0,r,s} + \theta_{0,2,0}^{0,r,s}).$$

*Step 4.*

295 We assume again  $\mathbb{K} = \mathbb{C}$ . Therefore, we fix a Cartan subalgebra  $\mathfrak{h} \subset S$ . We label  $\alpha_i$  (resp.  
296  $\gamma_j$ ) the corresponding roots system of the  $S$ -module  $\Delta_e \mathfrak{g} + R(\mathfrak{g}_e)$  (resp.  $\sum_{r+s=l-1} W^{1,r,s} \otimes$

297  $\tilde{\mathfrak{g}}_e + \sum_{r+s=l} W^{0,r,s} \otimes \tilde{\mathfrak{g}}_e$ ). Now, let  $\alpha_i, \alpha_{i'}$  and  $\gamma$  be roots. Taking into account the REDUCED  
 298 form  $\theta'$  of 2-cocycles, we denote by  $\theta'_{(\gamma)}$  the  $\gamma$ -component of  $\theta'$ . Then the Cartan formula  
 299 implies the following identities:

$$\gamma(H)\theta'_{(\gamma)}(u_{\alpha_i}, u_{\alpha_{i'}}) = (\alpha_i + \alpha_{i'})(H)\theta'_{(\gamma)}(u_{\alpha_i}, u_{\alpha_{i'}}), \forall \alpha_i, \alpha_{i'}, \gamma.$$

300 Of course, those identities hold if and only if each  $\theta'_{(\gamma)}$  vanishes identically.

301 **5.2. Proof of theorem 4.** The notations used will be the ones previously defined. Fix an  
 302 homogeneous submodule  $W \subseteq S^+\mathfrak{g}$  and let  $k$  be a non negative integer. Owing to Remark  
 303 3, we have:

$$\text{Ext}_{\mathfrak{g}}^k(W, F) = \bigoplus_l \text{Ext}_{\mathfrak{g}}^k(W^l, F)$$

304 where  $l$  runs over the set of positive integers and  $F \in \{\Delta_e \mathfrak{g}, \mathfrak{g}, \tilde{\mathfrak{g}}_e\}$ . On one hand, (12) gives  
 305 rise the classical long cohomology exact sequence. In particular, the following sequence is  
 306 exact at the level  $H^2(\mathfrak{g}, \text{Hom}(W^l, \mathfrak{g}))$ :

$$H^2(\mathfrak{g}, \text{Hom}(W, \Delta_e \mathfrak{g})) \longmapsto H^2(\mathfrak{g}, \text{Hom}(W, \mathfrak{g})) \longmapsto H^2(\mathfrak{g}, \text{Hom}(W, \tilde{\mathfrak{g}}_e))$$

307 On the other hand, if  $F$  is a finite dimensional  $\mathfrak{g}$ -module, then we have the following  
 308 classical linear isomorphisms:

$$\text{Ext}_{\mathfrak{g}}^k(W^l, F) \sim H^k(\mathfrak{g}, \text{Hom}(W^l, F))$$

309 Thus, owing to Theorem (3), the exact sequence above yields

$$\text{Ext}_{\mathfrak{g}}^2(W, \mathfrak{g})$$

310 **5.3. Proof of theorem 5.** Before proceeding, let us recall the meaning of Theorem (5). Let  
 311  $\Pi$  be a smooth Poisson tensor defined in a  $m$ -dimensional smooth manifold  $M$ . Theorem  
 312 (5) expresses a local property of Lie algebroids given by Poisson structures (near to their  
 313 singularities). Thus, we shall suppose that  $M$  is an open neighborhood of the origin of  $\mathbb{K}^m$ .  
 314 Let us suppose thus that  $\Pi$  vanishes at the origin 0 of  $\mathbb{K}^m$ . Let:

$$\check{\Pi} = \sum_I \Pi^I$$

315 be the Taylor expansion at 0 of  $\Pi$ , where all of the  $I$  are multi-indices, namely  $I =$   
 316  $(i_1, \dots, i_m)$ . Let  $\Pi^1$  be the linear part of  $\Pi$ . Then,  $\Pi^1$  defines a Lie algebra structure in  
 317 the vector space of linear functions. The Taylor expansion at 0 of  $\Pi$  is regarded as a formal  
 318 deformation of  $\Pi^1$ . In fact, we have three Poisson structures in a small neighborhood of  
 319 0, namely, the smooth structure  $\Pi$ , the formal structure  $\check{\Pi}$  and the linear structure  $\Pi^1$ . So,  
 320 finally, the question boils down to know whether  $\Pi$  (resp.  $\check{\Pi}$ ) is smoothly (resp. formally)  
 321 isomorphic to  $\Pi^1$ .

322 We can now proceed to the proof of theorem 5.

323 *Proof.* Given a multi-index  $I = (i_1, \dots, i_p)$ , we shall set:

$$l(I) = \sum_{j=1}^p i_j$$

324 Let us denote by  $\mathfrak{g}$  the Lie algebra defined by  $\Pi^1$ . Then, each homogeneous component  $\Pi^l$   
 325 of the Taylor expansion at 0 of  $\Pi$  is an element of  $C^2(\mathfrak{g}, S^{l(l)\mathfrak{g}})$ . Let us consider the Lie  
 326 algebroid structure:

$$A_\pi = (T^*M, \mathfrak{h})$$

327 which is defined by  $(M, \Pi)$ . Let us recall the following facts. Firstly, the anchor map  $\mathfrak{h}$  is  
 328 the vector bundle map defined by:

$$\mathfrak{h}(\alpha) = i(\alpha)\Pi.$$

329 Secondly, given 2 smooth sections of  $T^*M$ , say  $\alpha, \beta$  their bracket is defined by:

$$[\alpha, \beta]_{\mathfrak{h}} = i_{\mathfrak{h}\alpha}d\beta - i_{\mathfrak{h}\beta}d\alpha + di_{\Pi}\alpha \wedge \beta.$$

