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Article

# Hyperbolicity of the Compact Leaves of Statistical Manifold Foliations

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**Abstract:** In this work, we show a construction of a Hessian foliation on a statistical manifold. Our results provide conditions, under which the compact hessian leaves (up to diffeomorphism) are quotients of homogeneous convex cones.

## 1. Introduction

The Hessian structures play important roles in Information geometry. In [1], Shima observes that Fisher information of all classical used statistical models are Hessian metrics. As Amari did he also proved that any locally flat statistical manifold is a Hessian manifold. These structures arised from works of many mathematicians (e.g. J.-L. Koszul, Y. Matsushima, A. Nijenhuis, E. B. Vinberg) in their attempt to solve the Gerstenhaber conjecture in the category of locally flat manifold. The geometry of locally flat hyperbolic manifold is also named Koszul geometry (see [2]), the conjecture of Gerstenhaber say that every restricted theory of deformation generates its proper theory of cohomology. This conjecture has been solved in 2006 (see [3]) by Michel Boyom. In [4][5] he also realize the homological version of the Hessian geometry. The aim of this paper is to study hyperbolic foliations in statistical manifolds . Here a hyperbolic foliation is a foliation whose leaves support Koszul geometry. The paper is organised as it follows. In section is this introduction. Section 2 is devoted to basics notions which are used. Section 3 is devoted to useful notions in the category of locally flat manifolds and their KV-cohomology. Section 4 is devoted to Hessian equation of Koszul connection in tangent bundle of statistical manifolds. In section 5 we focus on a cononical pair of locally flat foliations in statistical manifolds ,and we discuss conditions under which these foliations are hyperbolic .

## 2. Statistical manifold

### 2.1. Koszul connections

We recall in this subsection some basics notions. All differentiable manifold we deal with are paracompact. The class of differentiability is  $C^\infty$ . For a manifold  $M$  the real algebra of differentiable real valued function is denote by  $C^\infty(M)$ . The  $C^\infty(M)$ -module of differentiable vectors fields is denoted by  $\mathcal{X}(M)$ .

**Definition 1.** A Koszul connection on  $M$  is a  $\mathbb{R}$ -bilinear product

$$(X, Y) \in \mathcal{X}(M)^2 \longrightarrow \nabla_X Y \in \mathcal{X}(M)$$

satisfying the following identities for  $X, Y \in \mathcal{X}(M)$  and  $f \in C^\infty(M)$ :

$$\nabla_{fX}Y = f\nabla_XY$$

$$\nabla_X(fY) = df(X)Y + f\nabla_XY$$

**Definition 2.** Let  $\nabla$  be a Koszul connection. Its torsion tensor  $T^\nabla$  and curvature tensor  $R^\nabla$  are defined as follows:

$$T^\nabla(X, Y) = \nabla_XY - \nabla_YX - [X, Y]$$

$$R^\nabla(X, Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X, Y]}$$

## 2.2. Statistical Manifold

**Definition 3.** A statistical manifold is a triple  $(M, g, \nabla)$  formed of a symmetric gauge structure  $(M, \nabla)$  and a Riemannian structure  $(M, g)$  which are connected by the following relation:

$$(\nabla_Xg)(Y, Z) = (\nabla_Yg)(X, Z).$$

It is easy to verify that  $(M, g, \nabla^*)$  is also a statistical manifold, where the symmetric connection  $\nabla^*$  is defined by the following identity:

$$g(\nabla_X^*Y, Z) = X \cdot g(Y, Z) - g(Y, \nabla_XY).$$

*Example 1:* Any Riemannian manifold  $(M, g, \nabla^{LC})$  where  $\nabla^{LC}$  is the Levi-Civita connection of  $g$ . We will say that  $M$  is a trivial statistical manifold.

*Example 2:* Let  $(M, g, \nabla, \nabla^*)$  any statistical manifold, the family  $(M, g, \nabla^{(\alpha)}, \nabla^{(-\alpha)})_{\alpha \in \mathbb{R}}$  where  $\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*$  is a statistical manifold for  $\forall \alpha \in \mathbb{R}$ .

*Example 3:* Let  $(M, g)$  some Riemannian manifold and denote  $\nabla^{LC}$  the Levi-Civita connection with respect to  $g$ , and let  $E \in TM \setminus \{0\}$ . The triplet  $(M, g, \nabla, \nabla^*)$  is a statistical manifold where  $\nabla_XY = \nabla_X^{LC}Y + g(X, E)g(Y, E)E$  and  $\nabla_X^*Y = \nabla_X^{LC}Y - g(X, E)g(Y, E)E$ .

## 3. KV-cohomology of Locally Flat Manifold

### 3.1. Locally Flat Manifold

**Definition 4.** A locally flat manifold is a pair  $(M, \nabla)$  formed by a differentiable manifold  $M$  and a connection  $\nabla$  whose curvature tensor  $R^\nabla$  and torsion tensor  $T^\nabla$  vanish:

$$\nabla_XY - \nabla_YX - [X, Y] = 0$$

$$\nabla_X(\nabla_YZ) - \nabla_Y(\nabla_XZ) - \nabla_{[X, Y]}Z = 0,$$

$\forall X, Y, Z \in \Gamma(TM)$ .

**Definition 5.** An affinely flat structure in a  $m$ -dimensional manifold  $M$  is defined by an complete atlas:

$$\mathcal{A} = \{(U_i, \Phi_i)\}$$

whose local chart changes  $\Phi_i^{-1} \circ \Phi_j$  are the restrictions of affine transformations of the  $m$ -dimensional affine space  $\mathbb{R}^m$ .

**Theorem 1.** There is a one-to-one correspondence between the category of affinely flat structures in  $M$  and the category of locally flat structures in  $M$ .

