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► **To cite this version:**

Michel Boyom, Emmanuel Gnandi, Stéphane Puechmorel. Hyperbolicity of the Compact Leaves of Statistical Manifold Foliations. 2021. hal-03008361v2

**HAL Id: hal-03008361**

**<https://enac.hal.science/hal-03008361v2>**

Preprint submitted on 9 Feb 2021

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Article

# Hyperbolicity of the Compact Leaves of Statistical Manifold Foliations

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Version February 9, 2021 submitted to Journal Not Specified

**Abstract:** In this work we show a construction of a Hessian foliation on a statistical manifold. Our results provide conditions under which the compact hessian leaves are quotients of homogeneous convex cones, up to diffeomorphism.

## 1. Introduction

Hessian structures play an important role in information geometry. In [1], Shima observed that the Fisher information of most statistical models are Hessian metrics. Amari also proved in [2] that any locally flat statistical manifold is a Hessian manifold. These structures arised from works of many mathematicians (such as J.-L. Koszul, Y. Matsushima, A. Nijenhuis, E. B. Vinberg) in their attempt to solve the Gerstenhaber conjecture in the category of locally flat manifolds. The geometry of locally flat hyperbolic manifolds is also called Koszul geometry (see [3]). Gerstenhaber's conjecture says that every restricted theory of deformation generates its proper theory of cohomology. This conjecture has been solved in 2006 by Michel Boyom (see [4]). In [5] and [6], Boyom also realizes the homological version of the Hessian geometry. The aim of this paper is to study hyperbolic foliations in statistical manifolds. In this context, a hyperbolic foliation is a foliation whose leaves handle Koszul geometry. The paper is organised as follows: section 2 is devoted to basic notions which are used later in the paper; section 3 covers useful notions about the category of locally flat manifolds and their KV-cohomology. It is also devoted to Hessian equations of Koszul connections in the tangent bundle of a statistical manifold; section 4 aims at focusing on canonical pairs of locally flat foliations in a statistical manifold, before discussing conditions under which these foliations are hyperbolic. Finally, quotients of convex cones and exponential models are discussed in section 5.

## 2. Basic notions

For the sake of completeness and to make this paper as much self-contained as possible, the basic notions which link Koszul geometry to information geometry will be recalled. Our goal is to make this note as self-contained as possible.

23 **2.1. Koszul connections.**

Let  $M$  be a differentiable manifold,  $C^\infty(M)$  the vector space of real valued functions defined in  $M$  and  $\mathcal{X}(M)$  the vector space of differentiable vector fields in  $M$ . Let  $(V, \pi, M)$  be a vector fiber bundle over  $M$ . Loosely speaking, we have a submersion

$$V \ni e \rightarrow \pi(e) \in M,$$

24 whose fibers  $\pi^{-1}(x)$  are vector spaces.

25 The space of sections of  $V$  is denoted as  $\mathcal{V}$ . The cotangent bundle of  $M$  is denoted as  $T^*M$ . The tensor product  
26  $T^*M \otimes V$  is canonically identified with the vector bundle  $\text{Hom}(TM, V)$ .

**Definition 2.1.** A Koszul connection on a vector bundle  $V$  is a first order differential operator

$$\mathcal{V} \ni s \rightarrow \nabla s \in T^*M \otimes \mathcal{V}$$

which satisfies the following requirement

$$\nabla(fs) = f\nabla s + df \otimes s, \forall f \in C^\infty(M). \quad (2.1)$$

Let

$$\nabla_X s = (\nabla s)(X). \quad (2.2)$$

The above requirement means, given  $(X, s) \in \mathcal{X}(M) \times \mathcal{V}$  and  $f \in C^\infty(M)$ , that

$$(\nabla(fs))(X) = f\nabla_X s + df(X)s. \quad (2.3)$$

The curvature (of  $\nabla$ ),  $R^\nabla \in \otimes^2 T^*M \otimes \mathcal{V}^* \otimes \mathcal{V}$  is defined as follows:

$$R^\nabla(X, Y)(s) = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X, Y]}s. \quad (2.4)$$

27 **Definition 2.2.** A vector fiber bundle is called a flat bundle if it admits a Koszul connection whose curvature vanishes  
28 identically.

29 When dealing with Riemannian manifolds, there is an distinguished connection that plays a particular role in the  
30 study of the metric.

**Definition 2.3.** Let  $(M, g)$  be a Riemannian manifold. A connection  $\nabla$  is said to be metric if for any triple  $(X, Y, Z)$  of vector fields:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

31 **Proposition 2.4.** On a Riemannian manifold, it exists a unique metric connection without torsion, called the  
32 Levi-Civita connection and denoted by  $\nabla^{\text{lc}}$

33 **2.2. Locally flat manifolds.**

Koszul connections on tangent vector bundles  $TM$  are usually called linear connections on  $M$ . The torsion of a linear Koszul connection  $\nabla$ ,  $T^\nabla \in \otimes^2 T^*M \otimes TM$  is defined as follows

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \quad (2.5)$$

**Definition 2.5.** A locally flat manifold is a pair  $(M, \nabla)$  where  $\nabla$  is a linear Koszul connection (on  $TM$ ) whose curvature  $R^\nabla$  and torsion  $T^\nabla$  both vanish identically.

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0, \quad (2.6)$$

$$\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]} Z = 0 \quad \forall X, Y, Z \in \Gamma(TM). \quad (2.7)$$

*Example 5:* Let  $\mathbb{R}^n = \{(x_1, \dots, x_n) / x_i \in \mathbb{R}\}$  be endowed with the linear connection  $\nabla^F$  defined as

$$\nabla^F \frac{\partial}{\partial x_i} = 0$$

34  $\forall i = 1, 2, \dots, n$  and  $\forall X \in \mathcal{X}(M)$

35

*Example 6:* Let  $\mathbb{R}^2$  and  $\forall X = f \frac{\partial}{\partial x_1} + g \frac{\partial}{\partial x_2} \in \mathcal{X}(M)$  be endowed with a linear connection defined as

$$\nabla \frac{\partial}{\partial x_1} = f \frac{\partial}{\partial x_1} \text{ and } \nabla \frac{\partial}{\partial x_2} = 0$$

36

37  $(\mathbb{R}^2, \nabla)$  is a locally flat manifold.

38 **2.3. The developing map of a locally flat manifold.**

Set a point  $x_0 \in M$  and consider the pairs  $\{0, [0, 1]\}$  and  $\{x_0, M\}$ . We now consider the space of pointed differentiable paths

$$c: \{0, [0, 1]\} \rightarrow \{x_0, M\}.$$

With the above notation meaning that

$$c(0) = x_0.$$

39 We will also have to consider differentiable paths in the tangent bundle. To distinguish them from paths in the  
40 manifold, a capital letter will be used.

41 **Definition 2.6.** Let  $c: [0, 1] \rightarrow M$  be a differentiable path. A path  $Y: [0, 1] \rightarrow TM$  is said to be a lift of  $c$  if  $\pi \circ Y = c$   
42 where  $\pi: TM \rightarrow M$  is the canonical projection.

**Definition 2.7.** Let  $\nabla$  be a Koszul connection. A path  $Y: [0, 1] \rightarrow TM$  is said to be the horizontal if:

$$\nabla \dot{c}(t) Y(t) = 0, \quad t \in [0, 1]$$

43 with  $c = \pi \circ Y$ .