330 Thus, one easily sees that a Poisson tensor and the associated Lie algebroid have the  
 331 same singularities and the same linear part at their singular points. Let  $\check{A}_\pi$ , be the Taylor  
 332 expansion at 0 of the anchor map  $\mathfrak{h}$ . It may be regarded as a formal deformation of the  
 333 linear Lie algebroid:

$$\mathfrak{g} = (T_0^*M, \Pi^1) \times T_0M$$

334 Now,  $\tilde{x}$  let be a smooth vector field defined near 0, let  $\tilde{\alpha}$  be a smooth section of  $T^*M$ . Let  
 335 us set:

$$\alpha = \tilde{\alpha}(0)$$

336 and:

$$x = \tilde{x}(0)$$

337 Then, the expression:

$$\rho(\alpha)x = [\mathfrak{h}(\tilde{\alpha}), \tilde{x}](0)$$

338 is well defined. It is easy to verify that:

$$\rho([\alpha, \beta]_{\mathfrak{h}}) = [\rho(\alpha), \rho(\beta)].$$

339 Therefore, the dual vector space:

$$\mathfrak{g}^* = T_0M$$

340 is a  $\mathfrak{g}$ -module, but the action of  $\mathfrak{g}$  is not the coadjoint action. The (graded)  $\mathfrak{g}$ -module structure  
 341 of  $S^+\mathfrak{g} \otimes \mathfrak{g}^*$  is inherited from the tensor product  $\text{ad}_{\mathfrak{g}} \otimes \rho$ . Moreover, the coefficients of the  
 342 Taylor expansion at 0 of  $\mathfrak{h}$  may be regarded as elements of the Spencer prolongations of the  
 343 linear subalgebra  $\rho(\mathfrak{g}) \subset \mathfrak{gl}(T_0) \subset \dots$ , [17]. From this viewpoint, the algebroid  $(T^*M, \mathfrak{h})$   
 344 is formally linearizable at 0 if the first prolongation of  $\rho(\mathfrak{g})$  is zero. So it is the case when  
 345  $\rho(\mathfrak{g})$  is compact or when the fundamental form

$$B_\rho(\alpha, \beta) = \text{Tr}(\rho(\alpha)\rho(\beta))$$

346 is non degenerate, [17]. What is just said above is nothing but the algebraic deformation  
 347 theoretic point of view applied to the linearization problem for the anchor map  $\mathfrak{h}$ . We can  
 348 also examine the linearization of  $(T^*M, \mathfrak{h})$  from the abstract algebra viewpoint. Indeed, let  
 349 us denote by  $\Omega(M)$  the vector space of smooth sections of  $T^*M$ . For every non negative  
 350 integer 1, we set:

$$\omega^{(l)} = \{\alpha \in \Omega(M) \mid j_0^l \alpha = 0\}$$

351 Every  $\Omega^{(l)}$  is an ideal of the Lie algebra structure associated to the Lie algebroid  $(T^*M, \mathfrak{h})$ .

352 Therefore, we get a natural filtration:

$$\Omega(M) \supset \Omega^{(1)}(M) \supset \Omega^{(2)}(M) \supset \dots$$

353 From now on, we shall keep in mind this filtered Lie algebra structure in  $\Omega(M)$ , [19]. It is  
 354 easily seen that the quotient space  $\Omega^{(l)}/\Omega^{(l+1)}$  may be canonically identified with:

$$S^l T_0^* M \otimes T_0^* M = \text{Hom} \left( S^l T_0 M, T_0^* M \right).$$

355 Now, let us denote by  $\text{grd } \Omega$  the graded Lie algebra whose  $l$ -th homogeneous subspace is:

$$\text{grd } {}^l \Omega = \Omega^{(l)}/\Omega^{(l+1)}.$$

356 In fact, the positive part of  $\text{grd } \Omega$ , namely:

$$\text{grd } {}^+ \Omega = \bigoplus_{l \geq 1} \Omega^{(l)}/\Omega^{(l+1)}$$

357 is an ideal of  $\text{grd } \Omega$ . Moreover, the quotient Lie algebra:

$$\text{grd } \Omega / \text{grd } {}^+ \omega$$

358 is nothing but the Lie algebra:

$$\left( T_0^* M, [\cdot, \cdot]_{\mathfrak{h}} \right) = \left( T_0^* M, \Pi^1 \right).$$

359 So, we have the following exact sequence of Lie algebras

$$(18) \quad 0 \rightarrow \text{grd } {}^+ \Omega \rightarrow \text{grd } \Omega \rightarrow \left( T_0^* M, [\cdot, \cdot]_{\mathfrak{h}} \right) \rightarrow 0$$