*Example 5:* Let  $\mathbb{R}^n = \{(x_1, \dots, x_n) / x_i \in \mathbb{R}\}$  endowed with linear connection  $\nabla^F$  defined by

$$\nabla^F \frac{\partial}{\partial x_i} = 0$$

$\forall i = 1, 2, \dots, n$  and  $\forall X \in \mathcal{X}(M)$

*Example 6:* Let  $\mathbb{R}^2$  and  $\forall X = f \frac{\partial}{\partial x_1} + g \frac{\partial}{\partial x_2} \in \mathcal{X}(M)$ , endowed by a linear connection defined by

$$\nabla \frac{\partial}{\partial x_1} = f \frac{\partial}{\partial x_1} \text{ and } \nabla \frac{\partial}{\partial x_2} = 0$$

43

44  $(\mathbb{R}^2, \nabla)$  is locally flat manifold.

45

To a locally flat manifold  $(M, \nabla)$  we assigned the algebra  $\mathcal{A}$  whose underlying vector space is the vectors fields, namely  $\mathcal{X}(M)$ , and the product  $X \cdot Y$  is defined by the Koszul connection  $\nabla$  as:

$$X \cdot Y = \nabla_X Y$$

**Definition 6.** An algebra  $\mathcal{B}$  is Koszul-Vinberg algebra (or KV in short) algebra if for any triple  $(a, b, c)$  in  $\mathcal{B}^3$  the following identity is satisfy:

$$a(bc) - (ab)c - b(ac) + (ba)c = 0.$$

46 **Proposition 1.** The algebra  $A = (\mathcal{X}(M), \nabla)$  of a locally flat manifold  $(M, \nabla)$  is a Koszul-Vinberg algebra.

**Proof.** Since the curvature tensor vanishes, we get for any triple  $(X, Y, Z)$  of vector fields that:

$$\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) = \nabla_{[X, Y]} Z;$$

but  $\nabla$  is torsion-less, so the right hand side is:

$$\nabla_{\nabla_X Y} Z - \nabla_{\nabla_Y X} Z.$$

This yields:

$$\nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z) + \nabla_{\nabla_Y X} Z = 0$$

or

$$X(YZ) - (XY)Z - Y(XZ) + (YX)Z = 0$$

47 which proves the claim.  $\square$ 

**Definition 7.** A two-sided module of an algebra  $\mathcal{B}$  is the real vector space  $W$  which is endowed with the two-sided linear mappings

$$\mathcal{B} \times W \ni (a, w) \rightarrow a \cdot w \in W \quad \text{and} \quad W \times \mathcal{B} \ni (w, a) \rightarrow w \cdot a \in W. \quad (1)$$

In what follows, we use the following notations:

$$\begin{aligned} (a, b, w) &= (a \cdot b) \cdot w - a \cdot (b \cdot w), \\ (a, w, b) &= (a \cdot w) \cdot b - a \cdot (w \cdot b), \\ (w, a, b) &= (w \cdot a) \cdot b - w \cdot (a \cdot b). \end{aligned} \quad (2)$$

**Definition 8.** A two-sided module  $W$  is Koszul-Vinberg module of a Koszul-Vinberg algebra  $\mathcal{B}$  if it satisfies the following identities :

$$\begin{aligned} (a, b, w) &= (b, a, w), \\ (a, w, b) &= (w, a, b). \end{aligned} \tag{3}$$

**Definition 9.** Given a two-sided module KV module  $W$  on  $\mathcal{A}$ , the subspace of Jacobi elements of  $W$ , namely  $J(W)$  is defined as follows:

$$J(W) = \{w \in W : (a, b, w) = 0 \forall a, b \in \mathcal{A}\}$$

*Remark 1.* The vector space  $C^\infty(M)$  is a left KV module of  $\mathcal{A}$  whose left action is defined by:

$$X \cdot f = \nabla_X f.$$

### 48 3.2. The Scalar KV valued cohomology

The scalar KV complex is the  $\mathbb{Z}$ -graded vector space  $C^\infty(\mathcal{A}, \mathbb{R}) = \bigoplus_q C^q(\mathcal{A}, C^\infty(M))$  by the following homogeneous subspaces :

$$\begin{aligned} C^q(\mathcal{A}, \mathbb{R}) &= 0 \text{ for } q < 0 \\ C^0(\mathcal{A}, \mathbb{R}) &= \{f \in C^\infty(M) : \nabla^2 f = 0\} \\ C^q(\mathcal{A}, \mathbb{R}) &= \text{Hom}_{\mathbb{R}}(\mathcal{A}^{\otimes q}, C^\infty(M)) \end{aligned}$$

The coboundary operator is defined as it follows :

$$\delta_{KV}^\nabla f = -f$$

For  $p > 0$ , let  $F \in C^p(\mathcal{A}, C^\infty(M))$ :

$$\delta_{KV}^\nabla F(\xi) = \sum_{i \leq q} \nabla_{X_i} F(\partial_i \xi) - F(\nabla_{X_i} \partial_i \xi).$$

The cohomology space of this cochain is denoted by

$$H_{KV}(\mathcal{A}, C^\infty(M)) = \bigoplus_q H_{KV}^q(\mathcal{A}, C^\infty(M)).$$

49 **Definition 10.** A Hessian metric in a locally flat manifold  $(M, \nabla)$  is a non degenerate symmetric  
50 2-cocyclic  $g \in Z_{KV}^2(\mathcal{A}, C^\infty(M))$ . It's cohomology class  $[g] \in H_{KV}^2(\mathcal{A}, C^\infty(M))$  is called a non  
51 degenerate Hessian class.

**Theorem 2.** Let  $g$  be a metric tensor in a locally flat manifold  $(M, \nabla)$ . The following statements are equivalent.

(1)  $(M, g, \nabla)$  is Hessian manifold.

