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Hyperbolicity of the Compact Leaves of Statistical Manifold Foliations

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Abstract: In this work we show a construction of a Hessian foliation on a statistical manifold. Our results provide conditions under which the compacts hessian leaves are quotients of homogeneous convex cones, up to diffeomorphism.

1. Introduction

Hessian structures play an important role in information geometry. In [1], Shima observed that the Fisher information of most statistical models are Hessian metrics. Amari also proved in [2] that any locally flat statistical manifold is a Hessian manifold. These structures arised from works of many mathematicians (such as J.-L. Koszul, Y. Matsushima, A. Nijenhuis, E. B. Vinberg) in their attempt to solve the Gerstenhaber conjecture in the category of locally flat manifolds. The geometry of locally flat hyperbolic manifolds is also called Koszul geometry (see [3]). Gerstenhaber's conjecture says that every restricted theory of deformation generates its proper theory of cohomology. This conjecture has been solved in 2006 by Michel Boyom (see [4]). In [5] and [6], Boyom also realizes the homological version of the Hessian geometry. The aim of this paper is to study hyperbolic foliations in statistical manifolds. In this context, a hyperbolic foliation is a foliation whose leaves handle Koszul geometry. The paper is organised as follows: section 2 is devoted to basic notions which are used later in the paper; section 3 covers useful notions about the category of locally flat manifolds and their KV-cohomology. It is also devoted to Hessian equations of Koszul connections in the tangent bundle of a statistical manifold; section 4 aims at focusing on canonical pairs of locally flat foliations in a statistical manifold, before discussing conditions under which these foliations are hyperbolic. Finally, quotients of convex cones and exponential models are discussed in section 5.

2. Basic notions

For the sake of completeness and to make this paper as much self-contained as possible, the basic notions which link Koszul geometry to information geometry will be recalled. Our goal is to make this note as self-contained as possible.

2.1. ***Koszul connections.***

Let M be a differentiable manifold, $C^\infty(M)$ the vector space of real valued functions defined in M and $\mathcal{X}(M)$ the vector space of differentiable vector fields in M. Let (V,π,M) be a vector fiber bundle over M. Loosely speaking, we have a submersion

$$V \ni e \to \pi(e) \in M,$$

whose fibers $\pi^{-1}(x)$ are vector spaces.
The space of sections of V is denoted as $\mathcal{V}$. The cotangent bundle of M is denoted as T^*M. The tensor product $T^*M \otimes V$ is canonically identified with the vector bundle $\mathrm{Hom}(TM,V)$.

Definition 2.1. *A Koszul connection on a vector bundle V is a first order differential operator*

$$\mathcal{V} \ni s \to \nabla s \in T^*M \otimes \mathcal{V}$$

which satisfies the following requirement

$$\nabla(fs) = f\nabla s + df \otimes s, \forall f \in C^\infty(M). \tag{2.1}$$

Let

$$\nabla_X s = (\nabla s)(X). \tag{2.2}$$

The above requirement means, given $(X,s) \in \mathcal{X}(M) \times \mathcal{V}$ and $f \in C^\infty(M)$, that

$$(\nabla(fs))(X) = f\nabla_X s + df(X)s. \tag{2.3}$$

The curvature (of ∇), $R^\nabla \in \otimes^2 T^*M \otimes \mathcal{V}^* \otimes \mathcal{V}$ is defined as follows:

$$R^\nabla(X,Y)(s) = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s. \tag{2.4}$$

Definition 2.2. *A vector fiber bundle is called a flat bundle if it admits a Koszul connection whose curvature vanishes identically.*

When dealing with Riemannian manifolfds, there is an distinguished connection that plays a particular role in the study of the metric.

Definition 2.3. *Let (M,g) be a Riemannian manifold. A connection ∇ is said to be metric if for any triple (X,Y,Z) of vector fields:*

$$X\left(g(Y,Z)\right) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right)$$

Proposition 2.4. *On a Riemannian manifold, it exists a unique metric connection without torsion, called the Levi-Civita connection and denoted by* ∇^{lc}

2.2. ***Locally flat manifolds.***

Koszul connections on tangent vector bundles TM are usually called linear connections on M. The torsion of a linear Koszul connection ∇, $T^{\nabla} \in \otimes^2 T^* M \otimes TM$ is defined as follows

$$T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{2.5}$$

Definition 2.5. *A locally flat manifold is a pair (M,∇) where ∇ is a linear Koszul connection (on TM) whose curvature R^{∇} and torsion T^{∇} both vanish identically.*

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0, \tag{2.6}$$
$$\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z = 0 \quad \forall X,Y,Z \in \Gamma(TM). \tag{2.7}$$

Example 5: Let $\mathbb{R}^n = \{(x_1, ..., x_n)/x_i \in \mathbb{R}\}$ be endowed with the linear connection ∇^F defined as

$$\nabla^F \frac{\partial}{\partial x_i} = 0$$

$\forall i = 1,2,..,n$ and $\forall X \in \mathcal{X}(M)$

Example 6: Let $\mathbb{R}^2$ and $\forall X = f\frac{\partial}{\partial x_1} + g\frac{\partial}{\partial x_2} \in \mathcal{X}(M)$ be endowed with a linear connection defined as

$$\nabla \frac{\partial}{\partial x_1} = f\frac{\partial}{\partial x_1} \text{ and } \nabla \frac{\partial}{\partial x_2} = 0$$

$(\mathbb{R}^2, \nabla)$ is a locally flat manifold.

2.3. ***The developing map of a locally flat manifold.***

Set a point $x_0 \in M$ and consider the pairs $\{0,[0,1]\}$ and $\{x_0, M\}$. We now consider the space of pointed differentiable paths

$$c : \{0,[0,1]\} \to \{x_0, M\}.$$

With the above notation meaning that

$$c(0) = x_0.$$

We will also have to consider differentiable paths in the tangent bundle. To distinguish them from paths in the manifold, a capital letter will be used.

Definition 2.6. *Let $c\colon [0,1] \to M$ be a differentiable path. A path $Y\colon [0,1] \to TM$ is said to be a lift of c if $\pi \circ Y = c$ where $\pi\colon TM \to M$ is the canonical projection.*

Definition 2.7. *Let ∇ be a Koszul connection. A path $Y\colon [0,1] \to TM$ is said to be the horizontal if:*

$$\nabla \dot{c}(t) Y(t) = 0,\, t \in [0,1]$$

with $c = \pi \circ Y$.

Using the Cauchy-Lipschitz theorem, it is easy to obtain the next result:

Proposition 2.8. *Let c be a C^1-path in M, ∇ a Koszul connection and let $V \in T_{c(0)}M$ be given. It exists a unique horizontal $Y \to TM$ lifting c.*

Definition 2.9. *The parallel transport along the C^1-path $c\colon [0,1] \to M$, $c(0) = x_0$, $c(1) = x_1$ is the linear mapping:*

$$\tau\colon V \in T_{x_0}M \to \tau(V) = Y(1) \in T_{x_1}M$$

where Y is the unique horizontal lift of c.