360 The problem to know whether the sequence above (Lie) splits is the formal linearization  
 361 problem for the Poisson structure  $(M, \Pi)$ , see [27]. The study of this problem involves  
 362 the cohomology space  $H^2(\mathfrak{g}, \text{grd } {}^+ \Omega)$ , [23, 25]. So, by the virtue of Theorem (4), the exact  
 363 sequence of Lie algebras 18 splits. In other words, there exists a Lie algebra homomorphism  
 364  $\varrho$  from  $\left( T_0^* M, [\cdot, \cdot]_{\mathfrak{h}} \right)$  to the formal Poisson algebra  $(j_0^\infty(C^\infty(M, \mathbb{R})), [\cdot, \cdot])$  such that  
 365  $d_0 \varrho(\lambda) = \lambda, \forall \lambda \in T_0^* M$ . Of course, the bracket on power series is induced by the Poisson  
 366 structure  $(M, \Pi)$ , viz:

$$[j_0^\infty f, j_0^\infty g] = j_0^\infty (\Pi(f, g))$$

367 Since  $\Pi(f, h)$  depends only on the differential of the both  $f$  and  $h$ , we may restrict our  
 368 attention to the set  $C_0^\infty(M, \mathbb{R})$  of real valued smooth or analytic functions which vanish at the  
 369 point  $0 \in M$ . The differentiation at  $0 \in M$ , say  $d_0$ , induces a Lie algebra homomorphism  
 370 from  $C_0^\infty(M, \mathbb{R})$  onto  $\left( T_0^* M, [\cdot, \cdot]_{\mathfrak{h}} \right)$ . So, the smooth or analytic linearization problem for  
 371  $\Pi$  is to know whether the following exact sequence of Lie algebra splits:

$$0 \longrightarrow \ker(d_0) \longrightarrow C_0^\infty(M, \mathbb{R}) \longrightarrow \left( T_0^* M, [\cdot, \cdot]_{\mathfrak{h}} \right) \longrightarrow 0$$

372 On the other side, from the deformation theory viewpoint, a sufficient condition for the  
 373 formal linearization of  $(T^* M, \mathfrak{h})$  (resp.  $\Pi$ ) is the vanishing of the cohomology space  
 374  $H^2(\mathfrak{g}, \text{Hom}(W, \mathfrak{g}))$  (resp.  $H^2(\mathfrak{g}, W^*)$ ) for every homogeneous submodule  $W$  of the graded  
 375  $\mathfrak{g}$ -module  $S^+(g^*)$ . Now, let us suppose the Lie algebra:

$$\mathfrak{g} = (T_0^* M, \Pi^1)$$

376 to have an affinelinear Lie algebra structure, namely  $(\mathfrak{g}, e)$ . Let us denote by  $\mathfrak{g}_f$  the formal  
 377 deformation of  $\mathfrak{g}$  defined by the Taylor expansion at 0 of the anchor map  $\mathfrak{h}$ . Its bracket is in  
 378 a basis  $(dx_1, \dots, dx_m)$  is defined as follows:

$$[dx_i, dx_j] = \sum_k \left( \sum_I c_{ij}^{I,k}(0) x^I \right) dx_k$$



379 Where  $c_{ij}^{l,k}(0)$  is regarded as an element of  $\text{Hom}(S^{l(l)}\mathfrak{g}^*, \mathfrak{g})$ . Keeping the notations of the  
 380 previous sections, we deduce from Theorem (4) that

$$H^2(\mathfrak{g}, \text{Hom}(W, \mathfrak{g})) = 0.$$

381 Thus, the formal Lie algebra structure  $\mathfrak{g}_f$  is formally isomorphic to its linear part, namely  
 382  $(T_0^*M, \Pi^1)$  □

383 *Remark 4.* When the Lie algebra  $(T^*M, \mathfrak{h})$  is an affineline Lie algebra, it is more relevant  
 384 to deal with the splitting problem of 18 rather than to deal with formal diffeomorphisms as  
 385 Conn (resp. Nguyen Tien Zung) does in the case where  $\mathfrak{g}$  is semi-simple. Indeed, when  $\mathfrak{g}$   
 386 is semi-simple, one has Casimir elements which permit to construct an explicit homotopy  
 387 operator. That operator yields an explicit formal diffeomorphism linearizing formally the  
 388 Poisson structure. If  $\mathfrak{g}$  is an non solvable affineline Lie algebra, one proceeds as it follows.  
 389 For every non negative integer  $k$ , let  $S_k = S_k^\infty(M, \mathbb{R})$  be the set of formal power series  
 390 whose orders are at least equal to  $k + 1$ . Every  $S_k$  is an ideal of the Lie algebra  $S_0$  and the  
 391 quotient Lie algebra  $\mathfrak{g}_k = S_0/S_k$  may be identified with the vector space  $\sum_{1 \leq j \leq k} S^j(\mathfrak{g})$ . In  
 392 particular one has  $\mathfrak{g}_0 = \mathfrak{g}$ . Moreover, one has the following exact sequence of Lie algebras:

$$0 \longrightarrow S^{k+1}(\mathfrak{g}) \longrightarrow \mathfrak{g}_k \longrightarrow \mathfrak{g}_{k-1} \longrightarrow 0$$

393 If  $j < k$ , the canonical projection of  $\mathfrak{g}_k$  onto  $\mathfrak{g}_j$  is denoted by  $p_{jk}$ . It is easily seen that  
 394  $p_{ij}p_{jk} = p_{ik}$ . Thereby, one has the projective system  $(\mathfrak{g}_k, p_{jk})$ . In particular the following  
 395 exact sequence of Lie algebras splits:

$$0 \longrightarrow S^2(\mathfrak{g}) \longrightarrow \mathfrak{g}_1 \longrightarrow \mathfrak{g} \longrightarrow 0$$

396 Thus, one constructs inductively an Lie algebra monomorphism  $\varrho_j: \mathfrak{g} \rightarrow \mathfrak{g}_j$  such that for  
 397  $j < k$ , one has  $p_{jk}\varrho_k = \varrho_j$ . Those considerations yield a Lie algebra monomorphism  $\varrho$   
 398 from  $\mathfrak{g}$  to the inverse limit of  $(\mathfrak{g}_k, p_{jk})$ , which is  $j_0^\infty(C_0^\infty(M, \mathbb{R}))$ . That is nothing but the  
 399 formal linearization of the Poisson structure, [27].