44 Using the Cauchy-Lipschitz theorem, it is easy to obtain the next result:

45 **Proposition 2.8.** *Let  $c$  be a  $C^1$ -path in  $M$ ,  $\nabla$  a Koszul connection and let  $V \in T_{c(0)}M$  be given. It exists a unique*  
 46 *horizontal  $Y \rightarrow TM$  lifting  $c$ .*

**Definition 2.9.** *The parallel transport along the  $C^1$ -path  $c: [0, 1] \rightarrow M$ ,  $c(0) = x_0$ ,  $c(1) = x_1$  is the linear mapping:*

$$\tau: V \in T_{x_0}M \rightarrow \tau(V) = Y(1) \in T_{x_1}M$$

47 *where  $Y$  is the unique horizontal lift of  $c$ .*

We fix  $\sigma \in [0, 1]$ , the parallel transport along the path

$$c: [0, \sigma] \rightarrow M$$

is denoted by  $\tau_\sigma$ ; this a one to one linear map of  $T_{x_0}M$  in  $T_{c(\sigma)}M$ . Thus we have

$$\tau_\sigma^{-1}\left(\frac{dc}{dt}(\sigma)\right) \in T_{x_0}M.$$

We define an application

$$c \rightarrow \mathcal{Q}(c) \in T_{x_0}M$$

by putting

$$\mathcal{Q}(c) = \int_0^1 \tau_\sigma^{-1}\left(\frac{dc}{dt}(\sigma)\right) d\sigma.$$

48 It is called the development of  $c$ .

**Definition 2.10.** *Let  $c$  and  $c^*$  be two pointed paths such that  $c(1) = c^*(1)$ . They are said to be homotopic if it exists a differential map:*

$$[0, 1] \times [0, 1] \ni (s, t) \rightarrow H(s, t) \in M$$

49 *satisfying:*

50 (1)  $H(s, 0) = x_0, \forall s;$

51 (2)  $H(s, 1) = c(1) = c^*(1) \forall s;$

52 (3)  $H(0, t) = c(t);$

53 (4)  $H(1, t) = c^*(t).$

Let us go back to the map

$$c \rightarrow \mathcal{Q}(c) \in T_{x_0}M.$$

54 The following assertion is a classical result, that can be obtained from the fact that a flat connection has a trivial  
 55 holonomy group.

**Proposition 2.11.** [7] *Let  $\nabla$  be a flat connection. If the paths  $c$  and  $c^*$  are homotopic then*

$$\mathcal{Q}(c) = \mathcal{Q}(c^*). \tag{2.8}$$

Given a locally flat structure  $(M, \nabla)$ , we fixe  $x_0 \in M$  we consider the space of pointed differentiable paths

$$\{(0, [0, 1]) \rightarrow (x_0, M)\}.$$

The set of their homotopy classes is the universal covering of  $M$ , denoted by  $\tilde{M}$ . The connection  $\nabla$  lifts to  $\tilde{M}$  and so deos the map  $\mathcal{Q}(c)$ :

$$\tilde{M} \ni [c] \rightarrow \mathcal{Q}([c]) \in T_{x_0}M.$$

56 This map is called the developing map of the locally flat manifold  $(M, \nabla)$ . We are in position to highlight the main  
57 question of this paper: **What does the image  $\mathcal{Q}(\tilde{M}) \subset T_{x_0}M$  looks like?**

58 The next definition is due to Koszul.

59 **Definition 2.12.** *A locally flat manifold  $(M, \nabla)$  is called hyperbolic if  $\mathcal{Q}(\tilde{M})$  is an open convex domain in the vector  
60 space  $T_{x_0}M$  not containing any straight line.*

61 **Theorem 2.13.** *[7] For  $(M, \nabla)$  being hyperbolic it is necessary that there exists a de Rham closed differential 1-form  
62  $\omega$  whose covariant derivative  $\nabla\omega$  is positive definite. If  $M$  is compact, then this condition is also sufficient.*

63 **Remark 2.14.** *The existence of the nonvanishing Koszul 1-form  $\omega$  proves that  $M$  is a radiant suspension and by  
64 using Tischler [8], we conclude that  $M$  is fibered over  $S^1$ . Topological consequences are that the Euler characteristic  
65  $\chi(M) = 0$  and the first Betti number  $b_1(M) > 0$ .*

66 This definition is the affine analogue to hyperbolic holomorphic manifolds following W.Kaup(see[9];[7]). The notion  
67 of hyperbolicity is among the fundamental notions of the geometry of Koszul. It may be addressed and studied  
68 form many perspectives. We are going to highlight the algebraic topology point of view, which is richer than the  
69 Riemannian geometry one. The reason is that the homological point of view highlights relationships with other  
70 research domains. An examples of these relations is the theory of affine representations of Lie groups which plays a  
71 key role in **Lie group theory of Heat** after Jean-Marie Souriau. In this paper we also perform the theory of affine  
72 representation to investigate the point set-topology of the image  $\mathcal{Q}(\tilde{M})$ .

73

#### 74 2.4. Algebraic topology of locally flat manifolds

75 **Definition 2.15.** *An affine structure in an  $m$ -dimensional differentiable manifold  $M$  is a pair  $(M, \mathcal{A})$  where  $\mathcal{A}$  is a  
76 complete atlas whose local coordinates changes coincide with affine transformations of  $\mathbb{R}^m$ .*

77 **Remark 2.16.** *There is a one to one correspondence between the category of locally flat structures in  $M$  and the  
78 category of affine structures in  $M$ .*

Many cochain complexes are associated with a locally flat manifold  $(M, \nabla)$ . We limit ourselves to what is used in this paper. readers interested in the general theory of cohomology of Koszul–Vinberg algebras are referred to Nguiffo

Boyom 2006, [10] and to Nijenhuis Albert 1968 . The group of integers is denoted by  $\mathbb{Z}$ .

$Aff(M, \nabla : \mathbb{R})$  stands for the vector space of  $f \in C^\infty(M)$  which are subject to the following requirement,

$$\nabla^2(f) = 0.$$

To investigate the structure of  $\mathcal{Q}((M, \nabla))$ , we introduce the  $\mathbb{Z}$ -graded vector space:

$$C(\nabla) = \bigoplus_q C^q(\nabla)$$

With homogeneous vector subspaces:

$$C^q(\nabla) = 0 \text{ if } q < 0;$$

$$C^0(\nabla) = Aff(M, \nabla : \mathbb{R});$$

$$C^q(\nabla) = Hom_{\mathbb{R}}(\otimes^q \mathcal{X}(M), C^\infty(M))$$

The coboundary operator

$$\delta : C^q(\nabla) \rightarrow C^{q+1}(\nabla)$$

79 is defined as it follows:

80

$$81 \quad (a) \delta f = df \forall f \in C^0(\nabla);$$

82

(b) let  $q > 0, f \in C^q(\nabla)$  and  $X_1 \otimes \dots \otimes X_{q+1} \in \otimes^{q+1} \mathcal{X}(M)$  then

$$\delta f(X_1 \otimes \dots \otimes X_{q+1}) = \sum_1^q (-1)^i [d(f(\dots \hat{X}_i \dots \otimes X_{q+1})) (X_i) - \sum_{j \neq i} f(\dots \hat{X}_i \dots \otimes \nabla_{X_i} X_j \otimes \dots)]$$

Remember that  $\hat{X}_i$  means that  $X_i$  is missing.

The pair  $(C(\nabla), \delta)$  is a differential vector space. This means that the linear  $\delta$  is subject to the requirement

$$\delta^2 = \delta \circ \delta = 0.$$

Elements of  $ker(\delta)$  are called cocycles; those of  $im(\delta)$  are called coboundaries.

The  $q$ -th cohomology space  $H^q(\nabla)$  is defined by:

$$H_{KV}^q(\nabla) = \frac{Ker(\delta)}{Im(\delta)}.$$

In this paper we are mainly concerned by the first terms of the complex:

$$\rightarrow C^1(\nabla) \rightarrow C^2(\nabla) \rightarrow C^3(\nabla) \rightarrow$$

83 Elements  $C^2(\nabla)$  are called KV 2-cochains. The coboundary of  $\partial \in C^2(\nabla)$  is computed as follows:

$$\begin{aligned} \delta \partial(X_1 \otimes X_2 \otimes X_3) &= -d(\partial(X_2 \otimes X_3))(X_1) + d\partial(X_1 \otimes X_3)(X_2) + \partial(\nabla_{X_1} X_2 \otimes X_3) + \partial(X_2 \otimes \nabla_{X_1} X_3) \\ &\quad - \partial(\nabla_{X_2} X_1 \otimes X_3) - \partial(X_1 \otimes \nabla_{X_2} X_3). \end{aligned}$$

84 **Remark 2.17.**  $d$  stands here for the usual exterior derivative.