We fix $\sigma \in [0,1]$, the parallel transport along the path

$$c : [0,\sigma] \to M$$

is denoted by τ_σ; this a one to one linear map of $T_{x_0}M$ in $T_{c(\sigma)}M$. Thus we have

$$\tau_\sigma^{-1}(\frac{dc}{dt}(\sigma)) \in T_{x_0}M.$$

We define an application

$$c \to Q(c) \in T_{x_0}M$$

by putting

$$Q(c) = \int_0^1 \tau_\sigma^{-1}(\frac{dc}{dt}(\sigma)d\sigma.$$

It is called the development of c.

Definition 2.10. *Let c and c^* be two pointed paths such that $c(1) = c^*(1)$. They are said to be homotopic if it exists a differential map:*

$$[0,1] \times [0,1] \ni (s,t) \to H(s,t) \in M$$

satisfying:
(1) $H(s,0) = x_0, \forall s$;
(2) $H(s,1) = c(1) = c*(1)\forall s$;
(3) H(0,t) = c(t);
(4) H(1,t) = c(t).*

Let us go back to the map

$$c \to Q(c) \in T_{x_0}M.$$

The following assertion is a classical result, that can be obtained from the fact that a flat connection has a trivial holonomy group.

Proposition 2.11. *[7] Let ∇ be a flat connection. If the paths c and $c*$ are homotopic then*

$$Q(c) = Q(c*). \tag{2.8}$$

Given a locally flat structure (M, ∇), we fixe $x_0 \in M$ we consider the space of pointed differentiable paths

$$\{(0, [0,1]) \to (x_0, M)\}.$$

The set of their homotopy classes is the universal covering of M, denoted by denoted by $\tilde{M}$. The connection ∇ lifts to $\tilde{M}$ and so deos the map $Q(c)$:

$$\tilde{M} \ni [c] \to Q([c]) \in T_{x_0}M.$$

This map is called the developing map of the locally flat manifold (M, ∇). We are in position to highlight the main question of this paper: **What does the image $Q(\tilde{M}) \subset T_{x_0}M$ looks like?**

The next definition is due to Koszul.

Definition 2.12. *A locally flat manifold (M, ∇) is called hyperbolic if $Q(\tilde{M})$ is an open convex domain in the vector space $T_{x_0}M$ not containing any straight line.*

Theorem 2.13. *[7] For (M, ∇) being hyperbolic it is necessary that there exists a de Rham closed differential 1-form ω whose covariant derivative $\nabla\omega$ is positive definite.If M is compact ,then this condition is also sufficient.*

Remark 2.14. *The existence of the nonvanishing Koszul 1-form ω proves that M is a radiant suspension and by using Tischler [8],we conclude that M is fibered over S^1.Topological consequences are that the Euler characteristic $\mathcal{X}(M) = 0$ and the first Betti number $b_1(M) > 0$.*

This definition is the affine analogue to hyperbolic holomorphic manifolds following W.Kaup(see[9];[7]). The notion of hyperbolicity is among the fundamental notions of the geometry of Koszul. It may be addressed and studied form many perspectives. We are going to highlight the algebraic topology point of view, which is richer than the Riemannian geometry one. The reason is that the homological point of view highlights relationships with other research domains. An examples of these relations is the theory of affine representations of Lie groups which plays a key role in **Lie group theory of Heat** after Jean-Marie Souriau. In this paper we also perform the theory of affine representation to investigate the point set-topology of the image $Q(\tilde{M})$.

2.4. Algebraic topology of locally flat manifolds

Definition 2.15. *An affine structure in an m-dimensional differentiable manifold M is a pair $(M, \mathcal{A})$ where $\mathcal{A}$ is a complete atlas whose local coordinates changes coincide with affine transformations of $\mathbb{R}^m$.*

Remark 2.16. *There is a one to one correspondence between the category of locally flat structures in M and the category of affine structures in M.*

Many cochain complexes are associated with a locally flat manifold (M, ∇). We limit ourselves to what is used in this paper. readers interested in the general theory of cohomology of Koszul–Vinberg algebras are referred to Nguiffo

Boyom 2006, [10] and to Nijenhuis Albert 1968 . The group of integers is denoted by $\mathbb{Z}$.
$Aff(M,\nabla:\mathbb{R})$ stands for the vector space of $f\in C^{\infty}(M)$ which are subject to the following requirement,

$$\nabla^2(f)=0.$$

To investigate the structure of $Q((M,\nabla))$, we introduce the $\mathbb{Z}$-graded vector space:

$$C(\nabla)=\oplus_q C^q(\nabla)$$

With homogeneous vector subspaces:

$$C^q(\nabla)=0 \text{ if } q<0;$$
$$C^0(\nabla)=Aff(M,\nabla:\mathbb{R});$$
$$C^q(\nabla)=Hom_{\mathbb{R}}(\otimes^q\mathcal{X}(M),C^{\infty}(M))$$

The coboundary operator

$$\delta:C^q(\nabla)\to C^{q+1}(\nabla)$$

is defined as it follows:

$(a)\delta f=df\,\forall f\in C^0(\nabla);$

(b) let $q>0, f\in C^q(\nabla)$ and $X_1\otimes..\otimes X_{q+1}\in\otimes^{q+1}\mathcal{X}(M)$ then

$$\delta f(X_1\otimes...\otimes X_{q+1})=\Sigma_1^q(-1)^i[d(f(..\otimes\hat{X}_i\otimes..\otimes X_{q+1}))(X_i)-\Sigma_{j\neq i}f(..\otimes\hat{X}_i\otimes..\otimes\nabla_{X_i}X_j\otimes..]$$

Remember that $\hat{X}_i$ means that X_i is missing.
The pair $(C(\nabla),\delta)$ is a differential vector space. This means that the linear δ is subject to the requirement

$$\delta^2=\delta\circ\delta=0.$$

Elements of $ker(\delta)$ are called cocycles; those of $im(\delta)$ are called coboundaries.
The q-th cohomology space $H^q(\nabla)$ is defined by:

$$H^q_{KV}(\nabla)=\frac{Ker(\delta)}{Im(\delta)}.$$

In this paper we are mainly concerned by the first terms of the complex:

$$\to C^1(\nabla)\to C^2(\nabla)\to C^3(\nabla)\to$$

Elements $C^2(\nabla)$ are called KV 2-cochains. The coboundary of $\partial\in C^2(\nabla)$ is computed as follows:

$$\delta\partial(X_1\otimes X_2\otimes X_3)=-d(\partial(X_2\otimes X_3))(X_1)+d\partial(X_1\otimes X_3)(X_2)+\partial(\nabla_{X_1}X_2\otimes X_3)+\partial(X_2\otimes\nabla_{X_1}X_3)$$
$$-\partial(\nabla_{X_2}X_1\otimes X_3)-\partial(X_1\otimes\nabla_{X_2}X_3).$$

Remark 2.17. *d stands here for the usual exterior derivative.*

Definition 2.18. *A 2nd cohomology class*

$$[\partial] \in H^2_{KV}(\nabla)$$

is called a Hessian (resp. Hessian non-degenerate, Hessian positive definite) class if it contains a symmetric (resp symmetric non-degenerate, symmetric positive definite) cocycle.