(2)  $\delta^\nabla g = 0$ .

(3) Every point has a neighborhood  $\mathcal{U}$  supporting a local smooth function  $f$  such that

$$g = \nabla^2 f$$

52 .

53 This theorem is a cohomological of proposition 2.1 as in [1]

54 **3.3. The notion of Hyperbolicity of locally flat manifold**

Given a locally flat manifold  $(M, \nabla)$ , we fix  $x_0 \in M$  and we consider the space of differentiable paths

$$\{(0, [0, 1]) \longrightarrow (x_0, M)\}$$

55 and we consider it's quotient modulo the fixed ends homotopy. This quotient set is denoted by  $\tilde{M}$ .

The map

$$\tilde{M} \ni [c] \rightarrow c(1) \in M$$

56 is a universal covering of  $M$ .

The universal covering of  $(M, \nabla)$  is denoted by  $(\tilde{M}, \tilde{\nabla})$ . We consider a path  $c$  and  $\theta \in [0, 1]$ . Let  $\tau_\theta$  the parallel transport:

$$\tau_\theta : T_{x_0}M \rightarrow T_{c(\theta)}M$$

The developing map is defined as it follows:

$$D([c]) = \int_0^1 \tau_\theta^{-1} \frac{dc}{dt}(\theta) d\theta.$$

57 **Definition 11.** A locally flat manifold  $(M, \nabla)$  is called hyperbolic if  $D$  is a diffeomorphism of  $\tilde{M}$  onto  
58 a convex domain not containing any straight line (Koszul [6]).

*Remark 2.* This definition is equivalent to the fact that  $M$  is diffeomorphic to the orbit space

$$M = \frac{C}{D}$$

59 where  $C$  is an open convex domain not containing any straight line in  $\mathbb{R}^n$  and  $D$  is a discrete subgroup  
60 of the Lie group  $Aff(n)$  (see J. L. Koszul [6,7]).

61 **Theorem 3.** (see [6]) For a locally flat manifold  $(M, \nabla)$  being hyperbolic, it is necessary that it exists a de  
62 Rham closed differential 1-form  $\omega$  on  $M$  whose covariant derivate  $\nabla\omega$  is positive definite. If  $M$  is compact then  
63 this condition is also sufficient.

64 Michel Boyom has given a cohomological analogue of the theorem above and his result states as  
65 follows:

66 **Theorem 4.** [8] For a compact Hessian manifold  $(M, g, \nabla)$ , the following assertions are equivalent:

- 67 1.  $[g] = [0] \in H_{KV}^2(\mathcal{X}(M), C^\infty(M))$ ,
- 68 2.  $(M, \nabla)$  is hyperbolic.

69 **4. The Hessian equation on a statistical manifold**

70 **4.1. Differential Operator**

Let  $(V, p, M)$  a vector bundle and let  $k$  be any non negative integer. The vector bundle of  $k$ -jets of sections of vector bundle of  $V$  is denoted  $J^k(V)$ . At a point  $x \in M$  the fiber  $J_x^k(V)$  is quotient vector space:

$$J_x^k(V) = \frac{\Gamma(V)}{I_x^{k+1}(M)\Gamma(V)}$$

71 where  $I_x(M) \subset C^\infty(M)$  is the ideal of differentiable functions that vanish at  $x \in M$ .

Let  $E, F$  two vectors bundles over  $M$  and  $\Gamma(E)$  and  $\Gamma(F)$  the spaces of their sections. For all  $k$  an nonnegative integer we denote  $j^k$  the map :

$$s \longrightarrow j^k(s)$$

72 from  $\Gamma(E)$  to  $\Gamma(J^k(E))$

**Definition 12.** The map

$$D : \Gamma(E) \longrightarrow \Gamma(F)$$

is  $k^{\text{th}}$  order differential operator if there exist an homomorphism of vector bundle

$$\mathcal{D} : J^k(E) \longrightarrow \Gamma(F)$$

such that

$$D(s) = \mathcal{D}(j^k s)$$

73  $\forall s \in \Gamma(E)$

74  $\mathcal{D}$  is called the principal symbol of  $D$ .

75

Let consider the short exact sequence

$$0 \rightarrow \check{J}^k(E) \xrightarrow{i} J^k(E) \xrightarrow{\pi^k} J^{k-1}(E) \rightarrow 0$$

**Definition 13.** The map

$$\sigma(D) = \mathcal{D} \circ i$$

76 from  $\check{J}^k(E)$  to  $\Gamma(F)$  is called the geometrical symbol of differential operator  $D$

77 Let  $V$  and  $W$  be an finite vector spaces ,let  $\rho$  be a subspace of the vector space  $Hom(V, W)$ .

**Definition 14.** The first prolongation of  $\rho$  is subspace  $\rho^1$  of  $Hom(V, \rho)$  which is  $\rho^1 = \rho \otimes V^* \cap W \otimes S^2(V^*)$ .By recurrence we define on  $p \in \mathbb{N}$ ,we defined the  $p$ th prolongation:

$$\rho^0 = \rho$$

$$\rho^p = V^* \otimes \rho^{p-1} \cap S^{p+1}V^* \otimes W$$

**Definition 15.** We consider the Koszul-Spencer complex of  $\rho$ :

$$\rightarrow Hom(\wedge^p V, \rho^q) \rightarrow Hom(\wedge^{p+1} V, \rho^{q-1}) \rightarrow$$

The differential operator  $\partial$  is defined by

$$\partial f(v_1 \wedge \dots \wedge v_{p+1} \otimes t_1 \dots t_q) = \sum_j (-1)^j f(v_1 \wedge \dots \wedge v_{p+1} \otimes v_j \cdot t_1 \dots t_q)$$

78 **Definition 16.**  $\rho$  is called involutive if  $H^{p,q}(\rho) = 0 \forall q, p$

79 4.2. Hessian equation of  $\nabla$

Given a gauge structure  $(M, \nabla)$ ,we define the following second order differential operator

$$X \in \mathcal{X}(M) \rightarrow \nabla^2 X \in T_1^2(M)$$

Here  $T_1^2(M)$  is the vector space of (2,1)-tensors in  $M$ .