400 Now, suppose the Poisson tensor  $\Pi$  to be analytic. Then, the corresponding Lie algebroid  
 401 is analytic as well. If  $\mathfrak{g} = (T_0^*M, [\cdot, \cdot])$  is semi-simple or if  $\mathfrak{g} = \mathfrak{g}_{n,1}$ , then the analytic  
 402 counterparts of Theorem 5 are nothing but Conn's linearization theorem of analytic Poisson  
 403 structure, [9, 10, 22] and the recent analytic non degeneracy theorem for  $\text{aff}(n)$  by Jean-Paul  
 404 Dufour and Nguyen Tien Zung, [13]. If  $\mathfrak{g}$  is an affineline Lie algebra, we know how to  
 405 construct a Lie algebra monomorphism from  $\mathfrak{g}$  to  $\mathfrak{g}_1$ . Indeed, let:

$$\sigma: \mathfrak{g} \mapsto \mathfrak{g}_1$$

406 be a linear monomorphism such that  $p_{01}\sigma$  is the identity endomorphism of  $\mathfrak{g}$ ; the bilinear  
 407 map:

$$\omega(a, b) = \sigma([a, b]) - [\sigma(a), \sigma(b)], \quad a, b \in \mathfrak{g}$$

408 is a  $S^2(\mathfrak{g})$ -valued 2-cocycle of  $\mathfrak{g}$  whose cohomology class doesn't depend on the choice of  
 409  $\sigma$ . On one side, the linear map:

$$\bar{\omega}(a) = (a - \sigma(p_{01}(a))) + p_{01}(a)$$

410 is a Lie algebra isomorphism from  $\mathfrak{g}_1$  onto the semi direct product  $S^2(\mathfrak{g}) \ltimes \mathfrak{g}$ . On the other  
 411 side, there is an element  $\theta \in \text{Hom}(\mathfrak{g}, S^2(\mathfrak{g}))$  such that:

$$\omega(a, b) = [\sigma(a), \theta(b)] - [\sigma(b), \theta(a)] - \theta([a, b]).$$

412 The map  $\varrho_1$  assigning to every  $a \in \mathfrak{g}$  the element  $\varrho_1(a) = \theta(a) + a$  is an Lie algebra  
 413 monomorphism from  $\mathfrak{g}$  to the semi-direct product  $S^2(\mathfrak{g}) \ltimes \mathfrak{g}$ . What is just done is nothing  
 414 but the first step yielding a Lie algebra monomorphism  $\varrho$  from  $\mathfrak{g}$  to  $j^\infty(C_0^\infty(M, \mathbb{R}))$ . Thus,  
 415 let  $x_1, \dots, x_n$  be a basis of  $\mathfrak{g}$ ; let  $c_{ij}^k$  be the structure constants of  $\mathfrak{g}$  in the basis  $(x_i)_{i=1 \dots n}$ ,  
 416 viz:

$$[x_i, x_j] = \sum_k c_{ij}^k x_k$$

417 then one has also:

$$[\varrho(x_i), \varrho(x_j)] = \sum_k c_{ij}^k \varrho(x_k)$$

418 So, the analytic linearization problem is to prove that one can choose the basic data  $\sigma$   
 419 and  $\theta$  to make sure that the formal series  $\varrho(x_1), \dots, \varrho(x_n)$  are convergent. By the virtue  
 420 of Conn's theorem (resp. Dufour-Nguen tien Zung's Theorem) such data can be chosen  
 421 whenever  $\mathfrak{g}$  is semi-simple (resp.  $\mathfrak{g}$  is the affine Lie algebra  $\mathfrak{aff}(n)$ ).

422 *Remark 5.* We conjecture that the same conclusion holds for non solvable affinlike Lie  
 423 algebras.

424 Finally, as a by product, we have the proof of theorem 1. In fact, this is the Poisson  
 425 counterpart of Theorem 5. Mutatis mutandis, the same arguments used in the proofs of  
 426 Theorem 5 yield:

$$H^2(\mathfrak{g}, S^+(\mathfrak{g})) = 0$$

427 for every non solvable affinlike Lie algebra  $(\mathfrak{g}, e)$ . Therefore, such an affinlike Lie algebra  
 428 is formally nondegenerate. Let us now give the proof of Theorem 2.

429 **5.4. Prof of 2.** We have already proved the equality:

$$H^2(\mathfrak{g}, \mathbb{K}) = H^2(\Delta\mathfrak{g}_e, \mathbb{K})$$

430 We denote the radical of  $\Delta_e\mathfrak{g}$  by  $R(\Delta\mathfrak{g}_e)$ . Lyndon-Hochschild-Serre spectral sequence  
 431 yields:

$$H^2(\Delta\mathfrak{g}_e, \mathbb{R}) = H^0(\Delta\mathfrak{g}_e, H^2(R(\Delta\mathfrak{g}_e), \mathbb{R}))$$