**Definition 2.18.** A 2nd cohomology class

$$[\partial] \in H_{KV}^2(\nabla)$$

85 is called a Hessian (resp. Hessian non-degenerate, Hessian positive definite) class if it contains a symmetric (resp.  
86 symmetric non-degenerate, symmetric positive definite) cocycle.

87 **Remark 2.19.** The Hessian geometry defined in [1] corresponds to the positive definite case.

88 A classical result states that a manifold  $M$  admits locally flat structures if and only if it admits affine structures. It is  
89 worth to notice that locally flat geometry is linked to the Cartan-Koszul geometry in  $TM$ , while the affine geometry is  
90 related to local analysis in  $M$ .

91 **Proposition 2.20.** [4] A locally flat manifold  $(M, \nabla)$  is Hessian if  $\delta\partial$  vanishes identically.

92 The next theorem relates hyperbolicity to KV cohomology.

93 **Theorem 2.21.** Let  $(M, g, \nabla)$  be a compact Hessian manifold then the following assertions are equivalent;

94 (a.1)  $[g] = 0 \in H_{KV}^2(\nabla)$ .

95 (a.2)  $(M, \nabla)$  is hyperbolic ♣

96 In order to characterize  $\mathcal{Q}(\tilde{M})$  and hyperbolicity of leaves, we will follow the next scheme. Firstly we are going to  
97 introduce some canonical dynamics in a statistical manifold whose orbits are locally flat manifolds. Then we study  
98 the question of whether an orbit  $M$  is hyperbolic or not using the Koszul characterisation, that is  $\mathcal{Q}(\tilde{M})$  is an open  
99 convex domain not containing a straight line. Finally, we investigate conditions under which  $\mathcal{Q}(\tilde{M})$  is an open convex  
100 cone. In such a case, the statistical manifold can be given an exponential statistical model based on the characteristic  
101 function of  $\mathcal{Q}(\tilde{M})$ . For details, readers may refer to the next references [11],[12],[13].

### 102 3. Hessian differential operators of gauge structures.

103 Throughout the paper we keep the notation of Nguiffo Boyom [14], [5].

104 We recall that a gauge structure in a manifold  $M$  is a pair  $(M, \nabla)$  where  $\nabla$  is a Koszul connection in the tangent  
105 bundle  $TM$ .

106

Let us introduce the Hessian differential operator:

$$\mathcal{X}(M) \ni X \rightarrow \nabla^2 X \in T_1^2(M)$$

107 with  $T_1^2(M)$  is the space of (2,1)-tensors in  $M$ .

108 The next proposition is an easy consequence of the definition of the covariant derivative of a tensor.

**Proposition 3.1.** For any triple  $(X, Y, Z)$  of vector fields:

$$\nabla^2 X(Y, Z) = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z.$$



Letting  $X.Y = \nabla_X Y$ , it reads as:

$$\nabla^2 X(Y, Z) = X.(Y.Z) - (X.Y).Z$$

109 showing that  $\nabla^2$  is the associator of the product defined above.

Let  $(x_1, \dots, x_m)$  be a system of local coordinate functions of  $M$  and let  $X \in \mathcal{X}(M)$ . We set

$$\partial i = \frac{\partial}{\partial x_i},$$

$$X = \sum_k X^k \partial k,$$

$$\nabla_{\partial i} \partial j = \sum_k \Gamma_{ij}^k \partial k.$$

110 The principal of symbol of the Hessian differential operator can be expressed as

$$(\nabla^2 X)(\partial i, \partial j) = \sum_k \Omega_{ij}^k \partial k \quad (3.1)$$

where

$$\Omega_{ij}^l = \frac{\partial^2 X^l}{\partial x_i \partial x_j} + \sum_k \left[ \Gamma_{ik}^l \frac{\partial X^k}{\partial x_j} + \Gamma_{jk}^l \frac{\partial X^k}{\partial x_i} - \Gamma_{ij}^k \frac{\partial X^l}{\partial x_k} \right] + \sum_k \left[ \frac{\partial \Gamma_{jk}^l}{\partial x_i} + \sum_m (\Gamma_{jk}^m \Gamma_{im}^k - \Gamma_{ij}^m \Gamma_{mk}^l) \right] \quad (3.2)$$

On  $(M, \nabla)$ , the Hessian equation is the following 2nd order differential equation, [14]

$$FE(\nabla) : \nabla^2 X = 0.$$

Its sheaf of solutions will be denoted by  $\mathcal{J}_\nabla(M)$  in the sequel. It is easily found to be a sheaf of associative algebras whose product of sections is given as

$$X.Y = \nabla_X Y.$$

111 The space of sections of  $\mathcal{J}_\nabla(M)$  will be denoted by  $\mathcal{J}_\nabla$ .

112 The pair  $(\mathcal{J}_\nabla, \nabla)$  is an associative algebra with commutator Lie algebra  $(\mathcal{J}_\nabla, [-, -]_\nabla)$  where the bracket  $[X, Y]_\nabla$   
113 is:

$$[X, Y]_\nabla = \nabla_X Y - \nabla_Y X.$$

Remember that  $(M, \nabla)$  is called symmetric gauge structure if the Koszul connection  $\nabla$  is symmetric, that is to say that the torsion of  $\nabla$  vanishes identically.

Therefore when  $(M, \nabla)$  is symmetric, the Lie algebra  $(\mathcal{J}_\nabla, [-, -]_\nabla)$  is a Lie subalgebra of the Lie algebra of vector fields

$$(\mathcal{X}(M), [-, -]);$$

114 According to [14] when  $(M, \nabla)$  is symmetric the Lie subalgebra  $\mathcal{J}_\nabla \subset \mathcal{X}(M)$  is finite dimensional over the field of  
115 real numbers.

116 **Proposition 3.2.** *Let  $(M, g)$  be a Riemman manifold and let Ric be its Ricci curvature tensor. We have*

$$\mathcal{J}_\nabla \subset \text{Ker}(\text{Ric}) \quad (3.3)$$

**Proof.** Let  $(X, Y)$  be a couple of vector fields and let  $\xi \in \mathcal{J}_\nabla$ . Then:

$$R^\nabla(Y, X)\xi = (\nabla^2\xi)(X, Y) - (\nabla^2\xi)(Y, X). \quad (3.4)$$

Taking the trace we deduce that:

$$\text{Ric}(X, \xi) = 0 \quad (3.5)$$

Then

$$\mathcal{J}_\nabla \subset \text{KerRic}$$

117     □

118 As a consequence, it may turn out that  $\mathcal{J}_\nabla$  is trivial:

**Corollary 3.3.** *If  $\nabla$  is the Levi-Civita connection of an Einstein Riemannian manifold, then*

$$\mathcal{J}_\nabla = \{0\}$$

119 Using the third theorem of Lie, it comes:

120 **Theorem 3.4.** *Up to a Lie group isomorphism there exists a unique simply connected Lie group  $G_\nabla$  whose Lie*  
121 *algebra is isomorphic to the Lie algebra  $\mathcal{J}_\nabla$ .*

122 Before pursuing we going to recall a few useful notions which will be involved in the sequel. Let  $G$  be a Lie group  
123 whose Lie algebra is denoted by  $\mathcal{G}$ .

124 **Definition 3.5.** *A Lie algebra homomorphism of  $\mathcal{G}$  in the Lie algebra  $\mathcal{X}(M)$  is called an infinitesimal differentiable*  
125 *action of  $G$  in  $M$ .*

126 *An infinitesimal differentiable action of  $G$  in  $M$  is called integrable if it is the derivative of an action of  $G$  in  $M$  [15].*

### 127 Example

128 In a compact manifold  $M$  every infinitesimal action of a finite dimensional Lie group is integrable. That is due to the  
129 fact every vector field  $X$  is complete in the meaning that  $X$  is a generator of a one parameter subgroup of the group of  
130 diffeomorphisms.

