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Article

Hyperbolicity of the Compact Leaves of Statistical Manifold Foliations

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Abstract: In this work, we show a construction of a Hessian foliation on a statistical manifold. Our results provide conditions, under which the compact hessian leaves (up to diffeomorphism) are quotients of homogeneous convex cones.

1. Introduction

The Hessian structures play important roles in Information geometry. In [1], Shima observes that Fisher information of all classical used statistical models are Hessian metrics. As Amari did he also proved that any locally flat statistical manifold is a Hessian manifold. These structures arised from works of many mathematicians (e.g. J.-L. Koszul, Y. Matsushima, A. Nijenhuis, E. B. Vinberg) in their attempt to solve the Gerstenhaber conjecture in the category of locally flat manifold. The geometry of locally flat hyperbolic manifold is also named Koszul geometry (see [2]), the conjecture of Gerstenhaber say that every restricted theory of deformation generates its proper theory of cohomology. This conjecture has been solved in 2006 (see [3]) by Michel Boyom. In [4][5] he also realize the homological version of the Hessian geometry. The aim of this paper is to study hyperbolic foliations in statistical manifolds . Here a hyperbolic foliation is a foliation whose leaves support Koszul geometry. The paper is organised as it follows. In section is this introduction. Section 2 is devoted to basics notions which are used. Section 3 is devoted to useful notions in the category of locally flat manifolds and their KV-cohomology. Section 4 is devoted to Hessian equation of Koszul connection in tangent bundle of statistical manifolds. In section 5 we focus on a cononical pair of locally flat foliations in statistical manifolds ,and we discuss conditions under which these foliations are hyperbolic .

2. Statistical manifold

2.1. Koszul connections

We recall in this subsection some basics notions. All differentiable manifold we deal with are paracompact. The class of differentiability is C^∞ . For a manifold M the real algebra of differentiable real valued function is denote by $C^\infty(M)$. The $C^\infty(M)$ -module of differentiable vectors fields is denoted by $\mathcal{X}(M)$.

Definition 1. A Koszul connection on M is a \mathbb{R} -bilinear product

$$(X, Y) \in \mathcal{X}(M)^2 \longrightarrow \nabla_X Y \in \mathcal{X}(M)$$

satisfying the following identities for $X, Y \in \mathcal{X}(M)$ and $f \in C^\infty(M)$:

$$\nabla_{fX}Y = f\nabla_XY$$

$$\nabla_X(fY) = df(X)Y + f\nabla_XY$$

Definition 2. Let ∇ be a Koszul connection. Its torsion tensor T^∇ and curvature tensor R^∇ are defined as follows:

$$T^\nabla(X, Y) = \nabla_XY - \nabla_YX - [X, Y]$$

$$R^\nabla(X, Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X, Y]}$$

2.2. Statistical Manifold

Definition 3. A statistical manifold is a triple (M, g, ∇) formed of a symmetric gauge structure (M, ∇) and a Riemannian structure (M, g) which are connected by the following relation:

$$(\nabla_Xg)(Y, Z) = (\nabla_Yg)(X, Z).$$

It is easy to verify that (M, g, ∇^*) is also a statistical manifold, where the symmetric connection ∇^* is defined by the following identity:

$$g(\nabla_X^*Y, Z) = X \cdot g(Y, Z) - g(Y, \nabla_XY).$$

Example 1: Any Riemannian manifold (M, g, ∇^{LC}) where ∇^{LC} is the Levi-Civita connection of g . We will say that M is a trivial statistical manifold.

Example 2: Let (M, g, ∇, ∇^*) any statistical manifold, the family $(M, g, \nabla^{(\alpha)}, \nabla^{(-\alpha)})_{\alpha \in \mathbb{R}}$ where $\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*$ is a statistical manifold for $\forall \alpha \in \mathbb{R}$.

Example 3: Let (M, g) some Riemannian manifold and denote ∇^{LC} the Levi-Civita connection with respect to g , and let $E \in TM \setminus \{0\}$. The triplet (M, g, ∇, ∇^*) is a statistical manifold where $\nabla_XY = \nabla_X^{LC}Y + g(X, E)g(Y, E)E$ and $\nabla_X^*Y = \nabla_X^{LC}Y - g(X, E)g(Y, E)E$.

3. KV-cohomology of Locally Flat Manifold

3.1. Locally Flat Manifold

Definition 4. A locally flat manifold is a pair (M, ∇) formed by a differentiable manifold M and a connection ∇ whose curvature tensor R^∇ and torsion tensor T^∇ vanish:

$$\nabla_XY - \nabla_YX - [X, Y] = 0$$

$$\nabla_X(\nabla_YZ) - \nabla_Y(\nabla_XZ) - \nabla_{[X, Y]}Z = 0,$$

$\forall X, Y, Z \in \Gamma(TM)$.

Definition 5. An affinely flat structure in a m -dimensional manifold M is defined by an complete atlas:

$$\mathcal{A} = \{(U_i, \Phi_i)\}$$

whose local chart changes $\Phi_i^{-1} \circ \Phi_j$ are the restrictions of affine transformations of the m -dimensional affine space \mathbb{R}^m .

Theorem 1. There is a one-to-one correspondence between the category of affinely flat structures in M and the category of locally flat structures in M .