432 Without loss of generality, we may suppose the ground field to be the field of complex  
 433 numbers. Let  $S \subset \Delta\mathfrak{g}_e$  a Levi sub-algebra of  $\Delta\mathfrak{g}_e$ , and let  $\mathfrak{h} \subset S$  be a Cartan sub-  
 434 algebra. Let  $\alpha$  be a weigh of the  $S$ -module  $R(\Delta\mathfrak{g}_e)$  and let us denote by  $r_\alpha \in R(\Delta\mathfrak{g}_e)$   
 435 be a non zero element such that  $[H, r_\alpha] = \alpha(H)r_\alpha, \forall H \in \mathfrak{h}$ . It is a straight consequence  
 436 of (alla 3) that  $R(\Delta\mathfrak{g}_e)$  is spanned by the elements  $r_\alpha$ . Thus, every cohomology class  
 437  $[\theta] \in H^0(\Delta\mathfrak{g}_e, H^2(R(\Delta\mathfrak{g}_e), \mathbb{R}))$  may be represented by a cocycle  $\theta \in \text{Hom}(R(\Delta\mathfrak{g}_e), \mathbb{R})$   
 438 such that  $\forall H \in \mathfrak{h}, L_H\theta$  is the differential of a  $\phi_H \in \text{Hom}(R(\Delta\mathfrak{g}_e), \mathbb{R})$ . . So, given two  
 439 elements  $r_\alpha, r_{\alpha'}$  as defined above, one has:

$$(\alpha + \alpha')(H)\theta(r_\alpha, r_{\alpha'}) = \phi_H([r_\alpha, r_{\alpha'}]).$$

440 Since  $R(\Delta\mathfrak{g}_e)$  is commutative, the last identity implies the following one:

$$(\alpha + \alpha')(H)\theta(r_\alpha, r_{\alpha'}) = 0.$$

441 Thus we see that  $H^2(R(\Delta\mathfrak{g}_e), \mathbb{R}) = 0$ . So, if  $\omega$  is a left invariant symplectic form on  $G$ , then  
 442  $\omega$  is the differential of a left invariant 1-form  $\theta$  Thereby, the orbit of  $\theta$  under the coadjoint  
 443 action of  $G$  is an open set in the dual vector space  $\mathfrak{g}^*$ . If the ground field  $\mathbb{K}$  is the field of

444 complex numbers, then  $\text{Ad}^*G(\theta)$  is the complement of an algebraic variety. Indeed, given  
 445 an element  $\eta \in \mathfrak{g}^*$ , let  $x_\eta$ , be the element of  $\mathfrak{g}$  defined by:

$$i(x_\eta)\omega = \eta.$$

446 Let  $\theta \in \mathfrak{g}^*$ , let us define the linear endomorphism  $\Phi_\theta$  of  $\mathfrak{g}^*$  by setting:

$$\Phi_\theta(\eta) = \text{ad}^*(x_\eta\theta).$$

447 The orbit of  $\theta$  is an open set in  $\mathfrak{g}^*$  if and only if  $\Phi_\theta$  is injective; in this case,  $\text{Ad}^*G(\theta)$  is an  
 448 open and dense orbit. Since we assumed the ground field to be  $\mathbb{C}$ , it is unique.

449 Theorem 2 yields the following statement.

450 **Theorem 6.** *Let  $(n, m)$  be a pair of positive integers. If  $n > 2$ , then one has:*

$$H^2(\mathfrak{g}_{n,m}, \mathbb{R}) = 0$$

451 *In particular, ([24]) those of the  $\mathfrak{g}_{n,m}$  which admit symplectic forms are Frobenius algebras.*

452

## 6. A-ALGEBROIDS

453 This section will be devoted to a relevant application of Theorem 4. To begin with,  
 454 let us recall the following normal forms theorem of Jean-Paul Dufour, [12] ( see also  
 455 [2, 5–7, 20]).

456 **Theorem 7.** *DUF2 Let  $(A, a)$  be a Lie algebroid of rank  $n$  over a smooth  $m$ - dimensional  
 457 manifold  $M$ . Let  $r$  be the rank of the anchor map  $a$  at the point  $p \in M$ . Then, on an open  
 458 neighborhood  $U$  of  $p$ , there exist:*

459 (1) *A basis of local sections of  $A$ , say  $\{\sigma_1, \dots, \sigma_r, \tau_1, \dots, \tau_{n-r}\}$ ,*

460 (2) *A system of local coordinate functions on  $M$ , say  $\{x_1, \dots, x_r, y_1, \dots, y_{m-r}\}$*

461 *Such that the next conditions are satisfied for any  $j = 1, \dots, r$ ;  $k, l = 1, \dots, n - r$ :*

462 i)  $a(\sigma_j) = \partial x_j$  and  $a(\tau_k)(p) = 0$ ,

463 ii)  $a(\tau_k)$  doesn't depend on  $x_1, \dots, x_r$ ,

464 iii)  $[\sigma_j, \tau_k] = 0$ ,

465 iv)  $[\tau_k, \tau_l] = \sum_v c_{kl}^v \tau_v$ , the functions  $c_{kl}^v$  depending only on  $y_1, \dots, y_{m-r}$ .

466 In [6], R. Wolak and the author given an another proof of Theorem 7 which works for  
 467 Koszul-Vinberg algebroids as well. Both proofs in [12] and in [6] are based in differential  
 468 geometry arguments. The author intends to read the theorem of J-P Dufour from the sheaf  
 469 theoretic point of view. Really, Theorem 7 is different from the local decomposition of  
 470 Poisson manifolds, [26].