A non trivial example of integrable infinitesimal action arises in our setting. Given a gauge structure  $(M, \nabla)$  remember that the elements of the Lie subalgebra

$$\text{aff}(M, \nabla) \subset \mathcal{X}(M)$$

131 are vector fields  $X$  satisfying the next identity:

$$[X, \nabla_Y Z] - \nabla_{[X, Y]} Z - \nabla_Y [X, Z] = 0, \forall (Y, Z) \in \mathcal{X}(M) \quad (3.6)$$

132 .

**Definition 3.6.** A path  $c(t)$  is a geodesic in  $(M, \nabla)$  if it is a solution of the following differential equation

$$\nabla_{\dot{c}(t)}(\dot{c}(t)) = 0.$$

<sup>133</sup>  $(M, \nabla)$  is called geodesically complete if all of its geodesics are defined in  $]-\infty, +\infty[$ .

<sup>134</sup> **Proposition 3.7.** If  $(M, \nabla)$  is geodesically complete then elements of  $\text{aff}(M, \nabla)$  are complete, see  
<sup>135</sup> Kobayashi-Nomizu, [16].

Let  $(M, \nabla)$  be a symmetric gauge structure. We keep the notation  $J_\nabla, G_\nabla$ .

The Lie subalgebra

$$J_\nabla \subset \mathcal{X}(M)$$

is a locally effective infinitesimal differentiable action of  $G_\nabla$  in  $M$ . If either  $\nabla$  is geodesically complete or  $M$  is compact, then  $J_\nabla$  is integrable. Thus it is the infinitesimal counterpart of a locally effective differentiable action

$$G_\nabla \times M \ni (\gamma, x) \rightarrow \gamma.x \in M.$$

<sup>136</sup> **Proposition 3.8.** ([14] Under the same assumptions, if  $J_\nabla$  is integrable then each orbit of  $G_\nabla$  is an homogeneuous  
<sup>137</sup> locally flat manifold.

### <sup>138</sup> 3.0.1. A canonical representaion of $G_\nabla$

Let  $G$  be a group and the  $W$  be a finite dimensional real vector space whose group of affine isomorphisms is denoted by  $\text{Aff}(W)$ . This group is a semi-direct product  $GL(W) * W$  whose underlying set is the Cartesian product

$$GL(W) \times W$$

and the composition rule is

$$(\gamma_1, w_1) * (\gamma_2, w_2) = (\gamma_1 \circ \gamma_2, \gamma_1(w_2) + w_1)$$

There is a natural affine representaion of the Lie algebra  $J_\nabla$  in itself as a vector space:

$$J_\nabla \ni X \rightarrow \rho(X) = (\nabla_X, X) \in gl(J_\nabla) \times J_\nabla = \text{aff}(J_\nabla).$$

The affine action is defined as:

$$\rho(X).Y = \nabla_X Y + X, \quad \forall Y \in J_\nabla.$$

By virtue of the universal property of simply connected finite dimensional Lie groups, there exist a unique continuous affine representation

$$G_\nabla \ni \gamma \rightarrow (f(\gamma), q(\gamma)) \in GL(J_\nabla) \times J_\nabla = \text{Aff}(J_\nabla),$$

<sup>139</sup> whose differential at the unit element is the representation  $\rho$ .

So  $e$  being the unit element of  $G_\nabla$  one has

$$[(df)(e)](X) = \nabla_X,$$

$$[(dq)(e)](X) = X.$$

140 In the sequel, we point out how this canonical affine representation impacts the Information Geometry in statistical  
141 manifolds.

### 142 3.1. *The radiant class of affine representations.*

We keep the notations  $G_{\nabla}, \mathcal{J}_{\nabla}, \rho, f, q$  as in the section above.

The couple  $(f, q)$  is a continuous homomorphism of the Lie group  $G_{\nabla}$  on the Lie group  $Aff(\mathcal{J}_{\nabla})$ . Therefore one has

$$[f(\gamma), q(\gamma)] \circ [f(\gamma^*), q(\gamma^*)] = [f(\gamma) \circ f(\gamma^*), f(\gamma)q(\gamma^*) + q(\gamma)] = [f(\gamma \cdot \gamma^*), q(\gamma \cdot \gamma^*)].$$

So  $f$  is a linear representation of  $G_{\nabla}$  in  $\mathcal{J}_{\nabla}$  and  $q$  is a  $\mathcal{J}_{\nabla}$  valued 1-cocycle of  $f$ . The cohomology class of  $q$  is denoted by

$$[q] \in H^1(G_{\nabla}, \mathcal{J}_{\nabla}).$$

Let

$$f_{\nabla} : G_{\nabla} \longrightarrow GL(\mathcal{J}_{\nabla})$$

$$q_{\nabla} : G_{\nabla} \longrightarrow \mathcal{J}_{\nabla}$$

Remind the cohomology of the Lie group  $G_{\nabla}$  value in his Lie algebra  $\mathcal{J}_{\nabla}$  is defined by :

$$\dots \rightarrow C^q(G_{\nabla}, \mathcal{J}_{\nabla}) \rightarrow C^{q+1}(G_{\nabla}, \mathcal{J}_{\nabla}) \rightarrow C^{q+2}(G_{\nabla}, \mathcal{J}_{\nabla}) \rightarrow$$

with differential operator  $D$  defined by :

$$D\partial(\gamma_1, \dots, \gamma_{q+1}) = f_{\nabla}(\gamma_1) \cdot \partial(\gamma_2, \dots, \gamma_{q+1}) + \sum_{i \leq q} (-1)^i \partial(\dots, \gamma_i \gamma_{i+1}, \dots) + (-1)^q \partial(\gamma_1, \dots, \gamma_q)$$

See Koszul[17], Cartan and Eilenberg [18] and Chevalley and Eilenberg [19], M. Boyom[20] for more details on Lie algebra cohomology.

It's easy to see that the condition :

$$q_{\nabla}(\gamma_1 \cdot \gamma_2) = f_{\nabla}(\gamma_1) q_{\nabla}(\gamma_2) + q_{\nabla}(\gamma_1)$$

$$\forall \gamma_1, \gamma_2 \in G_{\nabla}$$

143 is equivalent to  $q_{\nabla} \in Z^1(G_{\nabla}, \mathcal{J}_{\nabla})$ . Let  $[q_{\nabla}] \in H^1(G_{\nabla}, \mathcal{J}_{\nabla})$  be its cohomology class. It is called the radiant class of the  
144 affine representation  $(f, q)$ .

145 **Theorem 3.9.** ] *The following statements are equivalent :*

1. *The affine action*

$$G_{\nabla} \times \mathcal{J}_{\nabla} \ni (\gamma, X) \rightarrow f(\gamma) \cdot X + q(\gamma) \in \mathcal{J}_{\nabla}$$

146 *has a fixed point;*

147

148 2. *The cohomology class  $[q]$  vanishes;*

149

3. *The affine representation*

$$G_{\nabla} \ni \gamma \rightarrow (f(\gamma), q(\gamma)) \in Aff(\mathcal{J}_{\nabla})$$

is conjugated to the linear representation

$$G_{\nabla} \ni \gamma \rightarrow f(\gamma) \in GL(\mathcal{J}_{\nabla}) \clubsuit$$

**Proof.** (1) implies (2).

Let  $-Y_0$  be a fixed point of the affine action  $(f, q)$ , then

$$f(\gamma)(-Y_0) + q(\gamma) = -Y_0, \quad \forall \gamma \in G_{\nabla}.$$

Therefore one has

$$q(\gamma) = f(\gamma)(Y_0) - Y_0, \quad \forall \gamma \in G_{\nabla}.$$

So the cocycle  $q$  is exact.

(2) implies (3).

Consider the affine the affine isomorphism  $(e, Y_0)$ . It is nothing but the translation by  $Y_0$

$$X \rightarrow X + Y_0;$$

We calculate

$$(e, Y_0) \circ (f(\gamma), q(\gamma)) \circ (e, Y_0)^{-1} = (f(\gamma), O_{\nabla}),$$

Where  $O_{\nabla}$  stands for the zero element of the vector space  $\mathcal{J}_{\nabla}$ .

(3) implies (1).