Example 5: Let $\mathbb{R}^n = \{(x_1, \dots, x_n) / x_i \in \mathbb{R}\}$ endowed with linear connection ∇^F defined by

$$\nabla^F \frac{\partial}{\partial x_i} = 0$$

$\forall i = 1, 2, \dots, n$ and $\forall X \in \mathcal{X}(M)$

Example 6: Let \mathbb{R}^2 and $\forall X = f \frac{\partial}{\partial x_1} + g \frac{\partial}{\partial x_2} \in \mathcal{X}(M)$, endowed by a linear connection defined by

$$\nabla \frac{\partial}{\partial x_1} = f \frac{\partial}{\partial x_1} \text{ and } \nabla \frac{\partial}{\partial x_2} = 0$$

43

44 (\mathbb{R}^2, ∇) is locally flat manifold.

45

To a locally flat manifold (M, ∇) we assigned the algebra \mathcal{A} whose underlying vector space is the vectors fields, namely $\mathcal{X}(M)$, and the product $X \cdot Y$ is defined by the Koszul connection ∇ as:

$$X \cdot Y = \nabla_X Y$$

Definition 6. An algebra \mathcal{B} is Koszul-Vinberg algebra (or KV in short) algebra if for any triple (a, b, c) in \mathcal{B}^3 the following identity is satisfy:

$$a(bc) - (ab)c - b(ac) + (ba)c = 0.$$

46 **Proposition 1.** The algebra $A = (\mathcal{X}(M), \nabla)$ of a locally flat manifold (M, ∇) is a Koszul-Vinberg algebra.

Proof. Since the curvature tensor vanishes, we get for any triple (X, Y, Z) of vector fields that:

$$\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) = \nabla_{[X, Y]} Z;$$

but ∇ is torsion-less, so the right hand side is:

$$\nabla_{\nabla_X Y} Z - \nabla_{\nabla_Y X} Z.$$

This yields:

$$\nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z) + \nabla_{\nabla_Y X} Z = 0$$

or

$$X(YZ) - (XY)Z - Y(XZ) + (YX)Z = 0$$

47 which proves the claim. \square

Definition 7. A two-sided module of an algebra \mathcal{B} is the real vector space W which is endowed with the two-sided linear mappings

$$\mathcal{B} \times W \ni (a, w) \rightarrow a \cdot w \in W \quad \text{and} \quad W \times \mathcal{B} \ni (w, a) \rightarrow w \cdot a \in W. \quad (1)$$

In what follows, we use the following notations:

$$\begin{aligned} (a, b, w) &= (a \cdot b) \cdot w - a \cdot (b \cdot w), \\ (a, w, b) &= (a \cdot w) \cdot b - a \cdot (w \cdot b), \\ (w, a, b) &= (w \cdot a) \cdot b - w \cdot (a \cdot b). \end{aligned} \quad (2)$$

Definition 8. A two-sided module W is Koszul-Vinberg module of a Koszul-Vinberg algebra \mathcal{B} if it satisfies the following identities :

$$\begin{aligned} (a, b, w) &= (b, a, w), \\ (a, w, b) &= (w, a, b). \end{aligned} \tag{3}$$

Definition 9. Given a two-sided module KV module W on \mathcal{A} , the subspace of Jacobi elements of W , namely $J(W)$ is defined as follows:

$$J(W) = \{w \in W : (a, b, w) = 0 \forall a, b \in \mathcal{A}\}$$

Remark 1. The vector space $C^\infty(M)$ is a left KV module of \mathcal{A} whose left action is defined by:

$$X \cdot f = \nabla_X f.$$

48 3.2. The Scalar KV valued cohomology

The scalar KV complex is the \mathbb{Z} -graded vector space $C^\infty(\mathcal{A}, \mathbb{R}) = \bigoplus_q C^q(\mathcal{A}, C^\infty(M))$ by the following homogeneous subspaces :

$$\begin{aligned} C^q(\mathcal{A}, \mathbb{R}) &= 0 \text{ for } q < 0 \\ C^0(\mathcal{A}, \mathbb{R}) &= \{f \in C^\infty(M) : \nabla^2 f = 0\} \\ C^q(\mathcal{A}, \mathbb{R}) &= \text{Hom}_{\mathbb{R}}(\mathcal{A}^{\otimes q}, C^\infty(M)) \end{aligned}$$

The coboundary operator is defined as it follows :

$$\delta_{KV}^\nabla f = -f$$

For $p > 0$, let $F \in C^p(\mathcal{A}, C^\infty(M))$:

$$\delta_{KV}^\nabla F(\xi) = \sum_{i \leq q} \nabla_{X_i} F(\partial_i \xi) - F(\nabla_{X_i} \partial_i \xi).$$

The cohomology space of this cochain is denoted by

$$H_{KV}(\mathcal{A}, C^\infty(M)) = \bigoplus_q H_{KV}^q(\mathcal{A}, C^\infty(M)).$$

49 **Definition 10.** A Hessian metric in a locally flat manifold (M, ∇) is a non degenerate symmetric
50 2-cocyclic $g \in Z_{KV}^2(\mathcal{A}, C^\infty(M))$. It's cohomology class $[g] \in H_{KV}^2(\mathcal{A}, C^\infty(M))$ is called a non
51 degenerate Hessian class.

Theorem 2. Let g be a metric tensor in a locally flat manifold (M, ∇) . The following statements are equivalent.

(1) (M, g, ∇) is Hessian manifold.

(2) $\delta^\nabla g = 0$.

(3) Every point has a neighborhood \mathcal{U} supporting a local smooth function f such that

$$g = \nabla^2 f$$

52 .

53 This theorem is a cohomological of proposition 2.1 as in [1]

54 **3.3. The notion of Hyperbolicity of locally flat manifold**

Given a locally flat manifold (M, ∇) , we fix $x_0 \in M$ and we consider the space of differentiable paths

$$\{(0, [0, 1]) \longrightarrow (x_0, M)\}$$

55 and we consider it's quotient modulo the fixed ends homotopy. This quotient set is denoted by \tilde{M} .