471 Let  $(A, a)$  be a smooth or analytic algebroid over a smooth or analytic manifold  $M$ . Let us  
 472 denote by  $\Gamma(A)$  (resp.  $\Gamma(TM)$ ) the sheaf of local smooth sections of  $A$  (resp.  $TM$ ). Because  
 473 the anchor map  $a$  may have non constant rank, the following exact sequence:

$$\ker(a) \longrightarrow A \longrightarrow TM$$

474 is not a sequence a vector bundles over  $M$ . However, the anchor map induces a sheaf  
 475 homomorphism:

$$\Gamma(A) \rightarrow \Gamma(TM)$$

476 which is still denoted by  $a$ . The kernel of the last sheaf homomorphism is a sub-sheaf  
 477 of  $\Gamma(A)$ , denoted  $\text{Ker}(a)$ . It is not locally trivial. However,  $\Gamma(A)$  is a sheaf of  $\text{Ker}(a)$ -  
 478 modules. Let  $\tau \in \Gamma(A)$ ,  $X \in \Gamma(TM)$ . Setting  $\tau.X = [a(\tau), X]$  makes  $\Gamma(TM)$  inherit  
 479 the trivial  $\text{Ker}(a)$ -module structure. So, one points out the following exact sequence of  
 480  $\text{Ker}(a)$ -modules:

$$0 \longrightarrow \text{Ker}(a) \longrightarrow \Gamma(A) \longrightarrow a(\Gamma(A)) \longrightarrow 0$$

481 Now, let us point out the following observation. The algebraic counterpart of Theorem 7  
 482 is that the cohomology class in  $H^1(\text{Ker}(a), \text{Hom}(a(\Gamma(A)), \text{Ker}(a)))$  which is represented  
 483 by the extension above is zero. So, in some particular cases, that algebraic counterpart of  
 484 Theorem 7 can be proved directly. In the next, we intend to give such a proof for stratified  
 485  $a$ -algebroids. Let us consider the Leibniz axiom of  $(A, a)$ . Then, given two sections  $s, s'$  of  
 486  $A$  and a smooth function  $f$ , one has:

$$[s, fs'] = f[s, s'] + (a(s)f)s'$$

487 It is a straight consequence of the identity above that the bracket of sections of  $A$  induces a  
 488 Lie algebra structure on the vector space:

$$\text{ker}(a)(p) = \text{span}\{\tau_1(p), \dots, \tau_{n-r}(p)\}$$

489 **Definition 7.** A lie algebroid  $(A, a)$  over a smooth manifold  $M$  is called  $a$ -algebroid (resp.  
 490 non solvable  $a$ -algebroid) if  $\forall p \in M, \text{ker}(a)(p)$  is an affinlike Lie algebra (resp. non  
 491 solvable affinlike Lie algebra).

492 Our aim is to show how in the case of non solvable  $a$ -algebroids, one can use the vanishing  
 493 theorems of Section 5 to supply an algebraic proof of Theorem 7. Now, let  $(A, a)$  be an  
 494  $a$ -algebroid over the manifold  $M$ ; For all  $p \in M$ , the bracket of sections of  $A$  induces an  
 495 affinlike Lie algebra structure on  $\text{ker}(a)(p)$ . Let  $r(p) = \text{rank}(a(p))$ . We put:

$$r_1(a) = \max_p(r(p)).$$

496 The singular points of  $(A, a)$  are elements of the subset  $\Sigma(a)$  which consists of those  $p \in M$   
 497 where  $r(p) < r_1$ . Of course  $M - \Sigma(a)$  is an open subset of  $M$ . Let  $p \in M - \Sigma(a)$  and let  
 498  $V \subset M - \Sigma(a)$  be an connected open neighborhood of  $p$ . Then, over the sub-manifold  $V$ ,  
 499 we get the following exact sequence of vector bundles:

$$0 \longrightarrow \text{ker}(a) \longrightarrow A \longrightarrow a(A) \longrightarrow 0$$

500 On one side, the sub-vector bundle  $a(A) \subset TV$  defines a regular foliation on  $V$ . On the  
 501 other side,  $\text{ker}(a)$  is a locally trivial bundle (over  $V$ ) of non solvable affinlike Lie algebras.  
 502 Indeed, we know that  $\forall p \in V$  one has:

$$H^2(\text{ker}(a)(p), \text{ker}(a)(p)) = 0.$$

503 So, all of the Lie algebras  $\text{ker}(a)(p)$  are rigid, [3, 23]. Thus,  $\forall p, p' \in V$ , the Lie algebras  
 504  $\text{ker}(a)(p)$  and  $\text{ker}(a)(p')$  are isomorphic. Over the open set  $V$ , the action of  $\text{Ker}(a) =$   
 505  $\Gamma(\text{ker}(a))$  on  $X(V)$  is trivial. Restricting ourself to  $V$ , we regard  $a(A)$  and  $TV$  as bundles  
 506 of  $\text{Ker}(a)$ -modules. For our purpose, we shall need the following result.

507 **Theorem 8.** *If  $T$  is a trivial module of a non solvable affinlike Lie algebra  $(\mathfrak{g}, e)$ , then*  
 508  $\text{Ext}_{\Delta\mathfrak{g}}^1(T, \mathfrak{g}) = 0$ .