Remark 2.19. *The Hessian geometry defined in [1] corresponds to the positive definite case.*

A classical result states that a manifold M admits locally flat structures if and only if it admits affine structures. It is worth to notice that locally flat geometry is linked to the Cartan-Koszul geometry in TM, while the affine geometry is related to local analysis in M.

Proposition 2.20. *[4] A locally flat manifold (M,∇) is Hessian if $\delta\partial$ vanishes identically.*

The next theorem relates hyperbolicity to KV cohomology.

Theorem 2.21. *Let (M,g,∇) be a compact Hessian manifold then the following assertions are equivalent;*
(a.1) $[g] = 0 \in H^2_{KV}(\nabla)$.
(a.2) (M,∇) *is hyperbolic* ♣

In order to characterize $Q(\tilde{M})$ and hyperbolicity of leaves, we will follow the next scheme. Firstly we are going to introduce some canonical dynamics in a statistical manifold whose orbits are locally flat manifolds.Then we study the question of whether an orbit M is hyperbolic or not using the Koszul characterisation, that is $Q(\tilde{M})$ is an open convex domain not containing a straight line. Finally, we investigate conditions under which $Q(\tilde{M})$ is an open convex cone. In such a case, the statistical manifold can be given an exponential statistical model based on the characteristic function of $Q(\tilde{M})$. For details, readers may refer to the next references [11],[12],[13].

3. Hessian differential operators of gauge structures.

Throughout the paper we keep the notation of Nguiffo Boyom [14], [5].
We recall that a gauge structure in a manifold M is a pair (M,∇) where ∇ is a Koszul connection in the tangent bundle TM.

Let us introduce the Hessian differential operator:

$$\mathcal{X}(M) \ni X \to \nabla^2 X \in T^2_1(M)$$

with $T^2_1(M)$ is the space of (2,1)-tensors in M.

The next proposition is an easy consequence of the definition of the covariant derivative of a tensor.

Proposition 3.1. *For any triple (X,Y,Z) of vector fields:*

$$\nabla^2 X(Y,Z) = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z.$$

Letting $X.Y = \nabla_X Y$, it reads as:

$$\nabla^2 X(Y,Z) = X.(Y.Z) - (X.Y).Z$$

showing that ∇^2 is the associator of the product defined above.

Let $(x_1,..,x_m)$ be a system of local coordinate functions of M and let $X \in \mathcal{X}(M)$. We set

$$\partial i = \frac{\partial}{\partial x_i},$$

$$X = \Sigma_k X^k \partial k,$$

$$\nabla_{\partial i}\partial j = \Sigma_k \Gamma^k_{i,j}\partial k.$$

The principal of symbol of the Hessian differential operator can be expressed as

$$(\nabla^2 X)(\partial i, \partial j) = \Sigma_k \Omega^k_{i,j}\partial k \tag{3.1}$$

where

$$\Omega^l_{ij} = \frac{\partial^2 X^l}{\partial x_i \partial x_j} + \sum_k \left[\Gamma^l_{ik}\frac{\partial X^k}{\partial x_j} + \Gamma^l_{jk}\frac{\partial X^k}{\partial x_i} - \Gamma^k_{ij}\frac{\partial X^l}{\partial x_k}\right] + \sum_k \left[\frac{\partial \Gamma^l_{jk}}{\partial x_i} + \sum_m \left(\Gamma^m_{jk}\Gamma^k_{im} - \Gamma^m_{ij}\Gamma^l_{mk}\right)\right] \tag{3.2}$$

On (M,∇), the Hessian equation is the following 2nd order differential equation, [14]

$$FE(\nabla) : \nabla^2 X = 0.$$

Its sheaf of solutions will be denoted by $\mathcal{J}_\nabla(M)$ in the sequel. It is easily found to be a sheaf of associative algebras whose product of sections is given as

$$X.Y = \nabla_X Y.$$

The space of sections of $\mathcal{J}_\nabla(M)$ will be denoted by J_∇.

The pair (J_∇, ∇) is an associative algebra with commutator Lie algebra $(J_\nabla, [-,-]_\nabla)$ where the bracket $[X,Y]_\nabla$ is:

$$[X,Y]_\nabla = \nabla_X Y - \nabla_Y X.$$

Remember that (M,∇) is called symmetric gauge structure if the Koszul connection ∇ is symmetric, that is to say that the torsion of ∇ vanishes identically.
Therefore when (M,∇) is symmetric, the Lie algebra $(J_\nabla, [-,-]_\nabla)$ is a Lie subalgebra of the Lie algebra of vector fields

$$(\mathcal{X}(M), [-,-]);$$

According to [14] when (M,∇) is symmetric the Lie subalgebra $J_\nabla \subset \mathcal{X}(M)$ is finite a dimensional over the field of real numbers.

Proposition 3.2. *Let (M,g) be a Riemmian manifold and let Ric be its Ricci curvature tensor.We have*

$$J_\nabla \subset Ker(Ric) \tag{3.3}$$

Proof. Let (X,Y) be a couple of vector fields and let $\xi \in J_\nabla$. Then:

$$R^\nabla(Y,X)\xi = (\nabla^2\xi)(X,Y) - (\nabla^2\xi)(Y,X). \tag{3.4}$$

Taking the trace we deduce that:

$$Ric(X,\xi) = 0 \tag{3.5}$$

Then

$$J_\nabla \subset KerRic$$

□

As a consequence, it may turn out that J_∇ is trivial:

Corollary 3.3. *If ∇ is the Levi-Civita connection of an Einstein Riemanian manifold ,then*

$$J_\nabla = \{0\}$$

Using the third theorem of Lie, it comes:

Theorem 3.4. *Up to a Lie group isomorphism there exists a unique simply connected Lie group G_∇ whose Lie algebra is isomorphic to the Lie algebra J_∇.*

Before pursuing we going to recall a few useful notions which will be involved in the sequel. Let G be a Lie group whose Lie algebra is denoted by $\mathcal{G}$.

Definition 3.5. *A Lie algebra homomorphism of $\mathcal{G}$ in the Lie algebra $\mathcal{X}(M)$ is called an infinitesimal differentiable action of G in M.*
An infinitesimal differentiable action of G in M is called integrable if it is the derivative of an action of G in M[15].

Example

In a compact manifold M every infinitesimal action of a finite dimensional Lie group is integrable. That is due to the fact every vector field X is complete in the meaning that X is a generator of a one parameter subgroup of the group of diffeomorphisms.

A non trivial example of integrable infinitesimal action arises in our setting. Given a gauge structure (M,∇) remember that the elements of the Lie subalgebra

$$aff(M,\nabla) \subset \mathcal{X}(M)$$

are vector fields X satisfying the next identity:

$$[X,\nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X,Z] = 0, \forall (Y,Z) \subset \mathcal{X}(M) \tag{3.6}$$

.