The map

$$\tilde{M} \ni [c] \rightarrow c(1) \in M$$

56 is a universal covering of M .

The universal covering of (M, ∇) is denoted by $(\tilde{M}, \tilde{\nabla})$. We consider a path c and $\theta \in [0, 1]$. Let τ_θ the parallel transport:

$$\tau_\theta : T_{x_0}M \rightarrow T_{c(\theta)}M$$

The developing map is defined as it follows:

$$D([c]) = \int_0^1 \tau_\theta^{-1} \frac{dc}{dt}(\theta) d\theta.$$

57 **Definition 11.** A locally flat manifold (M, ∇) is called hyperbolic if D is a diffeomorphism of \tilde{M} onto
58 a convex domain not containing any straight line (Koszul [6]).

Remark 2. This definition is equivalent to the fact that M is diffeomorphic to the orbit space

$$M = \frac{C}{D}$$

59 where C is an open convex domain not containing any straight line in \mathbb{R}^n and D is a discrete subgroup
60 of the Lie group $Aff(n)$ (see J. L. Koszul [6,7]).

61 **Theorem 3.** (see [6]) For a locally flat manifold (M, ∇) being hyperbolic, it is necessary that it exists a de
62 Rham closed differential 1-form ω on M whose covariant derivate $\nabla\omega$ is positive definite. If M is compact then
63 this condition is also sufficient.

64 Michel Boyom has given a cohomological analogue of the theorem above and his result states as
65 follows:

66 **Theorem 4.** [8] For a compact Hessian manifold (M, g, ∇) , the following assertions are equivalent:

- 67 1. $[g] = [0] \in H_{KV}^2(\mathcal{X}(M), C^\infty(M))$,
68 2. (M, ∇) is hyperbolic.

69 **4. The Hessian equation on a statistical manifold**

70 **4.1. Differential Operator**

Let (V, p, M) a vector bundle and let k be any non negative integer. The vector bundle of k -jets of sections of vector bundle of V is denoted $J^k(V)$. At a point $x \in M$ the fiber $J_x^k(V)$ is quotient vector space:

$$J_x^k(V) = \frac{\Gamma(V)}{I_x^{k+1}(M)\Gamma(V)}$$

71 where $I_x(M) \subset C^\infty(M)$ is the ideal of differentiable functions that vanish at $x \in M$.

Let E, F two vectors bundles over M and $\Gamma(E)$ and $\Gamma(F)$ the spaces of their sections. For all k an nonnegative integer we denote j^k the map :

$$s \longrightarrow j^k(s)$$

72 from $\Gamma(E)$ to $\Gamma(J^k(E))$

Definition 12. The map

$$D : \Gamma(E) \longrightarrow \Gamma(F)$$

is k^{th} order differential operator if there exist an homomorphism of vector bundle

$$\mathcal{D} : J^k(E) \longrightarrow \Gamma(F)$$

such that

$$D(s) = \mathcal{D}(j^k s)$$

73 $\forall s \in \Gamma(E)$

74 \mathcal{D} is called the principal symbol of D .

75

Let consider the short exact sequence

$$0 \rightarrow \check{J}^k(E) \xrightarrow{i} J^k(E) \xrightarrow{\pi^k} J^{k-1}(E) \rightarrow 0$$

Definition 13. The map

$$\sigma(D) = \mathcal{D} \circ i$$

76 from $\check{J}^k(E)$ to $\Gamma(F)$ is called the geometrical symbol of differential operator D

77 Let V and W be an finite vector spaces ,let ρ be a subspace of the vector space $Hom(V, W)$.

Definition 14. The first prolongation of ρ is subspace ρ^1 of $Hom(V, \rho)$ which is $\rho^1 = \rho \otimes V^* \cap W \otimes S^2(V^*)$.By recurrence we define on $p \in \mathbb{N}$,we defined the p th prolongation:

$$\rho^0 = \rho$$

$$\rho^p = V^* \otimes \rho^{p-1} \cap S^{p+1}V^* \otimes W$$

Definition 15. We consider the Koszul-Spencer complex of ρ :

$$\rightarrow Hom(\wedge^p V, \rho^q) \rightarrow Hom(\wedge^{p+1} V, \rho^{q-1}) \rightarrow$$

The differential operator ∂ is defined by

$$\partial f(v_1 \wedge \dots \wedge v_{p+1} \otimes t_1 \dots t_q) = \sum_j (-1)^j f(v_1 \wedge \dots \wedge v_{p+1} \otimes v_j \cdot t_1 \dots t_q)$$

78 **Definition 16.** ρ is called involutive if $H^{p,q}(\rho) = 0 \forall q, p$

79 4.2. Hessian equation of ∇

Given a gauge structure (M, ∇) ,we define the following second order differential operator

$$X \in \mathcal{X}(M) \rightarrow \nabla^2 X \in T_1^2(M)$$

Here $T_1^2(M)$ is the vector space of (2,1)-tensors in M .

We recall the definition of $\nabla^2 X$,

$$(\nabla^2 X)(Y, Z) = \nabla_Y(\nabla_Z X) - \nabla_{\nabla_Y Z} X$$

Definition 17. The Hessian equation of (M, ∇) is the following second differential equation

$$\nabla^2 X = 0$$

Let $\mathcal{J}_\nabla(M)$ the sheaf of solutions of Hessian equation $\nabla^2 X = 0$ and $J_\nabla(M)$ be the vector space of global section of the sheaf $\mathcal{J}_\nabla(M)$, it is well in [8] that \mathcal{J}_∇ is Associative Algebra sheaf(AAS).