We recall the definition of  $\nabla^2 X$ ,

$$(\nabla^2 X)(Y, Z) = \nabla_Y(\nabla_Z X) - \nabla_{\nabla_Y Z} X$$

**Definition 17.** The Hessian equation of  $(M, \nabla)$  is the following second differential equation

$$\nabla^2 X = 0$$

Let  $\mathcal{J}_\nabla(M)$  the sheaf of solutions of Hessian equation  $\nabla^2 X = 0$  and  $J_\nabla(M)$  be the vector space of global section of the sheaf  $\mathcal{J}_\nabla(M)$ , it is well in [8] that  $\mathcal{J}_\nabla$  is Associative Algebra sheaf(AAS).

Let one evaluate the principal principal symbol of  $X \rightarrow \nabla^2 X$ . the principal symbol of the operator  $\nabla^2$  is defined by

$$(\nabla^2(X))\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_k \Omega_{ij}^k \frac{\partial}{\partial x_k}$$

where

$$\Omega_{ij}^l = \frac{\partial^2 X^l}{\partial x_i \partial x_j} + \sum_k \left[ \Gamma_{ik}^l \frac{\partial X^k}{\partial x_j} + \Gamma_{jk}^l \frac{\partial X^k}{\partial x_i} - \Gamma_{ij}^k \frac{\partial X^l}{\partial x_k} \right] + \sum_k \left[ \frac{\partial \Gamma_{jk}^l}{\partial x_i} + \sum_m \left( \Gamma_{jk}^m \Gamma_{im}^k - \Gamma_{ij}^m \Gamma_{mk}^l \right) \right]$$

The geometrical symbol of  $\nabla^2$  denoted by  $\sigma(\nabla^2)$  is

$$\left[ \frac{\partial^2 X^k}{\partial x_i \partial x_j} \right]_{i,j} = [a_{i,j}^k]$$

80  $\sigma(\nabla^2)$  is equal to  $S^2 \mathbb{R}^{*n} \otimes \mathbb{R}^n$ , the first prolongation of  $\sigma(\nabla^2)$  denoted  $\sigma(\nabla^2)^1 = S^3 \mathbb{R}^{*n} \otimes \mathbb{R}^n$ . It is easy  
81 to see that  $\sigma(\nabla^2)$  admit a quasi regular basis then it is involutive, this is equivalent to  $H^{p,q}(\sigma(\nabla^2)) =$   
82  $0, \forall p, \forall q > 1$  (see Jean Pierre Serre in appendix of [9]). We conclude that Hessian equation is involutive  
83 and finite type in sense of Elie Cartan [9], so we conclude that  $(J_\nabla, \nabla)$  finite associative algebra and  
84  $(J_\nabla, [\cdot, \cdot]_\nabla)$  is finite Lie algebra.

85 **Proposition 2.** Let  $(M, g)$  be a Riemman manifold and let Ric be its Ricci curvature. We have

$$J_\nabla \subset \text{Ker Ric} \tag{4}$$

86 **Proof.** Let  $\xi \in J_\nabla$ , by using the Bianchi identity we have:

$$\begin{aligned} R^\nabla(\xi, X)Y &= R^\nabla(\xi, Y)X \quad \forall X, Y \in \chi(M) \\ g(R^\nabla(\xi, X)Y, Z) &= g(R^\nabla(\xi, Y)X, Z) \\ &= -g(X, R^\nabla(\xi, Y)Z) \\ &= g(X, R^\nabla(\xi, Z)Y) \\ &= -g(R^\nabla(\xi, Z)X, Y) \\ &= g(R^\nabla(\xi, X)Z, Y) \\ &= -g(R^\nabla(\xi, X)Y, Z) \end{aligned}$$

Then

$$J_\nabla \subset \text{Ker Ric}^\nabla$$

87  $\square$

**Corollary 1.** If  $\nabla$  is the Levi-Civita connection of an Einstein Riemannian manifold, then

$$J_\nabla = \{0\}$$

**Definition 18.** A gauge structure  $(M, \nabla)$  is called special if

$$J_\nabla \neq \{0\}$$

88 Example 7: Every locally flat manifold  $(M, \nabla)$  is special gauge structure.

89



90 Example 9: For any positive integer  $m$ ,  $B^m = \{X \in \mathbb{R}^m / \|X\| < 1\}$  equipped with the Euclidean  
 91 Levi-Civita connection is special gauge structure.  
 92 In the rest of this work we assume that our gauge structure are special.

93  
 94 Let  $(M, g, \nabla, \nabla^*)$  be a statistical manifold, consider the two Hessian operators  $\nabla^2$  and  $\nabla^{*2}$  and  
 95 we denote by  $(J_\nabla(M), [, ]_\nabla)$  and  $(J_{\nabla^*}(M), [, ]_{\nabla^*})$  the finite Lie algebra of global sections of sheaf of  
 96 solutions of  $\nabla^2 X = 0$  and  $\nabla^{*2} X = 0$ .  
 97 Let  $G_\nabla$  (resp  $G_{\nabla^*}$ ) a simply connected Lie group whose Lie algebra is denoted by  $J_\nabla$  (resp  $J_{\nabla^*}$ )  
 98 according to the third theorem of Lie.