Definition 3.6. *A path c(t) is a geodesic in (M,∇) if it is a solution of the following differential equation*

$$\nabla_{\dot{c}(t)}(\dot{c}(t)) = 0.$$

(M,∇) is called geodesically complete if all of its geodesics are defined in $]-\infty,+\infty[$.

Proposition 3.7. *If (M,∇) is geodesically complete then elements of $\mathfrak{aff}(M,\nabla)$ are complete, see Kobayashi-Nomizu,[16].*

Let (M,∇) be a symmetric gauge structure. We keep the notation J_∇, G_∇.
The Lie subalgebra

$$J_\nabla \subset \mathcal{X}(M)$$

is a locally effective infinitesimal differentiable action of G_∇ in M. If either ∇ is geodesically complete or M is compact, then J_∇ is integrable. Thus it is the infinitesimal counterpart of a locally effective differentiable action

$$G_\nabla \times M \ni (\gamma, x) \to \gamma.x \in M.$$

Proposition 3.8. *([14] Under the same assumptions, if J_∇ is integrable then each orbit of G_∇ is an homogenoeous locally flat manifold.*

3.0.1. **A canonical representaion of G_∇**

Let G be a group and the W be a finite dimensional real vector space whose group of affine isomorphisms is denoted by Aff(W). This group is a semi-direct product $GL(W) * W$ whose underlying set is the Cartesian product

$$GL(W) \times W$$

and the composition rule is

$$(\gamma_1, w_1) * (\gamma_2, w_2) = (\gamma_1 \circ \gamma_2, \gamma_1(w_2) + w_1)$$

There is a natural affine representaion of the Lie algebra J_∇ in itself as a vector space:

$$J_\nabla \ni X \to \rho(X) = (\nabla_X, X) \in gl(J_\nabla) \times J_\nabla = \mathfrak{aff}(J_\nabla).$$

The affine action is defined as:

$$\rho(X).Y = \nabla_X Y + X, \quad \forall Y \in J_\nabla.$$

By virtue of the universal property of simply connected finite dimensional Lie groups, there exist a unique continuous affine representation

$$G_\nabla \ni \gamma \to (f(\gamma), q(\gamma)) \in GL(J_\nabla) \times J_\nabla) = Aff(J_\nabla),$$

whose differential at the unit element is the representation ρ.

So e being the unit element of G_∇ one has

$$[(df)(e)](X) = \nabla_X,$$

$$[(dq)(e)](X) = X.$$

In the sequel, we point out how this canonical affine representation impacts the Information Geometry in statistical manifolds.

3.1. *The radiant class of affine representations.*

We keep the notations $G_\nabla, J_\nabla, \rho, f, q$ as in the section above.
The couple (f,q) is a continuous homomorphism of the Lie group G_∇ on the Lie group $Aff(J_\nabla)$. Therefore one has

$$[f(\gamma),q(\gamma)]\circ[f(\gamma*),q(\gamma*)]=[f(\gamma)\circ f(\gamma*),f(\gamma)q(\gamma*)+q(\gamma)]=[f(\gamma.\gamma*),q(\gamma.\gamma*)].$$

So f is a linear representation of G_∇ in J_∇ and q is a J_∇ valued 1-cocycle of f. The cohomology class of q is denoted by

$$[q]\in H^1(G_\nabla,J_\nabla).$$

Let

$$f_\nabla : G_\nabla \longrightarrow GL(J_\nabla)$$

$$q_\nabla : G_\nabla \longrightarrow J_\nabla$$

Remind the cohomology of the Lie group G_∇ value in his Lie algebra J_∇ is defined by :

$$...\to C^q(G_\nabla,J_\nabla)\to C^{q+1}(G_\nabla,J_\nabla)\to C^{q+2}(G_\nabla,J_\nabla)\to$$

with differential operator D defined by :

$$D\partial(\gamma_1,......,\gamma_{q+1})=f_\nabla(\gamma_1).\partial(\gamma_2,.......,\gamma_{q+1})+\sum_{i\leqslant q}(-1)^i\partial(..,\gamma_i\gamma_{i+1},...)+(-1)^q\partial(\gamma_1,....,\gamma_q)$$

See Koszul[17] ,Cartan and Eleinberg [18] and Chevalley and Eleinberg [19],M.Boyom[20] for more details on Lie algebra cohomology.
It's easy to see that the condition :

$$q_\nabla(\gamma_1.\gamma_2)=f_\nabla(\gamma_1)q_\nabla(\gamma_2)+q_\nabla(\gamma_1)$$

$$\forall\gamma_1,\gamma_2\in G_\nabla$$

is equivalent to $q_\nabla\in Z^1(G_\nabla,J_\nabla)$. Let $[q_\nabla]\in H^1(G_\nabla,J_\nabla)$ be its cohomology class. It is called the radiant class of the affine representation (f,q).

Theorem 3.9. *] The following statements are equivalent :*

1. *The affine action*

$$G_\nabla\times J_\nabla\ni(\gamma,X)\to f(\gamma).X+q(\gamma)\in J_\nabla$$

has a fixed point;

2. *The cohomology class [q] vanishes;*

3. *The affine representation*

$$G_\nabla\ni\gamma\to(f(\gamma),q(\gamma)\in Aff(J_\nabla)$$

is conjugated to the linear representation

$$G_\nabla \ni \gamma \to f(\gamma) \in GL(J_\nabla)\clubsuit$$

Proof. *(1) implies (2).*
Let $-Y_0$ be a fixed point of the affine action (f,q), then

$$f(\gamma)(-Y_0) + q(\gamma) = -Y_0, \quad \forall \gamma \in G_\nabla.$$

Therefore one has

$$q(\gamma) = f(\gamma)(Y_0) - Y_0, \quad \forall \gamma \in G_\nabla.$$

So the cocycle q is exact.
(2) implies (3).
Consider the affine the affine isomorphism (e, Y_0). It is nothing but the translation by Y_0

$$X \to X + Y_0;$$

We calculate

$$(e, Y_0) \circ (f(\gamma), q(\gamma)) \circ (e, Y_0)^{-1} = (f(\gamma), 0_\nabla),$$

Where O_∇ stands for the zero element of the vector space J_∇.
(3) implies (1).
The assertion means that there exists an affine isomorphism

$$J_\nabla \ni Y \to L(Y) + X_0 \in J_\nabla$$

such that

$$(L, X_0) \circ (f(\gamma), q(\gamma)) \circ (L, X_0)^{-1} = (f(\gamma), 0_\nabla), \quad \forall \gamma \in G_\nabla$$

The calculation of the left member yields the following identities,

$$(a): \quad L \circ f(\gamma) \circ L = f(\gamma),$$

$$(b): \quad L(q(\gamma)) + X_0 - [L \circ (f(\gamma) \circ L^{-1}](X_0) = 0_\nabla.$$

The identity (b) yields

$$q(\gamma) = f(\gamma)(L^{-1}(X_0)) - L^{-1}(X_0), \quad \forall \gamma \in G_\nabla.$$

Taking into account the identity (a), we obtain the following identity:

$$q(\gamma) = f(\gamma)(X_0) - X_0, \quad \forall \gamma \in G_\nabla.$$

So the vector $-X_0$ is a fixed point of the affine representaion (f,q).