509 *Proof.* Keeping the notations of section 2, we consider the following exact sequence of  
510  $\Delta_{\mathfrak{g}}$ -modules:

$$(19) \quad 0 \longrightarrow \text{Hom}(T, \Delta_{\mathfrak{g}}) \longrightarrow \text{Hom}(T, \mathfrak{g}) \longrightarrow \text{Hom}(T, \mathfrak{g}/\Delta_{\mathfrak{g}}) \longrightarrow 0$$

*Step 1.*

511 Let us show that:

$$H^1(\Delta_{\mathfrak{g}}, \text{Hom}(T, \Delta_{\mathfrak{g}})) = 0.$$

512 To do that, we assume again the ground field to be the field of complex numbers. Let us fix  
513 a Levi subalgebra  $S([\mathfrak{g}_e, \mathfrak{g}_e])$  and a Cartan subalgebra  $\mathfrak{h} \subset S$ . To  $(\mathfrak{h}, S)$  we shall also assign  
514 a fixed weights system  $\{\alpha, \beta, \dots\}$  (for the  $S$ -module  $\Delta_{\mathfrak{g}}$ ). Let  $\theta \in \text{Hom}(\Delta_{\mathfrak{g}}, \text{Hom}(T, \Delta_{\mathfrak{g}}))$   
515 be a cocycle of  $\Delta_{\mathfrak{g}}$ . There exists an element  $\xi \in \text{Hom}(T, \Delta_{\mathfrak{g}})$  such that  $\forall s \in S, \forall t \in T$ , one  
516 has:

$$(\theta(s))(t) = (L_s \xi)(t) = [s, \xi(t)].$$

517 Now, let  $(\alpha, \beta)$  be a couple of weights of the  $S$ -module  $\Delta_{\mathfrak{g}}$  and let  $r_{\alpha}$  be a non zero element  
518 of  $\Delta_{\mathfrak{g}}$  whose weight is  $\alpha$ . Pour tout  $t \in T$  et  $H \in \mathfrak{h}$ , we have:

$$[H, (\theta(r_{\alpha}))(t)] [r_{\alpha}, (\theta(H))(t)] = (\theta([H, r_{\alpha}])(t)).$$

519 Let us compute the  $\beta$ -component of the identity just set. The calculation yields the following  
520 identity:

$$(\beta - \alpha) \theta_b(r_{\alpha}) = [r_{\alpha}, (\theta(H))(t)]_{\beta} = [r_{\alpha}, [H, (\xi(t))_{\beta - \alpha}]] = (\beta - \alpha)(H) [r_{\alpha}, \xi_{\beta - \alpha}(t)]$$

521 The last identity will hold if and only if  $\forall \alpha, \forall t \in T$ , one has:

$$\theta(r_{\alpha})(t) = [r_{\alpha}, \xi(t)].$$

522 So we have to conclude that:

$$H^1(\Delta_{\mathfrak{g}}, \text{Hom}(T, \Delta_{\mathfrak{g}})) = 0.$$

*Step 2.*

523 It is convenient now to observe that the structure of  $\Delta_{\mathfrak{g}}$ -module of  $\text{Hom}(T, \mathfrak{g}, \Delta_{\mathfrak{g}})$  is  
524 trivial. Thus, we can identify  $H^1(\Delta_{\mathfrak{g}}, \text{Hom}(T, \mathfrak{g}/\Delta_{\mathfrak{g}}))$  with  $(H^1(\Delta_{\mathfrak{g}}, \mathbb{K}))^{\dim T}$ . Since the  
525 Lie algebra  $\Delta_{\mathfrak{g}}$  is perfect,  $H^1(\Delta_{\mathfrak{g}}, \mathbb{K}) = 0$ . Taking into account both *Step 1* and *Step 2*  
526 above, we deduce from the exact cohomology sequence given by the exact sequence 19 that  
527  $H^1(\Delta_{\mathfrak{g}}, \text{Hom}(T, \mathfrak{g})) = 0$ .  $\square$

*An algebraic proof of theorem 7*

528 *Proof.* Let us keep the notations already used above. Let us fix a connected open set  
529  $V \subset M - \Sigma(a)$ . From the sheaf theoretic point of view, one is interested in the following  
530 sheaf cohomology:

$$H^1(\Delta \text{Ker}(a), \Gamma(\text{Hom}(a(A), \text{Ker}(a)))).$$

531 By taking into account Theorem 8, one sees that:

$$H^1(\Delta \text{Ker}(a), \Gamma(\text{Hom}(a(A), \text{Ker}(a)))) = 0.$$

532 In other words, one has:

$$\text{Ext}_{\Delta \text{Ker}(a)}^1((a(\Gamma(A)), \text{Ker}(a))) = 0.$$

533 Thereby, the following exact sequence of  $\Delta \Gamma(\text{ker}(a))$ -modules splits:

$$0 \longrightarrow \text{Ker}(a) \longrightarrow \Gamma(A) \longrightarrow a(\Gamma(A)) \longrightarrow 0$$

534 The last assertion means that every point  $p \in V$  has an open neighborhood  $U$  on which the  
535 following claims hold:

536 (a) It exists a partial basis of local sections of  $A$ , say  $(\sigma_i)_{i=1\dots r}$ , such that:

$$\forall p \in U, \text{rank}\{a(\sigma_1)(p), \dots, a(\sigma_r)(p)\} = r;$$

537 (a) It exists a basis of local sections of  $\Delta\text{Ker}(a)$ , say  $(\tau_j)_{j=1\dots n-r-1}$  satisfying:

$$[\sigma_i, \tau_j] = 0.$$

538 To yield the normal forms theorem of [12], let us assume the domain  $U$  of  $(\sigma_i, \tau_j)$  to be  
539 also a domain of local coordinate functions of  $V$ , say  $(x_i, y_l)_{i=1\dots r, l=1\dots m-r}$  such that the  
540 connected components of the leaves of  $a(A)$  which are included in  $U$  are defined by the  
541 following systems:

$$y_l = \text{cte}, l = 1 \dots m - r$$

542 Thereby, one can choose the local basis  $(\sigma_i)$  such that:

$$a(\sigma_i) = \partial x_i, i = 1 \dots r.$$

543 The last conditions imply that:

$$[\sigma_i, \sigma_{i'}] = 0.$$

544 Therefore, the property  $[\sigma_i, \tau_l] = 0$  implies that:

$$[\tau_l, \tau_{l'}] = \sum_{l''} c_{ll''}^{\tau_{l'}}(y_1, \dots, y_{m-r}) \tau_{l''}.$$