**Proposition 3.** *Let  $(M, g, \nabla, \nabla^*)$  a statistical manifold and let  $\{J_\nabla, J_{\nabla^*}\}$  a pair of Lie algebras and  $\{G_\nabla, G_{\nabla^*}\}$  a pair of is Lie groups .We have the following properties:*

1)  $\{(J_\nabla, \nabla); (J_{\nabla^*}, \nabla^*)\}$  is a pair of KV-algebras,

2)  $\{(G_\nabla, \nabla); (G_{\nabla^*}, \nabla^*)\}$  pair of simply connected bi-invariant locally flat Lie groups.

99 **Definition 19.** An infinitesimal dynamical of  $G_\nabla$  is a manifold  $M$  is a lie algebra homomorphism from  
 100 his Lie algebra  $J_\nabla$  to the Lie algebra of vectors fields of  $M$ .

101

102 **Definition 20.** A locally effective action of lie group  $G_\nabla$  on a differentiable manifold  $M$  is a group  
 103 homomorphism from  $G_\nabla$  to the diffeomorphism group of  $M$  such that his kernel is discrete.

104 **Proposition 4.**  $J_\nabla$  is integrable if is induced by a locally effective action of the Lie group  $G_\nabla$ .

105 For more details see Richard S.Palais [10]

106 **Remark 3.** 1. If Any element of  $J_\nabla$  is complete then  $J_\nabla$  is integrable (If  $M$  is compact then  $J_\nabla$  is  
 107 integrable).

108 2. If  $(M, \nabla)$  a geodesically complete locally flat manifold then  $J_\nabla$  is integrable .

109 **Definition 21.** A locally flat foliation in a gauge structure  $(M, \nabla)$  is a  $\nabla$ -auto-parallel foliation  $\mathcal{F}$   
 110 whose leaves are locally flat submanifolds on  $(M, \nabla)$ .

111 **Definition 22.** A Hessian foliation  $\mathcal{F}$  in  $(M, g, \nabla)$  is a locally flat foliation in gauge structure  $(M, \nabla)$   
 112 such that the restriction of the Riemaniann metric on  $\mathcal{F}$  is Hessiann metric.

113 Let  $\mathcal{F}_\nabla$  (resp  $\mathcal{F}_{\nabla^*}$ ) denoted the foliation defined by the  $G_\nabla$ -orbits (resp  $G_{\nabla^*}$ ).

114 **Theorem 5.** *Let  $(M, g, \nabla, \nabla^*)$  a compact statistical manifold. We have*

115 (1) *The foliation  $\mathcal{F}_\nabla$  is Hessian foliation in  $(M, g, \nabla)$ ,*

116 (2) *The foliation  $\mathcal{F}_{\nabla^*}$  is Hessian foliation in  $(M, g, \nabla^*)$  .*

117 **Proof.** (1) By assumption  $M$  is compact then  $J_\nabla$  is integrable, we conclude that  $J_\nabla$  is infinitesimal action  
 118 of  $G_\nabla$  on  $M$ . The  $G_\nabla$ -orbits are  $\nabla$ -auto-parallel and satisfy  $R^\nabla(X, Y)J_\nabla = 0$  then  $\mathcal{F}_\nabla$  is locally flat  
 119 foliation in  $(M, \nabla)$ . The manifold  $(M, g, \nabla)$  is statistical manifold and the  $G_\nabla$ -orbits are  $\nabla$ -auto-parallel  
 120 then the leaves of  $\mathcal{F}_\nabla$  are Hessian submanifolds of  $(M, g, \nabla)$ .

121 (2) By using the same arguments for  $G_{\nabla^*}(M)$ , the theorem is proved.  $\square$

## 122 5. Hyperbolicity Of Compacts Orbits

### 123 5.1. Basics Notions

**Definition 23.** Let  $G$  be a Lie group . A pair  $(f, q)$  of a homomorphism  $f$  from  $G$  to  $GL(V)$  and a map  
 $q$  from  $G$  to  $V$  is said to be affine representation of  $G$  in  $V$  if it satisfies the following identity:

$$q(s_1 s_2) = f(s_1)q(s_2) + q(s_1)$$

$$124 \quad \forall s_1, s_2 \in G$$

**Definition 24.** A affine representation of Lie algebra from  $\mathcal{G}$  to  $V$  is homomorphism from the Lie algebra  $\mathcal{G}$  to  $aff(V)$ . The affine representation of  $\mathcal{G}$  to  $V$  is  $(f, q)$  such that  $f$  is linear map from  $\mathcal{G}$  to  $End(V)$  and  $q$  is linear map from  $\mathcal{G}$  to  $V$  satisfy the following identities:

$$q([X, Y]) = f(X)q(Y) - f(Y)q(X), \forall X, Y \in \mathcal{G}$$

125 5.2. The Canonical affine representation of affine Lie groups and is Lie algebra

Let  $G_\nabla = (G, \nabla)$  be a connected locally flat lie group whose (Left invariant) KV algebra is denote by  $J_\nabla$ . In fact, the KV algebra of  $G_\nabla$  is the real algebra

$$J_\nabla = (J, \nabla)$$

where  $J$  is Lie algebra of the lie group  $G$  and  $1_{J_\nabla}$  is the identity endomorphism of the vector space  $J_\nabla$ . The affine representation of Lie algebra  $J_\nabla$  in the vector space  $J_\nabla$  by

$$J_\nabla \ni X \longrightarrow (\nabla, 1_{J_\nabla})(X) = (\nabla_X, X) \in aff(J_\nabla)$$

The affine map  $(\nabla_X, X)$  is defined by

$$(\nabla_X, X)(Y) = \nabla_X Y + X, \forall Y \in J_\nabla$$

*Remark 4.*  $(\nabla, 1_{J_\nabla})$  is the differential at the unit element of a unique affine representation  $(f_\nabla, q_\nabla)$  of  $G_\nabla$  to  $J_\nabla$ , namely

$$\begin{aligned} (f_\nabla, q_\nabla) : G_\nabla &\longrightarrow aff(J_\nabla) \\ \gamma &\longrightarrow (f_\nabla(\gamma), q_\nabla(\gamma)) \end{aligned}$$