Let one evaluate the principal principal symbol of $X \rightarrow \nabla^2 X$. the principal symbol of the operator ∇^2 is defined by

$$(\nabla^2(X))\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_k \Omega_{ij}^k \frac{\partial}{\partial x_k}$$

where

$$\Omega_{ij}^l = \frac{\partial^2 X^l}{\partial x_i \partial x_j} + \sum_k \left[\Gamma_{ik}^l \frac{\partial X^k}{\partial x_j} + \Gamma_{jk}^l \frac{\partial X^k}{\partial x_i} - \Gamma_{ij}^k \frac{\partial X^l}{\partial x_k} \right] + \sum_k \left[\frac{\partial \Gamma_{jk}^l}{\partial x_i} + \sum_m \left(\Gamma_{jk}^m \Gamma_{im}^k - \Gamma_{ij}^m \Gamma_{mk}^l \right) \right]$$

The geometrical symbol of ∇^2 denoted by $\sigma(\nabla^2)$ is

$$\left[\frac{\partial^2 X^k}{\partial x_i \partial x_j} \right]_{i,j} = [a_{i,j}^k]$$

80 $\sigma(\nabla^2)$ is equal to $S^2 \mathbb{R}^{*n} \otimes \mathbb{R}^n$, the first prolongation of $\sigma(\nabla^2)$ denoted $\sigma(\nabla^2)^1 = S^3 \mathbb{R}^{*n} \otimes \mathbb{R}^n$. It is easy
81 to see that $\sigma(\nabla^2)$ admit a quasi regular basis then it is involutive, this is equivalent to $H^{p,q}(\sigma(\nabla^2)) =$
82 $0, \forall p, \forall q > 1$ (see Jean Pierre Serre in appendix of [9]). We conclude that Hessian equation is involutive
83 and finite type in sense of Elie Cartan [9], so we conclude that (J_∇, ∇) finite associative algebra and
84 $(J_\nabla, [\cdot, \cdot]_\nabla)$ is finite Lie algebra.

85 **Proposition 2.** Let (M, g) be a Riemman manifold and let Ric be its Ricci curvature. We have

$$J_\nabla \subset \text{Ker Ric} \tag{4}$$

86 **Proof.** Let $\xi \in J_\nabla$, by using the Bianchi identity we have:

$$\begin{aligned} R^\nabla(\xi, X)Y &= R^\nabla(\xi, Y)X \quad \forall X, Y \in \chi(M) \\ g(R^\nabla(\xi, X)Y, Z) &= g(R^\nabla(\xi, Y)X, Z) \\ &= -g(X, R^\nabla(\xi, Y)Z) \\ &= g(X, R^\nabla(\xi, Z)Y) \\ &= -g(R^\nabla(\xi, Z)X, Y) \\ &= g(R^\nabla(\xi, X)Z, Y) \\ &= -g(R^\nabla(\xi, X)Y, Z) \end{aligned}$$

Then

$$J_\nabla \subset \text{Ker Ric}^\nabla$$

87 \square

Corollary 1. If ∇ is the Levi-Civita connection of an Einstein Riemannian manifold, then

$$J_\nabla = \{0\}$$

Definition 18. A gauge structure (M, ∇) is called special if

$$J_\nabla \neq \{0\}$$

88 Example 7: Every locally flat manifold (M, ∇) is special gauge structure.

89

90 Example 9: For any positive integer m , $B^m = \{X \in \mathbb{R}^m / \|X\| < 1\}$ equipped with the Euclidean
 91 Levi-Civita connection is special gauge structure.
 92 In the rest of this work we assume that our gauge structure are special.

93
 94 Let (M, g, ∇, ∇^*) be a statistical manifold, consider the two Hessian operators ∇^2 and ∇^{*2} and
 95 we denote by $(J_\nabla(M), [,]_\nabla)$ and $(J_{\nabla^*}(M), [,]_{\nabla^*})$ the finite Lie algebra of global sections of sheaf of
 96 solutions of $\nabla^2 X = 0$ and $\nabla^{*2} X = 0$.
 97 Let G_∇ (resp G_{∇^*}) a simply connected Lie group whose Lie algebra is denoted by J_∇ (resp J_{∇^*})
 98 according to the third theorem of Lie.

Proposition 3. *Let (M, g, ∇, ∇^*) a statistical manifold and let $\{J_\nabla, J_{\nabla^*}\}$ a pair of Lie algebras and $\{G_\nabla, G_{\nabla^*}\}$ a pair of is Lie groups. We have the following properties:*

1) $\{(J_\nabla, \nabla); (J_{\nabla^*}, \nabla^*)\}$ is a pair of KV-algebras,

2) $\{(G_\nabla, \nabla); (G_{\nabla^*}, \nabla^*)\}$ pair of simply connected bi-invariant locally flat Lie groups.

99 **Definition 19.** An infinitesimal dynamical of G_∇ is a manifold M is a lie algebra homomorphism from
 100 his Lie algebra J_∇ to the Lie algebra of vectors fields of M .

101

102 **Definition 20.** A locally effective action of lie group G_∇ on a differentiable manifold M is a group
 103 homomorphism from G_∇ to the diffeomorphism group of M such that his kernel is discrete.

104 **Proposition 4.** J_∇ is integrable if is induced by a locally effective action of the Lie group G_∇ .

105 For more details see Richard S. Palais [10]

106 **Remark 3.** 1. If Any element of J_∇ is complete then J_∇ is integrable (If M is compact then J_∇ is
 107 integrable).