□

Definition 3.10. *The affine representation (f,q) is called the canonical affine representation of the gauge structure (M,∇)*

Remark 3.11. *When the infinitesimal action J_∇ is integrable, the proposition above is a key tool to relate the canonical affine representaion of (M,∇) and the hyperbolicity problem for the orbits of G_∇.*

4. Statistical manifolds.

We go to restrict the attention Riemannian statistical manifolds (excluding the pseudo-Riemannian case) whose particular cases are nondegenerate Fisher information metrics of statistical models and their family of α-connections, $\alpha \in \mathbb{R}$.

We recall that a statistical manifold can be viewed as a triple (M,g,∇) formed of a Riemannian manifold (M,g) and a symmetric gauge structure (M,∇) which are linked by the following identity:

$$(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = 0, \quad \forall (X,Y,Z) \subset \mathcal{X}(M)$$

The g-dual, ∇^* is defined as:

$$g(\nabla^*_X Y, Z) = Xg(Y,Z) - g(Y,\nabla_X Z), \quad \forall (X,Y,Z).$$

Depending on need and aims we will use the next alternative definition:

Definition 4.1. *A statistical manifold is a quadruple (M,g,∇,∇^*) formed a positive Riemannian manifold (M,g) and pair of symmetric gauge structure (M,∇,∇^*) which are linked by the following identity*

$$Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla^*_X Z) = 0, \quad \forall (X,Y,Z).$$

Example 1: Any Riemannian manifold (M,g,∇^{LC}) where ∇^{LC} is the Levi-Civita connection of g, We will say that M is a trivial statistical manifold.

Example 2: Let (M,g,∇,∇^*) any statistical manifold, the family $(M,g,\nabla^{(\alpha)},\nabla^{(-\alpha)})_{\alpha\in\mathbb{R}}$ where $\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*$ is a statistical manifold for $\forall\alpha \in \mathbb{R}$.

Example 3: Let (M,g) some Riemannian manifold and denote ∇^{LC} the Levi-Civita connection with respect to g, and let $E \in TM \setminus \{0\}$. The triplet (M,g,∇,∇^*) is a statistical manifold where $\nabla_X Y = \nabla^{LC}_X Y + g(X,E)g(Y,E)E$ and $\nabla^*_X Y = \nabla^{LC}_X Y - g(X,E)g(Y,E)E$.

According to previous sections, a statistical manifold (M,g,∇,∇^*) yields a pair of finite dimensional simply connected Lie groups:

$$(G_\nabla, G_{\nabla^*}).$$

Their corresponding canonical affine representations will be denoted by (f,q) and (f^*,q^*) respectively. The Levi-Civita connection of (M,g) is denoted by ∇^{LC}.

From now, we assume that either both (M,∇), (M,∇^*) and (M,∇^{LC}) are geodesically complete or M is compact. Therefore J_∇, J_{∇^*} and $J_{\nabla^{LC}}$ are the infinitesimal counterpart of the following locally effective differentiable dynamical systems:

$$G_\nabla \times M \to M,$$

$$G_{\nabla^*} \times M \to M,$$

$$G_{\nabla^{LC}} \times M \to M.$$

Gathering things, a statistical manifold (M,g,∇,∇^*) supports the following families of dynamics:

$$\{G_\nabla, J_\nabla, (f,q),\},$$
$$\left\{G_{\nabla^*}, J_{\nabla^*}, \left(f^*, q^*\right)\right\},$$
$$\left\{G_{\nabla^{LC}}, J_{\nabla^{LC}}, \left(f^{LC}, q^{LC}\right)\right\}.$$

For information geometry the relevant data are the radiant classes:

$$[q] \in H^1(G_\nabla, J_\nabla),$$
$$[q^*] \in H^1(G_{\nabla^*}, J_{\nabla^*}).$$

From the Riemannian geometry viewpoint the orbits of G_∇^{LC} are flat Riemannian manifolds. Up to isometry and up to finite covering each such an n-dimensional orbit is a flat cylinder of over an Euclidean torus[21], namely

$$\left(\frac{\mathbb{T}^k}{\Gamma} \times \mathbb{R}^{n-k}, g_0\right)$$

with the flat metric g_0 induced from the ambient Euclidean metric of $\mathbb{R}^n$. The integer k is the first Betti number of the orbit and Γ is finite group of isometry.

To make simpler we adopt the following notation.

$\mathcal{F}_\nabla$ is the foliation whose leaves are orbits of the Lie group G_∇.

$\mathcal{F}_{\nabla^*}$ is the foliation whose leaves are orbits of the Lie group G_{∇^*}.

(f_∇, q_∇) is the canonical affine representation of G_∇.

4.1. ***Radiancy of $(f_{\nabla^*}, q_{\nabla^*})$ and hyperbolicity of $\mathcal{F}_\nabla$.***

For the sake of simplicity, we assume statistical manifolds

$$(M,g,\nabla,\nabla^*)$$

where both J_∇ and J_{∇^*} are integrable and we use the following notation: g_∇ is the restriction of g to $\mathcal{F}_\nabla$.

Therefore we can prove the following facts:

Theorem 4.2. *Let (M, g, ∇, ∇^*) a compact statistical manifold. We have:*

1. *The foliation $\mathcal{F}_\nabla$ is Hessian foliation in (M, g, ∇),*

2. *The foliation $\mathcal{F}_{\nabla^*}$ is Hessian foliation in (M, g, ∇^*) .*

Proof. (1) By assumption M is compact then J_∇ is integrable,we conclude that J_∇ is an infinitesimal action of G_∇ on M.We have $R^\nabla(X,Y)J_\nabla = 0$ then $\mathcal{F}_\nabla$ is locally flat foliation in (M, ∇). The G_∇-orbits are ∇-auto-parallel then $\delta_{KV} g_\nabla = 0$. The leaves of $\mathcal{F}_\nabla$ are Hessian submanifolds of (M, g, ∇).
(2) By using the same arguments for $G_{\nabla^*}(M)$, the theorem is proved. □

We thus have that:

$$[\mathcal{F}_\nabla, g_\nabla, \nabla]$$

and

$$[\mathcal{F}_{\nabla^*}, g_{\nabla^*}, \nabla^*]$$

are Hessian foliations in (M, g, ∇) and in (M, g, ∇^*) respectively.

It is important to notice that the leaves of $\mathcal{F}_\nabla$ are not ∇^*-auto-parallel. Since all of them are statistical manifolds w.r.t. (g_∇, ∇), in each of them ∇ admits a g_∇-dual which coincides with the g-orthogonal projection of ∇^*; by abuse of notation, this projection also is denoted by ∇^*.