Where

$$(f_\nabla(\gamma), q_\nabla(\gamma))$$

is defined by :

$$(f_\nabla(\gamma), q_\nabla(\gamma))(X) = f_\nabla(\gamma)X + q_\nabla(\gamma)$$

and satisfy these identities :

$$\frac{d}{dt}(f(\exp tX), q(\exp tX))|_{t=0} = (\nabla_X, X)$$

and

$$q_\nabla(\gamma_1 \cdot \gamma_2) = f_\nabla(\gamma_1)q_\nabla(\gamma_2) + q_\nabla(\gamma_1) \forall \gamma_1, \gamma_2 \in G_\nabla$$

*Remark 5.* Let

$$f_\nabla : G_\nabla \longrightarrow GL(J_\nabla)$$

$$q_\nabla : G_\nabla \longrightarrow J_\nabla$$

Remind the cohomology of the Lie group  $G_\nabla$  value in his Lie algebra  $J_\nabla$  define by :

$$\dots \rightarrow C^q(G_\nabla, J_\nabla) \rightarrow C^{q+1}(G_\nabla, J_\nabla) \rightarrow C^{q+2}(G_\nabla, J_\nabla) \rightarrow$$

with differential operator  $d$  defined by :

$$d\theta(\gamma_1, \dots, \gamma_{q+1}) = f_\nabla(\gamma_1) \cdot \theta(\gamma_2, \dots, \gamma_{q+1}) + \sum_{i \leq q} (-1)^i \theta(\dots, \gamma_i \gamma_{i+1}, \dots) + (-1)^q \theta(\gamma_1, \dots, \gamma_q)$$

See Koszul[11], Cartan and Eilenberg [12] and Chevalley and Eilenberg [13], M. Boyom[14] for more details on Lie algebra cohomology.

It's easy to see that the condition :

$$q_{\nabla}(\gamma_1 \cdot \gamma_2) = f_{\nabla}(\gamma_1)q_{\nabla}(\gamma_2) + q_{\nabla}(\gamma_1)$$

$$\forall \gamma_1, \gamma_2 \in G_{\nabla}$$

126 is equivalent to the fact that  $q_{\nabla} \in Z^1(G_{\nabla}, J_{\nabla})$ , we denote  $[q_{\nabla}] \in H^1(G_{\nabla}, J_{\nabla})$  is cohomology class

### 127 5.3. Vanishing Theorems and Hyperbolicity

128 Let  $(M, g, \nabla, \nabla^*)$  be a compact statistical manifold, We go to deal conditions under which the  
129 Hessian foliations  $\mathcal{F}_{\nabla}$  and  $\mathcal{F}_{\nabla^*}$  are hyperbolic.

**Theorem 6.** *Let  $(M, g, \nabla, \nabla^*)$  a compact statistical manifold, the following assertions are equivalents on the leaves of  $\mathcal{F}_{\nabla}$  (resp on those of  $\mathcal{F}_{\nabla^*}$ ):*

- 1)  $[g]_{\nabla} = 0 \in H_{KV}^2(J_{\nabla}, \mathbb{R})$
- 2)  $[q_{\nabla^*}] = 0 \in H^1(G_{\nabla^*}, J_{\nabla^*})$
- 3)  $(f_{\nabla^*}, q_{\nabla^*})$  : admit a fixed point
- 4)  $(f_{\nabla^*}, q_{\nabla^*}) \simeq f_{\nabla^*}$

**Proof.** (2)  $\rightarrow$  (1): If  $[q_{\nabla^*}] = 0$  then  $\exists \xi \in J_{\nabla^*}$  such that the following identities hold

$$q_{\nabla^*}(\gamma_t) = f_{\nabla^*}(\gamma_t)\xi - \xi, \forall \gamma_t \in G_{\nabla^*}$$

by derivate this terms we obtain the following identity:

$$\frac{d}{dt}(q_{\nabla^*}(\gamma_t))|_{t=0} = \frac{d}{dt}(f_{\nabla^*}(\gamma_t)\xi - \xi)|_{t=0}, \forall \gamma_t \in G_{\nabla^*}$$

$$X = \nabla_X^* \xi, \forall X = \frac{d(\gamma_t)}{dt}|_{t=0}$$

by using the conjugaison formula, we deduce the following identity:

$$g(X, Y) = X.g(\xi, Y) - g(\xi, \nabla_X Y)$$

$$g(X, Y) = (d_{KV}^{\nabla}(i_{\xi}g))(X, Y)$$

$$[g]_{\nabla} = 0$$

by the same arguments we prove that  $[q_{\nabla^*}] = 0$  then  $[g]_{\nabla} = 0$

(1)  $\rightarrow$  (2):

Assume that  $[g]_{\nabla} = 0$  then there exist a one form  $\Theta$  define by  $\Theta(X) = g(\xi, X)$  such that

$$g(X, Y) = -X\Theta(Y) + \Theta(\nabla_X Y)$$

by using the conjugaison we obtain the following identity

$$g(\nabla_X^* \xi, Y) = X.g(\xi, Y) - g(\xi, \nabla_X Y)$$

we deduce the following relation:

$$\nabla_X^*(-\xi) = X$$

$\forall \gamma_t \in G_{\nabla^*}$  such that  $q_{\nabla^*}(\gamma_t) = f_{\nabla^*}(\gamma_t)\xi - \xi$ , then we conclude that

$$[q_{\nabla^*}] = 0 \in H^1(G_{\nabla^*}, J_{\nabla^*})$$

$$2) \iff 3) \iff 4)$$

this well known in [15][16]

Hint (2)  $\rightarrow$  (3) Assume that (2)  $q_{\nabla^*} \in B^2(G_{\nabla^*}, J_{\nabla^*})$  then  $\exists \xi \in J_{\nabla^*}$  such that:

$$f_{\nabla^*}(\gamma)(-\xi) + q_{\nabla^*}(\gamma) = -\xi$$

130 then  $(f_{\nabla^*}, q_{\nabla^*})$  admit an fixed point.