108 2. If (M, ∇) a geodesically complete locally flat manifold then J_∇ is integrable.

109 **Definition 21.** A locally flat foliation in a gauge structure (M, ∇) is a ∇ -auto-parallel foliation \mathcal{F}
 110 whose leaves are locally flat submanifolds on (M, ∇) .

111 **Definition 22.** A Hessian foliation \mathcal{F} in (M, g, ∇) is a locally flat foliation in gauge structure (M, ∇)
 112 such that the restriction of the Riemaniann metric on \mathcal{F} is Hessiann metric.

113 Let \mathcal{F}_∇ (resp \mathcal{F}_{∇^*}) denoted the foliation defined by the G_∇ -orbits (resp G_{∇^*}).

114 **Theorem 5.** *Let (M, g, ∇, ∇^*) a compact statistical manifold. We have*

115 (1) *The foliation \mathcal{F}_∇ is Hessian foliation in (M, g, ∇) ,*

116 (2) *The foliation \mathcal{F}_{∇^*} is Hessian foliation in (M, g, ∇^*) .*

117 **Proof.** (1) By assumption M is compact then J_∇ is integrable, we conclude that J_∇ is infinitesimal action
 118 of G_∇ on M . The G_∇ -orbits are ∇ -auto-parallel and satisfy $R^\nabla(X, Y)J_\nabla = 0$ then \mathcal{F}_∇ is locally flat
 119 foliation in (M, ∇) . The manifold (M, g, ∇) is statistical manifold and the G_∇ -orbits are ∇ -auto-parallel
 120 then the leaves of \mathcal{F}_∇ are Hessian submanifolds of (M, g, ∇) .

121 (2) By using the same arguments for $G_{\nabla^*}(M)$, the theorem is proved. \square

122 5. Hyperbolicity Of Compacts Orbits

123 5.1. Basics Notions

Definition 23. Let G be a Lie group. A pair (f, q) of a homomorphism f from G to $GL(V)$ and a map
 q from G to V is said to be affine representation of G in V if it satisfies the following identity:

$$q(s_1 s_2) = f(s_1)q(s_2) + q(s_1)$$

$$124 \quad \forall s_1, s_2 \in G$$

Definition 24. A affine representation of Lie algebra from \mathcal{G} to V is homomorphism from the Lie algebra \mathcal{G} to $aff(V)$. The affine representation of \mathcal{G} to V is (f, q) such that f is linear map from \mathcal{G} to $End(V)$ and q is linear map from \mathcal{G} to V satisfy the following identities:

$$q([X, Y]) = f(X)q(Y) - f(Y)q(X), \forall X, Y \in \mathcal{G}$$

125 5.2. The Canonical affine representation of affine Lie groups and is Lie algebra

Let $G_\nabla = (G, \nabla)$ be a connected locally flat lie group whose (Left invariant) KV algebra is denote by J_∇ . In fact, the KV algebra of G_∇ is the real algebra

$$J_\nabla = (J, \nabla)$$

where J is Lie algebra of the lie group G and 1_{J_∇} is the identity endomorphism of the vector space J_∇ . The affine representation of Lie algebra J_∇ in the vector space J_∇ by

$$J_\nabla \ni X \longrightarrow (\nabla, 1_{J_\nabla})(X) = (\nabla_X, X) \in aff(J_\nabla)$$

The affine map (∇_X, X) is defined by

$$(\nabla_X, X)(Y) = \nabla_X Y + X, \forall Y \in J_\nabla$$

Remark 4. $(\nabla, 1_{J_\nabla})$ is the differential at the unit element of a unique affine representation (f_∇, q_∇) of G_∇ to J_∇ , namely

$$\begin{aligned} (f_\nabla, q_\nabla) : G_\nabla &\longrightarrow aff(J_\nabla) \\ \gamma &\longrightarrow (f_\nabla(\gamma), q_\nabla(\gamma)) \end{aligned}$$

Where

$$(f_\nabla(\gamma), q_\nabla(\gamma))$$

is defined by :

$$(f_\nabla(\gamma), q_\nabla(\gamma))(X) = f_\nabla(\gamma)X + q_\nabla(\gamma)$$

and satisfy these identities :

$$\frac{d}{dt}(f(\exp tX), q(\exp tX))|_{t=0} = (\nabla_X, X)$$

and

$$q_\nabla(\gamma_1 \cdot \gamma_2) = f_\nabla(\gamma_1)q_\nabla(\gamma_2) + q_\nabla(\gamma_1) \forall \gamma_1, \gamma_2 \in G_\nabla$$

Remark 5. Let

$$f_\nabla : G_\nabla \longrightarrow GL(J_\nabla)$$

$$q_\nabla : G_\nabla \longrightarrow J_\nabla$$

Remind the cohomology of the Lie group G_∇ value in his Lie algebra J_∇ define by :

$$\dots \rightarrow C^q(G_\nabla, J_\nabla) \rightarrow C^{q+1}(G_\nabla, J_\nabla) \rightarrow C^{q+2}(G_\nabla, J_\nabla) \rightarrow$$

with differential operator d defined by :

$$d\theta(\gamma_1, \dots, \gamma_{q+1}) = f_\nabla(\gamma_1) \cdot \theta(\gamma_2, \dots, \gamma_{q+1}) + \sum_{i \leq q} (-1)^i \theta(\dots, \gamma_i \gamma_{i+1}, \dots) + (-1)^q \theta(\gamma_1, \dots, \gamma_q)$$

See Koszul[11], Cartan and Eilenberg [12] and Chevalley and Eilenberg [13], M. Boyom[14] for more details on Lie algebra cohomology.