The Hessian classes of $\mathcal{F}_\nabla$ and $\mathcal{F}_{\nabla^*}$ are denoted as it follows

$$[g_\nabla] \in H^2_{KV}(\mathcal{F}_\nabla)$$

$$[g_{\nabla^*}] \in H^2_{KV}(\mathcal{F}_{\nabla^*})$$

In the next theorem we prove that the statistical manifold (M, g, ∇, ∇^*) is a leaf of $\mathcal{F}_\nabla$ endowed with its Hessian structure (g_∇, ∇).

Theorem 4.3. *In a statistical manifold (M, g, ∇, ∇^*) the following assertions are equivalent.*

1. $[g_\nabla] = 0 \in H^2_{KV}(\mathcal{F}_\nabla)$;

2. $[q_{\nabla^*}] = 0 \in H^1(G_\nabla, J_\nabla)$;

3. $(f_{\nabla^*}, q_{\nabla^*})$ *has a fixed point;*

4. $(f_{\nabla^*}, q_{\nabla^*})$ *is affinely conjugated to its linear component* f_{∇^*}♣.

Proof. According to Theorem 3.9 assertions (2), (3) and (4) are equivalent. Thereof, it is sufficient to prove that assertions (1) and (2) are equivalent
Let us demonstrate first that (1) implies (2)

Since the class $[g_\nabla]$ vanishes the exist a deRham closed differential 1-form ∂ suject the following identity,

$$g_\nabla(X,Y) = -X\partial(Y) + \partial(\nabla_X Y), \quad \forall (X,Y).$$

By the defining property of statistical manifolds, it comes:

$$Xg_\nabla(Y,Z) = g_\nabla(\nabla^*_X Y,Z) + g_\nabla(Y,\nabla_X Z), \quad \forall (X,Y,Z).$$

Let H be the unique vector field such that

$$\partial(X) = g_\nabla(H,X), \quad \forall X.$$

Using once again the defining property of statistical manifolds, we get:

$$Xg_\nabla(H,Y) - g_\nabla(\nabla^*_X H,Y) - g_\nabla(H,\nabla_X Y) = 0,$$

Since left hand member is tensorial, i.e. $C^\infty(M)$-multilinear we can assume that

$$H \in J_\nabla.$$

218 Thus we get the following identity

$$g_\nabla(\nabla^*_X H,Y) = Xg_\nabla(H,Y) - g_\nabla(H,\nabla_X Y) = g_\nabla(X,Y), \quad \forall (X,Y) \subset \mathcal{X}(M).$$

Thus one has

$$\nabla^*_X(-H) - X = 0_\nabla, \quad \forall X \in \mathcal{X}(M).$$

So the $-H$ is a fixed point of $(f_{\nabla^*}, q_{\nabla^*})$.
In other words one has

$$f_{\nabla^*}(\gamma)(-H) + q_{\nabla^*}(\gamma) = -H, \quad \forall \gamma \in G_{\nabla^*}.$$

Thus one has

$$q_{\nabla^*}(\gamma) = f_{\nabla^*}(\gamma)(H) - H, \quad \forall \gamma \in G_{\nabla^*}.$$

Let us demonstrate now that (2) implies (1).

Let us assume that $(f_{\nabla^*}, q_{\nabla^*})$ has a fixed point

$$Y_0 \in J_{\nabla^*}.$$

Therefore

$$f_{\nabla^*}(\gamma)(Y_0) + q_{\nabla^*}(\gamma) = Y_0, \quad \forall \gamma \in G_{\nabla^*}.$$

To make every obvious, to every

$$X \in J_{\nabla^*}$$

we assign the one parameter subgroup

$$\{Exp(tX), t \in \mathbb{R}\} \subset G_{\nabla^*}.$$

Wehave

$$f_{\nabla^*}(Exp(tX))(Y_0) + q_{\nabla^*}(Exp(tX)) = Y_0, \quad \forall t \in \mathbb{R}.$$

One calculate the derivative at $t = 0 \in \mathbb{R}$ one obtains

$$\nabla^*_X Y_0 + X = 0.$$

Now finally:

$$Xg_\nabla(Y_0,Y) = g_\nabla(\nabla^*_X Y_0, Y) + g_\nabla(Y_0, \nabla_X Y);$$

Thus taking into account that

$$\nabla^*_X Y_0 = -X, \forall X,$$

one obtains the following identity,

$$Xg_\nabla(Y_0,Y) = -g_\nabla(X,Y) + g_\nabla(Y_0, \nabla_X Y).$$

By putting

$$\partial(Y) = -g_\nabla(Y_0, Y)$$

One obtains

$$g_\nabla(X,Y) = X\partial(Y) - \partial(\nabla_X Y), \quad \forall (X,Y).$$

Then one has

$$[g_\nabla] = 0 \in H^2_{KV}(\nabla).$$

□

From Theorems 2.21 and 3.9 we obtain the next corollary:

Corollary 4.4. *Let (M, g, ∇, ∇^*) such that both J_∇ and J_{∇^*} are integrable. We assume that leaves of both $\mathcal{F}_\nabla$ and $\mathcal{F}_{\nabla^*}$ satisfy one among the assertions of Theorem 4.3. Then we can conclude as follows:*
(1) Every compact leaf of $\mathcal{F}_\nabla$ is hyperbolic.
(2) Every compact leaf of $\mathcal{F}_{\nabla^*}$ is hyperbolic ♣

5. When are the orbits quotients of convex cones.

Convex cones are examples of bounded domains. The studies of convex cones have been and continue to be among high standing subjects in geometry and in analysis. The pioneering works are those of Elie Cartan, but there is a wealth of subsequent works, see [22], [23], [24] and many others. Nowadays the analysis in convex cones plays interesting role in the information geometry, [12]. Theorem 3.3 provides conditions under which compact orbits of G_∇ are hyperbolic. We are going to demonstrate that up to difeomorphisms, compact leaves of $\mathcal{F}_\nabla$ are quotient homogeneous convex cones. According to our previous notation, if M is a compact leaf of $\mathcal{F}_\nabla$ then $Q(\tilde{M})$ is a homogeneous convex cone.

5.1. A theorem of Koszul.

For convenience, we introduce the following notion.

Definition 5.1. *A statistical manifold (M, g, ∇, ∇^*) is called integrable if both J_∇ and J_{∇^*} are integrable.*

Compact statistical manifolds and geodesically complete ones are integrable.