The assertion means that there exists an affine isomorphism

$$\mathcal{J}_{\nabla} \ni Y \rightarrow L(Y) + X_0 \in \mathcal{J}_{\nabla}$$

such that

$$(L, X_0) \circ (f(\gamma), q(\gamma)) \circ (L, X_0)^{-1} = (f(\gamma), O_{\nabla}), \quad \forall \gamma \in G_{\nabla}$$

150 The calculation of the left member yields the following identities,

$$(a): \quad L \circ f(\gamma) \circ L = f(\gamma),$$

$$(b): \quad L(q(\gamma)) + X_0 - [L \circ (f(\gamma) \circ L^{-1})](X_0) = O_{\nabla}.$$

The identity (b) yields

$$q(\gamma) = f(\gamma)(L^{-1}(X_0)) - L^{-1}(X_0), \quad \forall \gamma \in G_{\nabla}.$$

Taking into account the identity (a), we obtain the following identity:

$$q(\gamma) = f(\gamma)(X_0) - X_0, \quad \forall \gamma \in G_{\nabla}.$$

151 So the vector  $-X_0$  is a fixed point of the affine representaion  $(f, q)$ .

152  $\square$

153 **Definition 3.10.** The affine representation  $(f, q)$  is called the canonical affine representation of the gauge structure  
154  $(M, \nabla)$

155 **Remark 3.11.** *When the infinitesimal action  $J_{\nabla}$  is integrable, the proposition above is a key tool to relate the*  
 156 *canonical affine representaion of  $(M, \nabla)$  and the hyperbolicity problem for the orbits of  $G_{\nabla}$ .*

#### 157 4. Statistical manifolds.

158 We go to restrict the attention Riemannian statistical manifolds (excluding the pseudo-Riemannian case) whose  
 159 particular cases are nondegenerate Fisher information metrics of statistical models and their family of  $\alpha$ -connections,  
 160  $\alpha \in \mathbb{R}$ .

161 We recall that a statistical manifold can be viewed as a triple  $(M, g, \nabla)$  formed of a Riemannian manifold  $(M, g)$  and  
 162 a symmetric gauge structure  $(M, \nabla)$  which are linked by the following identity:

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = 0, \quad \forall (X, Y, Z) \subset \mathcal{X}(M)$$

The  $g$ -dual,  $\nabla^*$  is defined as:

$$g(\nabla_X^* Y, Z) = Xg(Y, Z) - g(Y, \nabla_X Z), \quad \forall (X, Y, Z).$$

163 Depending on need and aims we will use the next alternative definition:

**Definition 4.1.** *A statistical manifold is a quadruple  $(M, g, \nabla, \nabla^*)$  formed a positive Riemannian manifold  $(M, g)$   
 and pair of symmetric gauge structure  $(M, \nabla, \nabla^*)$  which are linked by the following identity*

$$Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X^* Z) = 0, \quad \forall (X, Y, Z).$$

164 *Example 1:* Any Riemannian manifold  $(M, g, \nabla^{LC})$  where  $\nabla^{LC}$  is the Levi-Civita connection of  $g$ , We will say that  $M$   
 165 is a trivial statistical manifold.

166  
 167 *Example 2:* Let  $(M, g, \nabla, \nabla^*)$  any statistical manifold, the family  $(M, g, \nabla^{(a)}, \nabla^{(-a)})_{a \in \mathbb{R}}$  where  $\nabla^{(a)} = \frac{1+a}{2}\nabla + \frac{1-a}{2}\nabla^*$   
 168 is a statistical manifold for  $\forall a \in \mathbb{R}$ .

169  
 170 *Example 3:* Let  $(M, g)$  some Riemannian manifold and denote  $\nabla^{LC}$  the Levi-Civita connection with respect to  $g$ ,  
 171 and let  $E \in TM \setminus \{0\}$ . The triplet  $(M, g, \nabla, \nabla^*)$  is a statistical manifold where  $\nabla_X Y = \nabla_X^{LC} Y + g(X, E)g(Y, E)E$  and  
 172  $\nabla_X^* Y = \nabla_X^{LC} Y - g(X, E)g(Y, E)E$ .

173  
 174 According to previous sections, a statistical manifold  $(M, g, \nabla, \nabla^*)$  yields a pair of finite dimensional simply connected  
 175 Lie groups:

$$(G_{\nabla}, G_{\nabla^*}).$$

176 Their corresponding canonical affine representations will be denoted by  $(f, q)$  and  $(f^*, q^*)$  respectively. The  
 177 Levi-Civita connection of  $(M, g)$  is denoted by  $\nabla^{LC}$ .

178

From now, we assume that either both  $(M, \nabla)$ ,  $(M, \nabla^*)$  and  $(M, \nabla^{LC})$  are geodesically complete or  $M$  is compact. Therefore  $J_\nabla$ ,  $J_{\nabla^*}$  and  $J_{\nabla^{LC}}$  are the infinitesimal counterpart of the following locally effective differentiable dynamical systems:

$$G_\nabla \times M \rightarrow M,$$

$$G_{\nabla^*} \times M \rightarrow M,$$

$$G_{\nabla^{LC}} \times M \rightarrow M.$$

179 Gathering things, a statistical manifold  $(M, g, \nabla, \nabla^*)$  supports the following families of dynamics:

$$\begin{aligned} & \{G_\nabla, J_\nabla, (f, q), \}, \\ & \{G_{\nabla^*}, J_{\nabla^*}, (f^*, q^*), \}, \\ & \{G_{\nabla^{LC}}, J_{\nabla^{LC}}, (f^{LC}, q^{LC}), \}. \end{aligned}$$

180 For information geometry the relevant data are the radiant classes:

$$\begin{aligned} [q] & \in H^1(G_\nabla, J_\nabla), \\ [q^*] & \in H^1(G_{\nabla^*}, J_{\nabla^*}). \end{aligned}$$

From the Riemannian geometry viewpoint the orbits of  $G_\nabla^{LC}$  are flat Riemannian manifolds. Up to isometry and up to finite covering each such an  $n$ -dimensional orbit is a flat cylinder of over an Euclidean torus[21], namely

$$\left( \frac{\mathbb{T}^k}{\Gamma} \times \mathbb{R}^{n-k}, g_0 \right)$$

181 with the flat metric  $g_0$  induced from the ambient Euclidean metric of  $\mathbb{R}^n$ . The integer  $k$  is the first Betti number of the  
182 orbit and  $\Gamma$  is finite group of isometry.

183 To make simpler we adopt the following notation.

184

185  $\mathcal{F}_\nabla$  is the foliation whose leaves are orbits of the Lie group  $G_\nabla$ .

186

187  $\mathcal{F}_{\nabla^*}$  is the foliation whose leaves are orbits of the Lie group  $G_{\nabla^*}$ .

188

189  $(f_\nabla, q_\nabla)$  is the canonical affine representation of  $G_\nabla$ .

190

191 **4.1. Radiancy of  $(f_{\nabla^*}, q_{\nabla^*})$  and hyperbolicity of  $\mathcal{F}_\nabla$ .**

For the sake of simplicity, we assume statistical manifolds

$$(M, g, \nabla, \nabla^*)$$

192 where both  $J_\nabla$  and  $J_{\nabla^*}$  are integrable and we use the following notation:  $g_\nabla$  is the restriction of  $g$  to  $\mathcal{F}_\nabla$ .

193 Therefore we can prove the following facts:

194 **Theorem 4.2.** *Let  $(M, g, \nabla, \nabla^*)$  a compact statistical manifold. We have:*

- 195 1. *The foliation  $\mathcal{F}_\nabla$  is Hessian foliation in  $(M, g, \nabla)$ ,*  
 196  
 197 2. *The foliation  $\mathcal{F}_{\nabla^*}$  is Hessian foliation in  $(M, g, \nabla^*)$ .*

198 **Proof.** (1) By assumption  $M$  is compact then  $J_\nabla$  is integrable, we conclude that  $J_\nabla$  is an infinitesimal action of  $G_\nabla$   
 199 on  $M$ . We have  $R^\nabla(X, Y)J_\nabla = 0$  then  $\mathcal{F}_\nabla$  is locally flat foliation in  $(M, \nabla)$ . The  $G_\nabla$ -orbits are  $\nabla$ -auto-parallel then  
 200  $\delta_{KV}g_\nabla = 0$ . The leaves of  $\mathcal{F}_\nabla$  are Hessian submanifolds of  $(M, g, \nabla)$ .  
 201 (2) By using the same arguments for  $G_{\nabla^*}(M)$ , the theorem is proved.  $\square$

We thus have that:

$$[\mathcal{F}_\nabla, g_\nabla, \nabla]$$

and

$$[\mathcal{F}_{\nabla^*}, g_{\nabla^*}, \nabla^*]$$

202 are Hessian foliations in  $(M, g, \nabla)$  and in  $(M, g, \nabla^*)$  respectively.