It's easy to see that the condition :

$$q_{\nabla}(\gamma_1 \cdot \gamma_2) = f_{\nabla}(\gamma_1)q_{\nabla}(\gamma_2) + q_{\nabla}(\gamma_1)$$

$$\forall \gamma_1, \gamma_2 \in G_{\nabla}$$

126 is equivalent to the fact that $q_{\nabla} \in Z^1(G_{\nabla}, J_{\nabla})$, we denote $[q_{\nabla}] \in H^1(G_{\nabla}, J_{\nabla})$ is cohomology class

127 5.3. Vanishing Theorems and Hyperbolicity

128 Let (M, g, ∇, ∇^*) be a compact statistical manifold, We go to deal conditions under which the
129 Hessian foliations \mathcal{F}_{∇} and \mathcal{F}_{∇^*} are hyperbolic.

Theorem 6. *Let (M, g, ∇, ∇^*) a compact statistical manifold, the following assertions are equivalents on the leaves of \mathcal{F}_{∇} (resp on those of \mathcal{F}_{∇^*}):*

- 1) $[g]_{\nabla} = 0 \in H_{KV}^2(J_{\nabla}, \mathbb{R})$
- 2) $[q_{\nabla^*}] = 0 \in H^1(G_{\nabla^*}, J_{\nabla^*})$
- 3) $(f_{\nabla^*}, q_{\nabla^*})$: admit a fixed point
- 4) $(f_{\nabla^*}, q_{\nabla^*}) \simeq f_{\nabla^*}$

Proof. (2) \rightarrow (1): If $[q_{\nabla^*}] = 0$ then $\exists \xi \in J_{\nabla^*}$ such that the following identities hold

$$q_{\nabla^*}(\gamma_t) = f_{\nabla^*}(\gamma_t)\xi - \xi, \forall \gamma_t \in G_{\nabla^*}$$

by derivate this terms we obtain the following identity:

$$\frac{d}{dt}(q_{\nabla^*}(\gamma_t))|_{t=0} = \frac{d}{dt}(f_{\nabla^*}(\gamma_t)\xi - \xi)|_{t=0}, \forall \gamma_t \in G_{\nabla^*}$$

$$X = \nabla_X^* \xi, \forall X = \frac{d(\gamma_t)}{dt}|_{t=0}$$

by using the conjugaison formula, we deduce the following identity:

$$g(X, Y) = X.g(\xi, Y) - g(\xi, \nabla_X Y)$$

$$g(X, Y) = (d_{KV}^{\nabla}(i_{\xi}g))(X, Y)$$

$$[g]_{\nabla} = 0$$

by the same arguments we prove that $[q_{\nabla^*}] = 0$ then $[g]_{\nabla} = 0$

(1) \rightarrow (2):

Assume that $[g]_{\nabla} = 0$ then there exist a one form Θ define by $\Theta(X) = g(\xi, X)$ such that

$$g(X, Y) = -X\Theta(Y) + \Theta(\nabla_X Y)$$

by using the conjugaison we obtain the following identity

$$g(\nabla_X^* \xi, Y) = X.g(\xi, Y) - g(\xi, \nabla_X Y)$$

we deduce the following relation:

$$\nabla_X^*(-\xi) = X$$

$\forall \gamma_t \in G_{\nabla^*}$ such that $q_{\nabla^*}(\gamma_t) = f_{\nabla^*}(\gamma_t)\xi - \xi$, then we conclude that

$$[q_{\nabla^*}] = 0 \in H^1(G_{\nabla^*}, J_{\nabla^*})$$

$$2) \iff 3) \iff 4)$$

this well known in [15][16]

Hint (2) \rightarrow (3) Assume that (2) $q_{\nabla^*} \in B^2(G_{\nabla^*}, J_{\nabla^*})$ then $\exists \xi \in J_{\nabla^*}$ such that:

$$f_{\nabla^*}(\gamma)(-\xi) + q_{\nabla^*}(\gamma) = -\xi$$

130 then $(f_{\nabla^*}, q_{\nabla^*})$ admit an fixed point.

131 Now we assume (3) \rightarrow (4)

$$\begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_{\nabla^*} & q_{\nabla^*} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\xi \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} f_{\nabla^*} & 0 \\ 0 & 1 \end{bmatrix} \quad (5)$$

(4) \rightarrow (2) Assume that $\langle f_{\nabla^*}, q_{\nabla^*} \rangle$ is linearisable then $\exists \xi \in J_{\nabla^*}$ such that

$$q_{\nabla^*}(\gamma) = f_{\nabla^*}(\gamma)\xi - \xi, \forall \gamma \in G_{\nabla^*}$$

$$q_{\nabla^*}(\gamma) = d(\xi)(\gamma)$$

132 we conclude that $[q_{\nabla^*}] = [0]$. \square

133 Let denoted by F the compact leaf of foliation \mathcal{F}_{∇} and F^* those of \mathcal{F}_{∇^*} .

134

135 **Corollary 2.** Let (M, g, ∇, ∇^*) be a compact statistical manifold. Under the hypotheses of Theorem 6, we have :

136 (1) The compact leaves (F, ∇) are hyperbolic manifolds,

137 (2) The compact leaves (F^*, ∇^*) are hyperbolic manifolds.