Let (N, g, ∇, ∇^*) be an integrable statistical manifold whose Hessian foliations are denoted by

$$\{\mathcal{F}_\nabla, g_\nabla, \nabla\}$$

and
$$\left\{\mathcal{F}_{\nabla^*}, g_{\nabla^*}, \nabla^*\right\}.$$

Henceforth, by abuse of notation, a leaf of
$$\{\mathcal{F}_{\nabla}, g_{\nabla}, \nabla\}$$
is an Hessian manifold that we denote by
$$(M, g_{\nabla}, \nabla).$$
Its g-dual is denoted by
$$(M, g_{\nabla}, \nabla^*).$$
Theorem 4.2 links both the Hessian of (M, g_{∇}, ∇), namely
$$[g_{\nabla}] \in H^2_{KV}(\nabla)$$
and the radiant class of (M, g, ∇^*), namely
$$[q_{\nabla^*}] \in H^1(G_{\nabla^*}, J_{\nabla^*}).$$
By Theorem 4.3 we know that if M is compact then
$$[g_{\nabla}] \cup [q_{\nabla^*}]$$
is a characteristic obstruction to (M, ∇) being hyperbolic.

Henceforth we assume these obstructions vanish.

We have already pointed out that $\tilde{M}$ a universal covering of M. Therefore it admits a unique locally flat structure $(\tilde{M}, \tilde{\nabla})$ such that the covering map
$$\pi : \tilde{M} \to M$$
is a gauge morphism between $\tilde{\nabla}$ and ∇.

Theorem 5.2. *[23] Let*
$$\{(M, g, \nabla), \quad [g_{\nabla}] \cup [q_{\nabla^*}]\}$$
being a compact Hessian manifold. If:
$$[g_{\nabla}] \cup [q_{\nabla^*}] = 0.$$
Then for $Q(\tilde{M})$ being a cone, it is sufficient that $\tilde{M}$ being homogeneous under group of transformation of $(\tilde{M}, \tilde{\nabla})$♣

We are going to show that this theorem applies in the category of integrable statistical manifolds.

Theorem 5.3. *Under the assumptions of Theorem 4.3 and Theorem 4.3 the (simply connected) locally flat manifold $(\tilde{M}, \tilde{\nabla})$ is homogeneous ♣*

Proof. Remind that $\mathbb{P}$ is the topological space of differentiable pointed paths
$$(0, [0, 1]) \ni t \to c(t) \in (x_0, M).$$

If

$$M \ni x \to f(x) \in M$$

is a differentiable map then $f \circ c$ is a differentiable path whose origin is $f(x_0)$. This to remind that every continuous map of M in M can be canonically left to a map of $\tilde{M}$ in itself.
Therefore the action

$$G_\nabla \times M \to M$$

gives rise to the action

$$G_\nabla \times \tilde{M} \to \tilde{M}.$$

At one side remember that (M, ∇) is homogeneous under the action of the Lie group G_∇.
At another side remember that (M, ∇) is the quotient of $(\tilde{M}, \tilde{\nabla})$ under the action of the fundamental group $\pi_1(M)$.
Therefore both $\pi_1(M)$ and G_∇ are subgroups of the group

$$Aff(\tilde{M}, \tilde{\nabla}).$$

It is easy to check that G_∇ is included in the normalizer of $\pi_1(M)$.
Since M is transitively acted on by G_∇, every orbit of G_∇ in $\tilde{M}$ is an open submanifold of $\tilde{M}$. Since $\tilde{M}$ is connected the action of G_∇ is $\tilde{M}$ is transitive.
□

Theorem 5.4. *Let (N, g, ∇) be a integrable statistical manifold and let $\mathcal{F}_\nabla$ be its canonical Hessian foliation.*
If the characteristic obstruction vanishes:

$$[g_\nabla] \cup [q_{\nabla^*}] = 0.$$

Then up to affine diffeomorphism every m-dimensional compact leaf of $\mathcal{F}_\nabla$ is the quotient of a convex cone $C \subset \mathbb{R}^m$ under a discrete subgroup of affine group $GL(\mathbb{R}^m) * \mathbb{R}^m$♣

6. Characteristic function and Formalism of Barbaresco.

Let C be a convex cone of E whose dual cone is denoted by C^* and is defined by

$$C^* = \{\Psi \mid (\Psi, x) > 0, \forall x \in \bar{C} - \{0\}\} \tag{6.1}$$

J-L Koszul and E.B. Vinberg have introduced a characteristic function ρ of a regular convex cone C,it is defined by

$$\rho_C(x) = \int_{C^*} e^{-<\Psi, x>} d\Psi \tag{6.2}$$

where $d\Psi$ is the Lebesgue mesure of E^*

Applying this construction to aforementioned convex cone yields a realisation of the statistical manifold as an exponential model.

7. Application to statistical models of measurable sets

We focus on local regular functional models. Consider a measurable set (Σ, Ω). Remember that an m-dimensional regular local functional model of (Σ, Ω) is a couple (Θ, P) formed of an connected open subset $\Theta \subset \mathbb{R}^m$ and P a non negative real valued function:

$$\Theta \times \Sigma \ni (\partial, \xi) \longrightarrow P(\partial, \xi) \in \mathbb{R}.$$

Further

(a) $P(\partial,\xi)$ is smooth w.r.t. ∂;

(b) $\Sigma_\xi P(\partial,\xi) = 1$, $\forall\partial$;

(c) the following symmetric bilinear form g(,Fisher information,) is positive definite,

$$g(\partial)(X,Y) = \Sigma_\xi P(\partial,\xi)[dlog(P)(X)dlog(P)(Y)](\partial,\xi).$$

$$(X,Y) \subset \mathcal{X}(\Theta)$$

8. α-connections

$\partial = (\partial_j, 1 \le j \le m)$ are coordinate functions in $\mathbb{R}^m$. For every $\alpha \in \mathbb{R}$ we put

$$\Gamma^\alpha_{ij;k} = \Sigma_\xi P(\partial,\xi)[[\frac{\partial^2 log(P)}{\partial_i\partial_j} + \frac{1+\alpha}{2}\partial_i log(P)\partial_j log(P)]\partial_k log(P)](\partial,\xi).$$

These functions $\Gamma^\alpha_{ij;k}$ are Christoffel symbols of a ∇^α is symmetric connection on Θ . Here

$$\partial_i log(P) = \frac{\partial log(P)}{\partial\partial_i}$$

Let $(\Theta, g, \nabla^\alpha, \nabla^{-\alpha})$ be a statistical manifold.Putting:

$$J_\alpha = J_{\nabla^\alpha}$$

the results of the previous sections can be applied directly, with:

- $\mathcal{F}_{\nabla^\alpha}$ is the foliation whose leaves are orbits of the Lie group G_{∇^α}.
- $\mathcal{F}_{\nabla^{-\alpha}}$ is the foliation whose leaves are orbits of the Lie group $G_{\nabla^{-\alpha}}$.
- g_{∇^α} is the restriction of the Fisher Information to $\mathcal{F}_{\nabla^\alpha}$.
- $g_{\nabla^{-\alpha}}$ is the restriction of the Fisher Information to $\mathcal{F}_{\nabla^{-\alpha}}$.
- $(f_{\nabla^\alpha}, q_{\nabla^\alpha})$ is the canonical affine representation of G_{∇^α}.
- $(f_{\nabla^{-\alpha}}, q_{\nabla^{-\alpha}})$ is the canonical affine representation of $G_{\nabla^{-\alpha}}$.