545 To end the "algebraic" proof of Theorem 7, let us remind that all of the fibers of  
546  $(\ker(a))(p), p \in V$ , are isomorphic to a fixed non solvable affineline Lie algebra  $(\mathfrak{g}, e)$ .  
547 So,  $\text{Ker}(a)$  may be regarded as a  $(\mathfrak{g}, e)$ -Current algebra over  $V$ , [16]. Now, to end the proof,  
548 let us assume that  $U$  is also a domain of trivialization of  $(A, a)$  as well. So, as vector bundle  
549 over  $U$ ,  $\ker(a)$  isomorphic to  $Ux(\mathfrak{g}, e)$ . Let  $\tau_0$  be the section of  $A$  defined by:

$$\tau_0(p) = (p, e), \forall p \in U.$$

Now, to every couple  $(u, s) \in \Delta_e \mathfrak{g} \times S$ , we assign the couple  $(v, \tau)$  of sections of  $A$  which  
is defined by:

$$v(p) = (p, u)$$

$$\tau(p) = (p, s)$$

550 Actually, one easily sees that for  $i = 1 \dots r$ ,  $[\tau_0, \sigma_i] \in \Gamma(\ker(a))$ . On one hand, the  
551 identity:

$$[\tau, [\tau_0, \sigma_i]] = 0$$

552 shows that  $[\tau_0, \alpha_i] \in \Delta\text{Ker}(a)$ . On the other hand, the identity:

$$[\tau, [\tau_0, \sigma_i]] = 0$$

553 shows that  $\forall p \in V$ ,  $[\tau_0, \sigma_i](p) \in H^0(S, \Delta\mathfrak{g})$ . Since  $(\mathfrak{g}, e)$  is non solvable, one has to  
554 conclude that:

$$[\tau_0, \sigma_i] = 0.$$

555 Thus, the system:

$$\{\tau_l, \sigma_i\}_{l=0\dots n-r, i=1\dots r}$$

556 is a basis of local sections of  $A$  satisfying the properties required in Theorem 7. To get the  
 557 general case, let us assume that the  $a$ -algebroid  $(A, a)$  is stratified in following sense. The  
 558 manifold  $M$  admits a filtration:

$$M = \Sigma_0 \supset \Sigma_1 \supset \dots \supset \Sigma_i \supset \dots$$

559 with the next property. Let  $(A_i, a_i; )$  be the pull back of  $(A, a)$  by the inclusion map of  $\Sigma_i$   
 560 in  $M$ . Then:

561 (Str 1)  $\Sigma_{i=1}$  consists of singular points of  $(A_i, a_i)$ .

562 (Str 2) The family  $\Sigma_i - \Sigma_{i+1}$  is locally finite.

563 Claim (Str 1) implies that  $a_i(A_i)$  defines a regular foliation on  $\Sigma_i - \Sigma_{i+1}$ . Our algebraic  
 564 proof of Theorem 7 works for the restriction  $\Sigma_i - \Sigma_{i+1}$  of  $(A_i, a_i)$ . Actually all of the  $\Sigma_i$  are  
 565 closed sub-manifolds of  $M$ . Let  $d_i$  be the dimension of  $\sigma_i$ . Every point of  $M$  has an open  
 566 neighborhood  $U$  which is the domaine of local coordinate functions of  $M$ , say  $(x_1, \dots, x_m)$   
 567 such that the connected components of  $U \cap \Sigma_i$  are defined by the system

$$x_{d_i+1} = \text{cte}, \dots, x_m = \text{cte}.$$

568 The last assertion yields Theorem 7 for stratified  $a$ -algebroid. □

569 Let us make the following observations. Firstly, a Lie algebroid  $(A, a)$  over the manifold  
 570  $M$  is called  $s$  — algebroid if  $\forall p \in M, \ker(a)(p)$  is a semi-simple Lie algebra. Now, we  
 571 observe that all of the cohomology arguments which are used in the algebraic proof of  
 572 Theorem 7 work for  $s$ -algebroids. Secondly, R. Wolak and the author have proved the  
 573 analogue to Theorem 7 for Koszul-Vinberg algebroids, [6]. Our analytic proof of normal  
 574 forms theorem for Koszul-Vinberg algebroids might have its algebraic counterpart in KV-  
 575 cohomology [3–5].

## 576 7. CONCLUSION

577 In information geometry, the divergence function defines a regular Poisson structure in the  
 578 neighborhood of the diagonal, which becomes a Lagrangian sub-manifold for it. Poisson  
 579 tensors are also encountered when dealing with antisymmetric  $(1, 1)$ -tensors satisfying a  
 580 gauge equation, as indicated in the introduction. It is thus relevant to study such structures,  
 581 especially from the standing viewpoint of linearization. Within this frame, as pointed out by  
 582 A. Weinstein, understanding the question of degeneracy of Lie algebra is a key ingredient.  
 583 Our treatment of the problem makes use of cohomological machineries culminating in  
 584 theorem 5 stating that any non solvable affinelike Lie algebra is non-degenerate. This result  
 585 is part of a larger program for understanding Poisson structures in information geometry,  
 586 that will be addressed in future works.

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