131 Now we assume (3)  $\rightarrow$  (4)

$$\begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_{\nabla^*} & q_{\nabla^*} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\xi \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} f_{\nabla^*} & 0 \\ 0 & 1 \end{bmatrix} \quad (5)$$

(4)  $\rightarrow$  (2) Assume that  $\langle f_{\nabla^*}, q_{\nabla^*} \rangle$  is linearisable then  $\exists \xi \in J_{\nabla^*}$  such that

$$q_{\nabla^*}(\gamma) = f_{\nabla^*}(\gamma)\xi - \xi, \forall \gamma \in G_{\nabla^*}$$

$$q_{\nabla^*}(\gamma) = d(\xi)(\gamma)$$

132 we conclude that  $[q_{\nabla^*}] = [0]$ .  $\square$

133 Let denoted by  $F$  the compact leaf of foliation  $\mathcal{F}_{\nabla}$  and  $F^*$  those of  $\mathcal{F}_{\nabla^*}$ .

134

135 **Corollary 2.** Let  $(M, g, \nabla, \nabla^*)$  be a compact statistical manifold. Under the hypotheses of Theorem 6, we have :

136 (1) The compact leaves  $(F, \nabla)$  are hyperbolic manifolds,

137 (2) The compact leaves  $(F^*, \nabla^*)$  are hyperbolic manifolds.

138

### 139 Comments

140 In [8] Boyom proves that the compact orbits of  $G_{\nabla^{LC}}$  in case of trivial statistical manifold

141  $(M, g, \nabla^{LC})$  are flat cylinder over Riemannian torus.  $(Orb(G_{\nabla^{LC}})(p), g) \simeq \left( \frac{T^{b_1(Orb(G_{\nabla^{LC}}))}}{K} \times \right.$

142  $\mathbb{R}^{r(\nabla^{LC})(p) - b_1(Orb(G_{\nabla^{LC}}))}, g_0)$ . Here  $r(\nabla^{LC})(p)$  is the dimension of the orbits on  $p$ ,  $g_0$  is Euclidean

143 (flat) metric of  $\mathbb{R}^{r(\nabla^{LC})(p)}$ ,  $b_1(Orb(G_{\nabla^{LC}}))$  is the first Betti number of  $Orb(G_{\nabla^{LC}})$ ,  $K$  is finite subgroup of

144 isometries of  $g_0$  ie  $(Iso(g_0) \simeq O(r(\nabla^0)p) \times \mathbb{R}^{r(\nabla^{LC})(p)})$  [17].

145 We prove the similar result with hyperbolic orbits of  $G_{\nabla}$   $(Orb(G_{\nabla})(p), \nabla) \simeq \left( \frac{U}{\Gamma}, \nabla^F \right)$ . Here

146  $U$  is open convex domain not containing any straight line in  $\mathbb{R}^{r(\nabla)(p)}$ ,  $\Gamma$  is discrete subgroup of

147  $Aff(\mathbb{R}^{r(\nabla)(p)})$  ie  $(GL(\mathbb{R}^{r(\nabla)(p)}) \times \mathbb{R}^{r(\nabla)(p)})$ ,  $\nabla^F$  the canonical flat connection on  $\mathbb{R}^{r(\nabla)(p)}$ .

148

149 Now we discuss about some conditions under which the universal covering of compact

150 hyperbolic orbits are convex cones. Let remind an important theorem of J.L.Koszul.

151 **Theorem 7.** [18] If the universal covering of a compact hyperbolic locally flat manifold is affinely homogeneous,  
152 then its universal covering is isomorphic to a convex cone.

153 **Theorem 8.** Let  $(M, g, \nabla, \nabla^*)$  a compact statistical manifold which satisfy one of the equivalent conditions of  
 154 theorem 6, we have the following assertions.

- 155 1) The compact leaves of  $\mathcal{F}_\nabla$  are isomorphic to the quotient of convex homogeneous cones by discrete subgroups  
 156 of  $Aff(\mathbb{R}^{r(\nabla)^p})$ ,  
 157 2) The compact leaves of  $\mathcal{F}_{\nabla^*}$  are isomorphic to quotient of convex homogeneous cones by discrete subgroups of  
 158  $Aff(\mathbb{R}^{r(\nabla^*)^q})$ .

**Proof.** By using the following formula on the  $G_\nabla$ -Orbit :

$$\mathcal{L}_\xi \nabla = i_\xi R^\nabla + \nabla^2 \xi$$

159 we conclude that  $G_\nabla = Aff(Orbit(G_\nabla), \nabla)$ . The Lie Group  $Aff(Orbit(G_\nabla), \nabla)$  is isomorphic to the  
 160 normalizer  $N(\pi_1(Orbit(G_\nabla)))$  of  $\pi_1(Orbit(G_\nabla))$  in  $Aff(Orbit(G_\nabla), \tilde{\nabla})$ , then we conclude that  $G_\nabla$  acts  
 161 affinely on  $(Orbit(G_\nabla), \tilde{\nabla})$ . The action of  $G_\nabla$  on  $Orbit(G_\nabla)$  is affinely transitive, then the orbits of  
 162 this action at  $\tilde{x}$  denote by  $G_\nabla(\tilde{x})$  is isomorphic to the  $Orbit(G_\nabla)$ , then  $\dim(G_\nabla(\tilde{x})) = \dim(Orbit(G_\nabla))$ . We  
 163 conclude that  $G_\nabla(\tilde{x})$  is an open subset of  $Orbit(G_\nabla)$ , but  $Orbit(G_\nabla)$  is connected then  $G_\nabla$  acts transitively  
 164 on  $Orbit(G_\nabla)$ . By using Theorem 7, we conclude that the compact orbits  $Orbit(G_\nabla)$  are isomorphic  
 165 to the Quotient of Convex cone by the discrete subgroup  $\Gamma$  of  $Aff(\mathbb{R}^{\dim(Orbit(G_\nabla))})$ .