203 It is important to notice that the leaves of  $\mathcal{F}_\nabla$  are not  $\nabla^*$ -auto-parallel. Since all of them are statistical manifolds w.r.t.  
 204  $(g_\nabla, \nabla)$ , in each of them  $\nabla$  admits a  $g_\nabla$ -dual which coincides with the  $g$ -orthogonal projection of  $\nabla^*$ ; by abuse of  
 205 notation, this projection also is denoted by  $\nabla^*$ .

The Hessian classes of  $\mathcal{F}_\nabla$  and  $\mathcal{F}_{\nabla^*}$  are denoted as it follows

$$[g_\nabla] \in H_{KV}^2(\mathcal{F}_\nabla)$$

$$[g_{\nabla^*}] \in H_{KV}^2(\mathcal{F}_{\nabla^*})$$

206 In the next theorem we prove that the statistical manifold  $(M, g, \nabla, \nabla^*)$  is a leaf of  $\mathcal{F}_\nabla$  endowed with its Hessian  
 207 structure  $(g_\nabla, \nabla)$ .

208 **Theorem 4.3.** *In a statistical manifold  $(M, g, \nabla, \nabla^*)$  the following assertions are equivalent.*

- 209 1.  $[g_\nabla] = 0 \in H_{KV}^2(\mathcal{F}_\nabla)$ ;  
 210  
 211 2.  $[q_{\nabla^*}] = 0 \in H^1(G_\nabla, J_\nabla)$ ;  
 212  
 213 3.  $(f_{\nabla^*}, q_{\nabla^*})$  has a fixed point;  
 214  
 215 4.  $(f_{\nabla^*}, q_{\nabla^*})$  is affinely conjugated to its linear component  $f_{\nabla^* \clubsuit}$ .

216 **Proof.** According to Theorem 3.9 assertions (2), (3) and (4) are equivalent. Thereof, it is sufficient to prove that  
 217 assertions (1) and (2) are equivalent

Let us demonstrate first that (1) implies (2)

Since the class  $[g_\nabla]$  vanishes there exist a deRham closed differential 1-form  $\partial$  subject the following identity,

$$g_\nabla(X, Y) = -X\partial(Y) + \partial(\nabla_X Y), \quad \forall(X, Y).$$



By the defining property of statistical manifolds, it comes:

$$Xg_{\nabla}(Y, Z) = g_{\nabla}(\nabla_X^* Y, Z) + g_{\nabla}(Y, \nabla_X Z), \quad \forall (X, Y, Z).$$

Let  $H$  be the unique vector field such that

$$\partial(X) = g_{\nabla}(H, X), \quad \forall X.$$

Using once again the defining property of statistical manifolds, we get:

$$Xg_{\nabla}(H, Y) - g_{\nabla}(\nabla_X^* H, Y) - g_{\nabla}(H, \nabla_X Y) = 0,$$

Since left hand member is tensorial, i.e.  $C^\infty(M)$ -multilinear we can assume that

$$H \in \mathcal{J}_{\nabla}.$$

218 Thus we get the following identity

$$g_{\nabla}(\nabla_X^* H, Y) = Xg_{\nabla}(H, Y) - g_{\nabla}(H, \nabla_X Y) = g_{\nabla}(X, Y), \quad \forall (X, Y) \in \mathcal{X}(M).$$

Thus one has

$$\nabla_X^*(-H) - X = 0_{\nabla}, \quad \forall X \in \mathcal{X}(M).$$

So the  $-H$  is a fixed point of  $(f_{\nabla^*}, q_{\nabla^*})$ .

In other words one has

$$f_{\nabla^*}(\gamma)(-H) + q_{\nabla^*}(\gamma) = -H, \quad \forall \gamma \in G_{\nabla^*}.$$

Thus one has

$$q_{\nabla^*}(\gamma) = f_{\nabla^*}(\gamma)(H) - H, \quad \forall \gamma \in G_{\nabla^*}.$$

Let us demonstrate now that (2) implies (1).

Let us assume that  $(f_{\nabla^*}, q_{\nabla^*})$  has a fixed point

$$Y_0 \in \mathcal{J}_{\nabla^*}.$$

Therefore

$$f_{\nabla^*}(\gamma)(Y_0) + q_{\nabla^*}(\gamma) = Y_0, \quad \forall \gamma \in G_{\nabla^*}.$$

To make every obvious, to every

$$X \in \mathcal{J}_{\nabla^*}$$

we assign the one parameter subgroup

$$\{\text{Exp}(tX), t \in \mathbb{R}\} \subset G_{\nabla^*}.$$

We have

$$f_{\nabla^*}(\text{Exp}(tX))(Y_0) + q_{\nabla^*}(\text{Exp}(tX)) = Y_0, \quad \forall t \in \mathbb{R}.$$

One calculate the derivative at  $t = 0 \in \mathbb{R}$  one obtains

$$\nabla_X^* Y_0 + X = 0.$$

Now finally:

$$Xg_{\nabla}(Y_0, Y) = g_{\nabla}(\nabla_X^* Y_0, Y) + g_{\nabla}(Y_0, \nabla_X Y);$$

Thus taking into account that

$$\nabla_X^* Y_0 = -X, \forall X,$$

219 one obtains the following identity,

$$Xg_{\nabla}(Y_0, Y) = -g_{\nabla}(X, Y) + g_{\nabla}(Y_0, \nabla_X Y).$$

By putting

$$\partial(Y) = -g_{\nabla}(Y_0, Y)$$

One obtains

$$g_{\nabla}(X, Y) = X\partial(Y) - \partial(\nabla_X Y), \quad \forall (X, Y).$$

Then one has

$$[g_{\nabla}] = 0 \in H_{KV}^2(\nabla).$$

220  $\square$

221 From Theorems 2.21 and 3.9 we obtain the next corollary:

222 **Corollary 4.4.** *Let  $(M, g, \nabla, \nabla^*)$  such that both  $J_{\nabla}$  and  $J_{\nabla^*}$  are integrable. We assume that leaves of both  $\mathcal{F}_{\nabla}$  and*  
 223  *$\mathcal{F}_{\nabla^*}$  satisfy one among the assertions of Theorem 4.3. Then we can conclude as follows:*

224 (1) *Every compact leaf of  $\mathcal{F}_{\nabla}$  is hyperbolic.*

225 (2) *Every compact leaf of  $\mathcal{F}_{\nabla^*}$  is hyperbolic  $\clubsuit$*

## 226 5. When are the orbits quotients of convex cones.

227 Convex cones are examples of bounded domains. The studies of convex cones have been and continue to be among  
 228 high standing subjects in geometry and in analysis. The pioneering works are those of Elie Cartan, but there is a  
 229 wealth of subsequent works, see [22], [23], [24] and many others. Nowadays the analysis in convex cones plays  
 230 interesting role in the information geometry, [12]. Theorem 3.3 provides conditions under which compact orbits  
 231 of  $G_{\nabla}$  are hyperbolic. We are going to demonstrate that up to diffeomorphisms, compact leaves of  $\mathcal{F}_{\nabla}$  are quotient  
 232 homogeneous convex cones. According to our previous notation, if  $M$  is a compact leaf of  $\mathcal{F}_{\nabla}$  then  $\mathcal{Q}(\bar{M})$  is a  
 233 homogeneous convex cone.