138

139 Comments

140 In [8] Boyom proves that the compact orbits of $G_{\nabla^{LC}}$ in case of trivial statistical manifold

141 (M, g, ∇^{LC}) are flat cylinder over Riemannian torus. $(Orb(G_{\nabla^{LC}})(p), g) \simeq \left(\frac{T^{b_1(Orb(G_{\nabla^{LC}}))}}{K} \times \right.$

142 $\mathbb{R}^{r(\nabla^{LC})(p) - b_1(Orb(G_{\nabla^{LC}}))}, g_0)$. Here $r(\nabla^{LC})(p)$ is the dimension of the orbits on p , g_0 is Euclidean

143 (flat) metric of $\mathbb{R}^{r(\nabla^{LC})(p)}$, $b_1(Orb(G_{\nabla^{LC}}))$ is the first Betti number of $Orb(G_{\nabla^{LC}})$, K is finite subgroup of

144 isometries of g_0 i.e. $(Iso(g_0) \simeq O(r(\nabla^0)p) \times \mathbb{R}^{r(\nabla^{LC})(p)})$ [17].

145 We prove the similar result with hyperbolic orbits of G_{∇} $(Orb(G_{\nabla}(p), \nabla) \simeq (\frac{U}{\Gamma}, \nabla^F))$. Here

146 U is open convex domain not containing any straight line in $\mathbb{R}^{r(\nabla)(p)}$, Γ is discrete subgroup of

147 $Aff(\mathbb{R}^{r(\nabla)(p)})$ i.e. $(GL(\mathbb{R}^{r(\nabla)(p)}) \times \mathbb{R}^{r(\nabla)(p)})$, ∇^F the canonical flat connection on $\mathbb{R}^{r(\nabla)(p)}$.

148

149 Now we discuss about some conditions under which the universal covering of compact

150 hyperbolic orbits are convex cones. Let remind an important theorem of J.L.Koszul.

151 **Theorem 7.** [18] If the universal covering of a compact hyperbolic locally flat manifold is affinely homogeneous,
152 then its universal covering is isomorphic to a convex cone.

153 **Theorem 8.** Let (M, g, ∇, ∇^*) a compact statistical manifold which satisfy one of the equivalent conditions of
154 theorem 6, we have the following assertions.

- 155 1) The compact leaves of \mathcal{F}_∇ are isomorphic to the quotient of convex homogeneous cones by discrete subgroups
156 of $Aff(\mathbb{R}^{r(\nabla)^p})$,
157 2) The compact leaves of \mathcal{F}_{∇^*} are isomorphic to quotient of convex homogeneous cones by discrete subgroups of
158 $Aff(\mathbb{R}^{r(\nabla^*)^q})$.

Proof. By using the following formula on the G_∇ -Orbit :

$$\mathcal{L}_\xi \nabla = i_\xi R^\nabla + \nabla^2 \xi$$

159 we conclude that $G_\nabla = Aff(Orbit(G_\nabla), \nabla)$. The Lie Group $Aff(Orbit(G_\nabla), \nabla)$ is isomorphic to the
160 normalizer $N(\pi_1(Orbit(G_\nabla)))$ of $\pi_1(Orbit(G_\nabla))$ in $Aff(Orbit(G_\nabla), \tilde{\nabla})$, then we conclude that G_∇ acts
161 affinely on $(Orbit(G_\nabla), \tilde{\nabla})$. The action of G_∇ on $Orbit(G_\nabla)$ is affinely transitive, then the orbits of
162 this action at \tilde{x} denote by $G_\nabla(\tilde{x})$ is isomorphic to $Orbit(G_\nabla)$, then $\dim(G_\nabla(\tilde{x})) = \dim(Orbit(G_\nabla))$. We
163 conclude that $G_\nabla(\tilde{x})$ is an open subset of $Orbit(G_\nabla)$, but $Orbit(G_\nabla)$ is connected then G_∇ acts transitively
164 on $Orbit(G_\nabla)$. By using Theorem 7, we conclude that the compact orbits $Orbit(G_\nabla)$ are isomorphic
165 to the Quotient of Convex cone by the discrete subgroup Γ of $Aff(\mathbb{R}^{\dim(Orbit(G_\nabla))})$.

166 We can do the same arguments for G_{∇^*} . \square

167 **Remark 6.** Under Barbaresco formalisms the universal coverings of the compact leaves of our
168 hyperbolic foliations are statistical models.

169 **Corollary 3.** If (M, g, ∇, ∇^*) is a statistical model (Θ, P) then it's foliated by leaves which are quotient of
170 exponential models.

171 **Example:**

We shall give an example of compact Hessian manifold whose orbits are quotient of statistical
models by discrete subgroups. Let \mathbb{R}^2 be a 2-dimensional real affine space with the natural flat affine
connection D and let $\{x, y\}$ be an affine coordinate system of \mathbb{R}^2 .

Let Ω be a domain defined by $x > 0$ and $y > 0$. Consider a Riemannian metric on Ω given by
 $g = \frac{1}{x^2} dx^2 + \frac{1}{y^2} dy^2$.

Then (D, g) is a Hessian structure on Ω .

Let Σ and χ be linear transformations on Ω defined by :

$$\Sigma : (x, y) \rightarrow (2x, y)$$

$$\chi : (x, y) \rightarrow (2x, 3y)$$

Then $\langle \Sigma, \chi \rangle$ leave the Hessian structure (D, g) invariant.

We denote Γ the group generated by $\{\Sigma, \chi\}$, we can also write Γ as $\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \rangle$.

Γ acts properly discontinuously on Ω and $\frac{\Omega}{\Gamma}$ is a compact Hessian manifold which is diffeomorphic to
a Torus.