166 We can do the same arguments for  $G_{\nabla^*}$ .  $\square$

167 **Remark 6.** Under Barbaresco formalisms the universal coverings of the compact leaves of our  
 168 hyperbolic foliations are statistical models.

169 **Corollary 3.** If  $(M, g, \nabla, \nabla^*)$  is a statistical model  $(\Theta, P)$  then it's foliated by leaves which are quotient of  
 170 exponential models.

171 **Example:**

We shall give an example of compact Hessian manifold whose orbits are quotient of statistical  
 models by discrete subgroups. Let  $\mathbb{R}^2$  be a 2-dimensional real affine space with the natural flat affine  
 connection  $D$  and let  $\{x, y\}$  be an affine coordinate system of  $\mathbb{R}^2$ .

Let  $\Omega$  be a domain defined by  $x > 0$  and  $y > 0$ . Consider a Riemannian metric on  $\Omega$  given by  
 $g = \frac{1}{x^2} dx^2 + \frac{1}{y^2} dy^2$ .

Then  $(D, g)$  is a Hessian structure on  $\Omega$ .

Let  $\Sigma$  and  $\chi$  be linear transformations on  $\Omega$  defined by :

$$\Sigma : (x, y) \rightarrow (2x, y)$$

$$\chi : (x, y) \rightarrow (2x, 3y)$$

Then  $\langle \Sigma, \chi \rangle$  leave the Hessian structure  $(D, g)$  invariant.

We denote  $\Gamma$  the group generated by  $\{\Sigma, \chi\}$ , we can also write  $\Gamma$  as  $\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \rangle$ .

$\Gamma$  acts properly discontinuously on  $\Omega$  and  $\frac{\Omega}{\Gamma}$  is a compact Hessian manifold which is diffeomorphic to  
 a Torus.

Let us denote by  $\pi$  the projection from  $\Omega$  to  $\frac{\Omega}{\Gamma}$  and by  $(D, g)$  the Hessian structure on  $\frac{\Omega}{\Gamma}$ . Since the space  
 of all  $\Gamma$ -invariant  $D$ -parallel 1-forms on  $\Omega$  is spanned by  $dx$  and  $dy$ . The space of all  $D$ -parallel 1-forms on  
 $\frac{\Omega}{\Gamma}$  is spanned by  $\omega$  and  $\Phi$  where  $dx = \pi^* \omega$  and  $dy = \pi^* \Phi$ . Let  $X$  and  $\tilde{X}$  be vector fields on  $\frac{\Omega}{\Gamma}$  defined  
 by  $\omega(Y) = g(X, Y)$  and  $\Phi(Y) = g(\tilde{X}, Y)$  for each vector field  $Y$  on  $\frac{\Omega}{\Gamma}$ . Then  $X = \pi_* \left( \frac{\partial}{\partial x} \right)$  and  $\tilde{X} = \pi_* \left( \frac{\partial}{\partial y} \right)$   
 and the vector space  $J_\nabla$  of all Hessian vector fields on  $\frac{\Omega}{\Gamma}$  is spanned by  $\langle X, \tilde{X} \rangle$ . Since  $T^2$  is compact  
 the  $J_\nabla$  is integrable. Let  $Exp(tX)$  and  $Exp(tY)$  be a 1-parameter group of transformations generated by  
 $X$  and  $Y$ . Let  $G_\nabla = \langle Exp(tX), Exp(tY) \rangle$ . Consider  $a > 0$  and  $b > 0$ , the compact homogeneous  
 orbit  $G_\nabla \pi(a, b) = \{ \gamma \cdot \pi(a, b), \gamma \in G_\nabla \}$  is a circle. Then we conclude that  $(S^1, D, g_{S^1})$  is a Hessian  
 manifold and satisfy  $D \frac{\partial}{\partial \theta} = 0$  and  $g_{S^1} = \frac{1}{\theta^2} d\theta^{\otimes 2}$ . By using conjugation we have  $D \frac{\partial}{\partial \theta} = -\frac{2}{\theta} \frac{\partial}{\partial \theta}$ . Let

$H^* = h(\theta) \frac{\partial}{\partial \theta}$  vectors fields on  $S^1$ .  $H^*$  is homothety vector fields of  $D^*$  if  $h$  is solution of differential equation  $y' - \frac{2}{\theta^2} y = 1$ . We deduce that

$$h(\theta) = Ae^{-\frac{2}{\theta}} + 2e^{-\frac{2}{\theta}} \int_{-\frac{2}{\theta}}^{+\infty} \frac{e^{-s}}{s} ds + \theta$$

where  $A$  is constant.

As  $D^*H^* = Id$  then  $(S^1, D)$  is hyperbolic. Now let  $p : \mathbb{R} \rightarrow S^1$  the covering map, and consider the map

$$(\mathbb{R}, \tilde{D}) \rightarrow (S^1, D)$$

172 .The compact orbits is  $G_{\nabla} \pi(a, b)$  is quotient  $\frac{\mathbb{R}^{>0}}{\Gamma}$ , here  $\Gamma$  is discrete subgroup of  $Gl(\mathbb{R}) \times \mathbb{R}$

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