### 234 5.1. A theorem of Koszul.

235 For convenience, we introduce the following notion.

236 **Definition 5.1.** *A statistical manifold  $(M, g, \nabla, \nabla^*)$  is called integrable if both  $J_{\nabla}$  and  $J_{\nabla^*}$  are integrable.*

237 Compact statistical manifolds and geodesically complete ones are integrable.

Let  $(N, g, \nabla, \nabla^*)$  be an integrable statistical manifold whose Hessian foliations are denoted by

$$\{\mathcal{F}_{\nabla}, g_{\nabla}, \nabla\}$$

and

$$\{\mathcal{F}_{\nabla^*}, g_{\nabla^*}, \nabla^*\}.$$

Henceforth, by abuse of notation, a leaf of

$$\{\mathcal{F}_{\nabla}, g_{\nabla}, \nabla\}$$

is an Hessian manifold that we denote by

$$(M, g_{\nabla}, \nabla).$$

Its  $g$ -dual is denoted by

$$(M, g_{\nabla}, \nabla^*).$$

Theorem 4.2 links both the Hessian of  $(M, g_{\nabla}, \nabla)$ , namely

$$[g_{\nabla}] \in H_{KV}^2(\nabla)$$

and the radiant class of  $(M, g, \nabla^*)$ , namely

$$[q_{\nabla^*}] \in H^1(G_{\nabla^*}, J_{\nabla^*}).$$

By Theorem 4.3 we know that if  $M$  is compact then

$$[g_{\nabla}] \cup [q_{\nabla^*}]$$

238 is a characteristic obstruction to  $(M, \nabla)$  being hyperbolic.

239

240 Henceforth we assume these obstructions vanish.

We have already pointed out that  $\tilde{M}$  a universal covering of  $M$ . Therefore it admits a unique locally flat structure  $(\tilde{M}, \tilde{\nabla})$  such that the covering map

$$\pi : \tilde{M} \rightarrow M$$

241 is a gauge morphism between  $\tilde{\nabla}$  and  $\nabla$ .

**Theorem 5.2.** [23] *Let*

$$\{(M, g, \nabla), [g_{\nabla}] \cup [q_{\nabla^*}]\}$$

*being a compact Hessian manifold. If:*

$$[g_{\nabla}] \cup [q_{\nabla^*}] = 0.$$

242 *Then for  $\mathcal{Q}(\tilde{M})$  being a cone, it is sufficient that  $\tilde{M}$  being homogeneous under group of transformation of  $(\tilde{M}, \tilde{\nabla})$  ♣*

243 We are going to show that this theorem applies in the category of integrable statistical manifolds.

244 **Theorem 5.3.** *Under the assumptions of Theorem 4.3 and Theorem 4.3 the (simply connected) locally flat manifold*  
245  *$(\tilde{M}, \tilde{\nabla})$  is homogeneous ♣*

**Proof.** Remind that  $\mathbb{P}$  is the topological space of differentiable pointed paths

$$(0, [0, 1]) \ni t \rightarrow c(t) \in (x_0, M).$$

If

$$M \ni x \rightarrow f(x) \in M$$

is a differentiable map then  $f \circ c$  is a differentiable path whose origin is  $f(x_0)$ . This to remind that every continuous map of  $M$  in  $M$  can be canonically left to a map of  $\tilde{M}$  in itself.

Therefore the action

$$G_{\nabla} \times M \rightarrow M$$

gives rise to the action

$$G_{\nabla} \times \tilde{M} \rightarrow \tilde{M}.$$

At one side remember that  $(M, \nabla)$  is homogeneous under the action of the Lie group  $G_{\nabla}$ .

At another side remember that  $(M, \nabla)$  is the quotient of  $(\tilde{M}, \tilde{\nabla})$  under the action of the fundamental group  $\pi_1(M)$ .

Therefore both  $\pi_1(M)$  and  $G_{\nabla}$  are subgroups of the group

$$\text{Aff}(\tilde{M}, \tilde{\nabla}).$$

246 It is easy to check that  $G_{\nabla}$  is included in the normalizer of  $\pi_1(M)$ .

247 Since  $M$  is transitively acted on by  $G_{\nabla}$ , every orbit of  $G_{\nabla}$  in  $\tilde{M}$  is an open submanifold of  $\tilde{M}$ . Since  $\tilde{M}$  is connected  
248 the action of  $G_{\nabla}$  in  $\tilde{M}$  is transitive.

249  $\square$

**Theorem 5.4.** *Let  $(N, g, \nabla)$  be a integrable statistical manifold and let  $\mathcal{F}_{\nabla}$  be its canonical Hessian foliation. If the characteristic obstruction vanishes:*

$$[g_{\nabla}] \cup [q_{\nabla}^*] = 0.$$

250 *Then up to affine diffeomorphism every  $m$ -dimensional compact leaf of  $\mathcal{F}_{\nabla}$  is the quotient of a convex cone  $C \subset \mathbb{R}^m$   
251 under a discrete subgroup of affine group  $GL(\mathbb{R}^m) * \mathbb{R}^m$ .*

## 252 6. Characteristic function and Formalism of Barbaresco.

*Let  $C$  be a convex cone of  $E$  whose dual cone is denoted by  $C^*$  and is defined by*

$$C^* = \{\Psi \mid (\Psi, x) > 0, \forall x \in \bar{C} - \{0\}\} \quad (6.1)$$

*J-L Koszul and E.B. Vinberg have introduced a characteristic function  $\rho$  of a regular convex cone  $C$ , it is defined by*

$$\rho_C(x) = \int_{C^*} e^{-\langle \Psi, x \rangle} d\Psi \quad (6.2)$$

253 *where  $d\Psi$  is the Lebesgue measure of  $E^*$*

254 *Applying this construction to aforementioned convex cone yields a realisation of the statistical manifold as an  
255 exponential model.*

## 256 7. Application to statistical models of measurable sets

We focus on local regular functional models. Consider a measurable set  $(\Sigma, \Omega)$ . Remember that an  $m$ -dimensional regular local functional model of  $(\Sigma, \Omega)$  is a couple  $(\Theta, P)$  formed of an connected open subset  $\Theta \subset \mathbb{R}^m$  and  $P$  a non negative real valued function:

$$\Theta \times \Sigma \ni (\partial, \xi) \longrightarrow P(\partial, \xi) \in \mathbb{R}.$$

257 Further

- 258 (a)  $P(\partial, \xi)$  is smooth w.r.t.  $\partial$ ;  
 259 (b)  $\sum_{\xi} P(\partial, \xi) = 1, \forall \partial$ ;  
 (c) the following symmetric bilinear form  $g$  (Fisher information,) is positive definite,

$$g(\partial)(X, Y) = \sum_{\xi} P(\partial, \xi) [d \log(P)(X) d \log(P)(Y)](\partial, \xi).$$

$$(X, Y) \subset \mathcal{X}(\Theta)$$

## 260 8. $a$ -connections

$\partial = (\partial_j, 1 \leq j \leq m)$  are coordinate functions in  $\mathbb{R}^m$ . For every  $a \in \mathbb{R}$  we put

$$\Gamma_{ij;k}^a = \sum_{\xi} P(\partial, \xi) \left[ \frac{\partial^2 \log(P)}{\partial_i \partial_j} + \frac{1+a}{2} \partial_i \log(P) \partial_j \log(P) \right] \partial_k \log(P)(\partial, \xi).$$

These functions  $\Gamma_{ij;k}^a$  are Christoffel symbols of a  $\nabla^a$  is symmetric connection on  $\Theta$ . Here

$$\partial_i \log(P) = \frac{\partial \log(P)}{\partial \partial_i}$$

Let  $(\Theta, g, \nabla^a, \nabla^{-a})$  be a statistical manifold. Putting:

$$J_a = \mathcal{J}_{\nabla^a}$$

261 the results of the previous sections can be applied directly, with:

- 262 •  $\mathcal{F}_{\nabla^a}$  is the foliation whose leaves are orbits of the Lie group  $G_{\nabla^a}$ .
- 263 •  $\mathcal{F}_{\nabla^{-a}}$  is the foliation whose leaves are orbits of the Lie group  $G_{\nabla^{-a}}$ .
- 264 •  $g_{\nabla^a}$  is the restriction of the Fisher Information to  $\mathcal{F}_{\nabla^a}$ .
- 265 •  $g_{\nabla^{-a}}$  is the restriction of the Fisher Information to  $\mathcal{F}_{\nabla^{-a}}$ .
- 266 •  $(f_{\nabla^a}, q_{\nabla^a})$  is the canonical affine representation of  $G_{\nabla^a}$ .
- 267 •  $(f_{\nabla^{-a}}, q_{\nabla^{-a}})$  is the canonical affine representation of  $G_{\nabla^{-a}}$ .