Theorems below are then easy consequences of the general results.

Theorem 8.1. *Let (Θ, P) a statistical model.We have the following statements.*
(1)$[\mathcal{F}_{\nabla^\alpha}, g_{\nabla^\alpha}, \nabla^\alpha]$ is exponential foliation in (Θ, P),
(2)$[\mathcal{F}_{\nabla^{-\alpha}}, g_{\nabla^{-\alpha}}, \nabla^{-\alpha}]$ is exponential foliation in (Θ, P).

Theorem 8.2. *In a statistical model (Θ, P) the following assertions are equivalent:*

1. $[g_{\nabla^\alpha}] = 0 \in H^2_{KV}(\mathcal{F}_{\nabla^\alpha})$;
2. $[q_{\nabla^{-\alpha}}] = 0 \in H^1(G_{\nabla^{-\alpha}}, J_{\nabla^{-\alpha}})$;
3. $(f_{\nabla^{-\alpha}}, q_{\nabla^{-\alpha}})$ *has a fixed point;*
4. $(f_{\nabla^{-\alpha}}, q_{\nabla^{-\alpha}})$ *is affinely conjugated to its linear component* $f_{\nabla^{-\alpha}}$♣.

Theorem 8.3. *Let (Θ, P) be a statistical model which satisfies one of the equivalent conditions of the theorem 7.3,we have the following assertions:*

1. *The compacts leaves of the exponential foliations $\mathcal{F}_{\nabla^{\alpha}}$ are isomorphic to the quotient of convex homogeneous cones by discrete subgroups of $GL(\mathbb{R}^s) \ltimes \mathbb{R}^s$;*
2. *The compacts leaves of the exponential foliations $\mathcal{F}_{\nabla^{-\alpha}}$ are isomorphic to the quotient of convex homogeneous cones by discrete subgroups of $GL(\mathbb{R}^s) \ltimes \mathbb{R}^l$.*

Corollary 8.4. *Let $(\Theta \subset \mathbb{R}^4, P)$ a statistical model satisfying the conditions of the Theorem 7.4. We have the following assertions.*

$$\textit{If } s = 1 \quad \Sigma = S^1 \tag{8.1}$$

$$\textit{If } s = 2 \quad \Sigma = T^2 \tag{8.2}$$

$$\textit{If } s = 3 \quad \Sigma = \begin{cases} \textbf{\textit{Hyperbolic Torus bundle}} \\ \textbf{\textit{Nilmanifold}} \\ \textbf{\textit{Seifert Manifold}} \end{cases} \tag{8.3}$$

We shall now give a example of compact hessian manifold which orbits are quotient of statistical models by discrete subgroup.Let $\mathbb{R}^2$ be a 2-dimensional real affine space with the natural flat affine connection D and let $\{x, y\}$ be an affine coordinate system of $\mathbb{R}^2$.
Let Ω be a domain define by $x > 0$ and $y > 0$.Consider a Riemannian metric on Ω given by $g = \frac{1}{x^2}dx^2 + \frac{1}{y^2}dy^2$.
Then (D,g) is Hessian structure on Ω.
Let Σ and χ be linear transformations on Ω defined by :

$$\Sigma : (x, y) \to (2x, y)$$

$$\chi : (x, y) \to (2x, 3y)$$

Then $<\Sigma,\chi>$ leave the Hessian structure (D, g) invariant.
We denote Γ the group generate by $\{\Sigma, \chi\}$, we can also write Γ as $< \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right); \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right) >$.
Γ acts properly discontinuously on Ω and $\frac{\Omega}{\Gamma}$ is is compact Hessian manifold which is diffeomorphic to a Torus.
Let us denote by π the projection from Ω to $\frac{\Omega}{\Gamma}$ and by (D, g) the Hessian structure on $\frac{\Omega}{\Gamma}$.Since the space of all Γ-invariant D parallel 1-forms on $\Omega$is spanned by dx and dy.The space of all D-parallel 1-forms on $\frac{\Omega}{\Gamma}$ is spanned by ω and Φ where $dx = \pi^*\omega$ and $dy = \pi^*\Phi$.Let X and $\tilde{X}$ be a vector field on $\frac{\Omega}{\Gamma}$ defined by $\omega(Y) = g(X, Y)$ and $\Phi(Y) = g(\tilde{X}, Y)$for each vector field Y on $\frac{\Omega}{\Gamma}$.Then $X = \pi_*(\frac{\partial}{\partial x})$ and $\tilde{X} = \pi_*(\frac{\partial}{\partial y})$ and the vector space J_∇ of all Hessian vector fields on $\frac{\Omega}{\Gamma}$ is spanned by $<X,\tilde{X}>$.Since T^2 is compact the J_∇ is integrable.Let $Exp(tX)$ and $Exp(tY)$ is a 1-parameter group of transformations generated by X and by Y.Let $G_\nabla =< Exp(tX), Exp(tY) >$.
Consider $a > 0$ and $b > 0$, the compact homogeneous the orbit $G_\nabla \pi(a, b) = \{\gamma.\pi(a, b), \gamma \in G_\nabla\}$ is a circle. Then we conclude that (S^1, D, g_{S^1}) is Hessian manifold and satisfies $D\frac{\partial}{\partial \eth} = 0$ and $g_{S^1} = \frac{1}{\eth^2}d\eth^{\otimes 2}$.By using conjugation we have $D^*_{\frac{\partial}{\partial \eth}}\frac{\partial}{\partial \eth} = -\frac{2}{\eth}\frac{\partial}{\partial \eth}$.Let $H^* = h(\eth)\frac{\partial}{\partial \eth}$ vectors fields on S^1.H^* is homothety vector fields of D^* if h is solution of differential equation $y' - \frac{2}{\eth^2}y = 1$.We deduce that

$$h(\eth) = Ae^{\frac{-2}{\eth}} + 2e^{\frac{-2}{\eth}} \int_{\frac{-2}{\eth}}^{+\infty} \frac{e^{-s}}{s} \mathrm{d}s + \eth$$

where A is constant.
As $D^{*}H^{*} = Id$ then (S^1, D) is hyperbolic.Now let $p : \mathbb{R} \to S^1$ the covering map, and consider the map

$$(\mathbb{R}, \tilde{D}) \to (S^1, D)$$

.The compacts orbits is $G_{\nabla}\pi(a, b)$ is quotient $\frac{\mathbb{R}^{>0}}{\Gamma}$, here Γ is discrete subgroup of $Gl(\mathbb{R}) \ltimes \mathbb{R}$

9. Conclusion

The theorems 3.9,4.3 give conditions for hyperbolicity of compact leaves and in turn allow an explicit construction of a statistical exponential model for a statistical manifold. While is it already known that any statistical manifold posseses such concrete realizations, there is no mechanism to construct them, which our results provide.In a future work, study of special cases will be done with the idea of finding new canonical densities on manifolds.

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