Let us denote by π the projection from Ω to $\frac{\Omega}{\Gamma}$ and by (D, g) the Hessian structure on $\frac{\Omega}{\Gamma}$. Since the space
of all Γ -invariant D -parallel 1-forms on Ω is spanned by dx and dy . The space of all D -parallel 1-forms on
 $\frac{\Omega}{\Gamma}$ is spanned by ω and Φ where $dx = \pi^* \omega$ and $dy = \pi^* \Phi$. Let X and \tilde{X} be vector fields on $\frac{\Omega}{\Gamma}$ defined
by $\omega(Y) = g(X, Y)$ and $\Phi(Y) = g(\tilde{X}, Y)$ for each vector field Y on $\frac{\Omega}{\Gamma}$. Then $X = \pi_* \left(\frac{\partial}{\partial x} \right)$ and $\tilde{X} = \pi_* \left(\frac{\partial}{\partial y} \right)$
and the vector space J_∇ of all Hessian vector fields on $\frac{\Omega}{\Gamma}$ is spanned by $\langle X, \tilde{X} \rangle$. Since T^2 is compact
the J_∇ is integrable. Let $Exp(tX)$ and $Exp(tY)$ be a 1-parameter group of transformations generated by
 X and Y . Let $G_\nabla = \langle Exp(tX), Exp(tY) \rangle$. Consider $a > 0$ and $b > 0$, the compact homogeneous
orbit $G_\nabla \pi(a, b) = \{ \gamma \cdot \pi(a, b), \gamma \in G_\nabla \}$ is a circle. Then we conclude that (S^1, D, g_{S^1}) is a Hessian
manifold and satisfy $D \frac{\partial}{\partial \theta} = 0$ and $g_{S^1} = \frac{1}{\theta^2} d\theta^{\otimes 2}$. By using conjugation we have $D \frac{\partial}{\partial \theta} = -\frac{2}{\theta} \frac{\partial}{\partial \theta}$. Let

$H^* = h(\theta) \frac{\partial}{\partial \theta}$ vectors fields on S^1 . H^* is homothety vector fields of D^* if h is solution of differential equation $y' - \frac{2}{\theta^2}y = 1$. We deduce that

$$h(\theta) = Ae^{-\frac{2}{\theta}} + 2e^{-\frac{2}{\theta}} \int_{-\frac{2}{\theta}}^{+\infty} \frac{e^{-s}}{s} ds + \theta$$

where A is constant.

As $D^*H^* = Id$ then (S^1, D) is hyperbolic. Now let $p : \mathbb{R} \rightarrow S^1$ the covering map, and consider the map

$$(\mathbb{R}, \tilde{D}) \rightarrow (S^1, D)$$

172 .The compact orbits is $G_{\nabla} \pi(a, b)$ is quotient $\frac{\mathbb{R}^{>0}}{\Gamma}$, here Γ is discrete subgroup of $Gl(\mathbb{R}) \times \mathbb{R}$

173 References

- 174 1. Shima, H. The geometry of Hessian structures; World Scientific, 2017.
- 175 2. Gerstenhaber, M. On the deformation of rings and algebras. Annals of Mathematics **1956**, pp. 59–103.
- 176 3. Nguiffo Boyom, M. The cohomology of Koszul–Vinberg algebras. Pacific journal of mathematics **2006**,
- 177 225, 119–153.
- 178 4. Boyom, M.N. Foliations-webs-hessian geometry-information geometry-entropy and cohomology. Entropy
- 179 **2016**, 18, 433.
- 180 5. Nguiffo Boyom, M. The Last Formula of Jean-Louis Koszul. preprint.
- 181 6. Koszul, J.L. Déformations de connexions localement plates. Annales de l’institut Fourier **1968**, 18, 103–114.
- 182 7. Vey, J. Sur une notion d’hyperbolicité des variétés localement plates. PhD thesis, Université Joseph-Fourier
- 183 - Grenoble I, 1969.
- 184 8. Boyom, M.N. Numerical properties of Koszul connections. arXiv preprint arXiv:1708.01106 **2017**.
- 185 9. Guillemin, V.W.; Sternberg, S. An algebraic model of transitive differential geometry. Bulletin of the
- 186 American Mathematical Society **1964**, 70, 16–47.
- 187 10. Palais, R.S. A global formulation of the Lie theory of transformation groups. Mem. Amer. Math. Soc. **1957**,
- 188 22.
- 189 11. Koszul, J.L. Homologie et cohomologie des algèbres de Lie. Bulletin de la Société Mathématique de France
- 190 **1950**, 78, 65–127.
- 191 12. Cartan, H.; Eilenberg, S. Homological algebra; Vol. 41, Princeton University Press, 1999.
- 192 13. Chevalley, C.; Eilenberg, S. Cohomology theory of Lie groups and Lie algebras. Transactions of the
- 193 American Mathematical society **1948**, 63, 85–124.
- 194 14. Boyom, N.B.; others. Sur les structures affines homotopes à zéro des groupes de Lie. Journal of Differential
- 195 Geometry **1990**, 31, 859–911.
- 196 15. Michel, B. Algèbre à associateur symétrique et algèbres de Lie réductives. Thèse 3 ième cycle univ.
- 197 Grenoble **1968**.
- 198 16. Fried, D.; Goldman, W.; Hirsch, M.W. Affine manifolds with nilpotent holonomy. Commentarii
- 199 Mathematici Helvetici **1981**, 56, 487–523.
- 200 17. A.Wolf, J. Space of constant curvature; Boston Mass, 1974.
- 201 18. Koszul, J.L. Domaines bornés homogenes et orbites de groupes de transformations affines. Bulletin de la
- 202 Société Mathématique de France **1961**, 89, 515–533.

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