268 Theorems below are then easy consequences of the general results.

269 **Theorem 8.1.** *Let  $(\Theta, P)$  a statistical model. We have the following statements.*

- 270 (1)  $[\mathcal{F}_{\nabla^a}, g_{\nabla^a}, \nabla^a]$  is exponential foliation in  $(\Theta, P)$ ,  
 271 (2)  $[\mathcal{F}_{\nabla^{-a}}, g_{\nabla^{-a}}, \nabla^{-a}]$  is exponential foliation in  $(\Theta, P)$ .

272 **Theorem 8.2.** *In a statistical model  $(\Theta, P)$  the following assertions are equivalent:*

- 273 1.  $[g_{\nabla^a}] = 0 \in H_{KV}^2(\mathcal{F}_{\nabla^a})$ ;  
 274 2.  $[q_{\nabla^{-a}}] = 0 \in H^1(G_{\nabla^{-a}}, \mathcal{J}_{\nabla^{-a}})$ ;  
 275 3.  $(f_{\nabla^{-a}}, q_{\nabla^{-a}})$  has a fixed point;  
 276 4.  $(f_{\nabla^{-a}}, q_{\nabla^{-a}})$  is affinely conjugated to its linear component  $f_{\nabla^{-a}} \clubsuit$ .

277 **Theorem 8.3.** Let  $(\Theta, P)$  be a statistical model which satisfies one of the equivalent conditions of the theorem 7.3, we  
278 have the following assertions:

- 279 1. The compact leaves of the exponential foliations  $\mathcal{F}_{\nabla^a}$  are isomorphic to the quotient of convex homogeneous  
280 cones by discrete subgroups of  $GL(\mathbb{R}^s) \ltimes \mathbb{R}^s$ ;
- 281 2. The compact leaves of the exponential foliations  $\mathcal{F}_{\nabla^{-a}}$  are isomorphic to the quotient of convex homogeneous  
282 cones by discrete subgroups of  $GL(\mathbb{R}^s) \ltimes \mathbb{R}^l$ .

**Corollary 8.4.** Let  $(\Theta \subset \mathbb{R}^4, P)$  a statistical model satisfying the conditions of the Theorem 7.4. We have the following assertions.

$$\text{If } s = 1 \quad \Sigma = S^1 \quad (8.1)$$

$$\text{If } s = 2 \quad \Sigma = T^2 \quad (8.2)$$

$$\text{If } s = 3 \quad \Sigma = \begin{cases} \text{Hyperbolic Torus bundle} \\ \text{Nilmanifold} \\ \text{Seifert Manifold} \end{cases} \quad (8.3)$$

We shall now give a example of compact hessian manifold which orbits are quotient of statistical models by discrete subgroup. Let  $\mathbb{R}^2$  be a 2-dimensional real affine space with the natural flat affine connection  $D$  and let  $\{x, y\}$  be an affine coordinate system of  $\mathbb{R}^2$ .

Let  $\Omega$  be a domain define by  $x > 0$  and  $y > 0$ . Consider a Riemannian metric on  $\Omega$  given by  $g = \frac{1}{x^2} dx^2 + \frac{1}{y^2} dy^2$ .

Then  $(D, g)$  is Hessian structure on  $\Omega$ .

Let  $\Sigma$  and  $\chi$  be linear transformations on  $\Omega$  defined by :

$$\Sigma : (x, y) \rightarrow (2x, y)$$

$$\chi : (x, y) \rightarrow (2x, 3y)$$

Then  $\langle \Sigma, \chi \rangle$  leave the Hessian structure  $(D, g)$  invariant.

We denote  $\Gamma$  the group generate by  $\{\Sigma, \chi\}$ , we can also write  $\Gamma$  as  $\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \rangle$ .

$\Gamma$  acts properly discontinuously on  $\Omega$  and  $\frac{\Omega}{\Gamma}$  is compact Hessian manifold which is diffeomorphic to a Torus.

Let us denote by  $\pi$  the projection from  $\Omega$  to  $\frac{\Omega}{\Gamma}$  and by  $(D, g)$  the Hessian structure on  $\frac{\Omega}{\Gamma}$ . Since the space of all  $\Gamma$ -invariant  $D$  parallel 1-forms on  $\Omega$  is spanned by  $dx$  and  $dy$ . The space of all  $D$ -parallel 1-forms on  $\frac{\Omega}{\Gamma}$  is spanned by  $\omega$  and  $\Phi$  where  $dx = \pi^* \omega$  and  $dy = \pi^* \Phi$ . Let  $X$  and  $\tilde{X}$  be a vector field on  $\frac{\Omega}{\Gamma}$  defined by  $\omega(Y) = g(X, Y)$  and  $\Phi(Y) = g(\tilde{X}, Y)$  for each vector field  $Y$  on  $\frac{\Omega}{\Gamma}$ . Then  $X = \pi_* \left( \frac{\partial}{\partial x} \right)$  and  $\tilde{X} = \pi_* \left( \frac{\partial}{\partial y} \right)$  and the vector space  $\mathcal{J}_{\nabla}$  of all Hessian vector fields on  $\frac{\Omega}{\Gamma}$  is spanned by  $\langle X, \tilde{X} \rangle$ . Since  $T^2$  is compact the  $\mathcal{J}_{\nabla}$  is integrable. Let  $Exp(tX)$  and  $Exp(tY)$  is a 1-parameter group of transformations generated by  $X$  and by  $Y$ . Let  $G_{\nabla} = \langle Exp(tX), Exp(tY) \rangle$ . Consider  $a > 0$  and  $b > 0$ , the compact homogeneous the orbit  $G_{\nabla} \pi(a, b) = \{\gamma \cdot \pi(a, b), \gamma \in G_{\nabla}\}$  is a circle. Then we conclude that  $(S^1, D, g_{S^1})$  is Hessian manifold and satisfies  $D \frac{\partial}{\partial \theta} = 0$  and  $g_{S^1} = \frac{1}{\partial^2} d\theta^{\otimes 2}$ . By using conjugation we have  $D^* \frac{\partial}{\partial \theta} = -\frac{2}{\partial} \frac{\partial}{\partial \theta}$ . Let  $H^* = h(\partial) \frac{\partial}{\partial \theta}$  vectors fields on  $S^1$ .  $H^*$  is homothety vector fields of  $D^*$  if  $h$  is solution of differential equation  $y' - \frac{2}{\partial^2} y = 1$ . We deduce that

$$h(\partial) = Ae^{-\frac{2}{\partial}} + 2e^{-\frac{2}{\partial}} \int_{-\frac{2}{\partial}}^{+\infty} \frac{e^{-s}}{s} ds + \partial$$

where  $A$  is constant.

As  $D^*H^* = Id$  then  $(S^1, D)$  is hyperbolic. Now let  $p: \mathbb{R} \rightarrow S^1$  the covering map, and consider the map

$$(\mathbb{R}, \tilde{D}) \rightarrow (S^1, D)$$

283 .The compact orbits is  $G_\nabla \pi(a, b)$  is quotient  $\frac{\mathbb{R}_{>0}}{\Gamma}$ , here  $\Gamma$  is discrete subgroup of  $Gl(\mathbb{R}) \times \mathbb{R}$

## 284 9. Conclusion

285 The theorems 3.9,4.3 give conditions for hyperbolicity of compact leaves and in turn allow an explicit construction  
 286 of a statistical exponential model for a statistical manifold. While is it already known that any statistical manifold  
 287 posseses such concrete realizations, there is no mechanism to construct them, which our results provide. In a future  
 288 work, study of special cases will be done with the idea of finding new canonical densities on manifolds.

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