

# Lifting dual connections to the cotangent bundle

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**Abstract:** Let  $(M, g)$  be a Riemannian manifold equipped with a pair of dual connections  $(\nabla, \nabla^*)$ . Such a structure is known as a statistical manifold since it was defined in the context of information geometry. This paper aims at defining the complete lift of such a structure to the cotangent bundle  $T^*M$  using the Riemannian extension of the Levi-Civita connection of  $M$ . In the first section, common tensors associated with pairs of dual connections, emphasizing the cyclic symmetry property of the so-called skewness tensor. In a second section, the complete lift of this tensor is obtained, allowing the definition of dual connections on  $TT^*M$  with respect to the Riemannian extension.

## 1. Introduction

Information geometry was originally dealing with parameter spaces of families of probability densities viewed as differentiable manifolds [1,2]. More specifically, let  $E$  be a measure space and let  $S = \{p_\theta, \theta \in M\}$  be a parameterized family of densities on  $E$  satisfying:

1.  $M$  is a topological manifold (in most of the case it is simply an open subset of  $\mathbb{R}^n$ ).
2. The topology of  $S$  induced by the  $L^1$  norm is compatible with the topology of  $M$ .
3. It exists a probability measure  $\mu$  on  $E$  such that for any  $\theta \in M$ ,  $p_\theta < \mu$ .
4.  $\theta \mapsto (x \in E \rightarrow p_\theta(x))$  is smooth uniformly in  $x$ .
5.  $\partial_\theta E_\mu[\log p(x, \theta)] = E_\mu[\partial_\theta \log p(x, \theta)]$ .
6. The moments up to order 3 of  $x \mapsto \partial_\theta \log p(x, \theta)$  exist and are smooth.
7. The matrix  $F$  with entries  $F_{ij}(\theta) = E_{p_\theta}[\partial_{\theta_i} \log p(x, \theta) \partial_{\theta_j} \log p(x, \theta)]$  is positive definite.

The last assumption allows to endow  $M$  with the structure of a Riemannian manifold with metric:

$$g_\theta(\partial_{\theta_i}, \partial_{\theta_j}) = F_{ij}(\theta) \quad (1)$$

Parameterized families of the so-called exponential type, whose densities can be written as:

$$p(x; \theta) = \exp(-\langle \theta, T(x) \rangle - \psi(\theta) + h(x))$$

play a special role in statistics and have a well behaved Riemannian structure. When  $T(x) = x$ , the family is said to be natural and is defined entirely by  $\psi$ . In such a case, the Fisher information matrix takes the form:

$$F_{ij}(\theta) = -E_{p_\theta} \left[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \right]$$

so that the Riemannian metric is Hessian. The structure of such manifolds has been thoroughly studied in [3]. Finally, from considerations arising in statistical estimation, a pair of dual connections  $\nabla, \nabla^*$  with respect to the Fisher metric can be constructed [4]. They possess vanishing torsion and are related by the skewness tensor:

$$g(\nabla_X Y, Z) - g(\nabla_X^* Y, Z) = T(X, Y, Z)$$

with:

$$T_{ijk} = E_{p_\theta} \left[ \partial_{\theta_i} \log p(x, \theta) \partial_{\theta_j} \log p(x, \theta) \partial_{\theta_k} \log p(x, \theta) \right]$$

As a generalization, a smooth Riemannian manifold  $(M, g)$  equipped with a pair  $(\nabla, \nabla^*)$  of torsionless dual connections is called a statistical manifold. It can be defined equivalently by  $(M, g, T)$  where  $T$  is a fully symmetric  $(0, 3)$ -tensor. It turns out [5] that any statistical manifold can be embedded as a statistical model, i.e. one related to a parameterized family of densities.

For a Riemannian manifold  $(M, g)$ , lifting geometric objects to the tangent bundle  $TM$  (resp. cotangent bundle  $T^*M$ ) is a classical problem [6–8] that relies most of the time on the Whitney sum  $TTM = HTM \oplus VTM$  (resp.  $TT^*M = HT^*M \oplus VT^*M$ ) with  $VTM$  the vertical bundle obtained from the kernel of the canonical projection  $d\pi: TTM \rightarrow TM$  (resp.  $d\pi: TT^*M \rightarrow T^*M$ ) and  $HTM$  the horizontal subspace arising from a fixed affine connection  $\nabla$ . In the tangent bundle, [8] introduces a lift based on horizontal and vertical lifts of vector fields and relies on a quasi-complex structure on  $TM$ . For  $T^*M$ , the preferred method involves complete lifts [9] and Riemann extensions [10], which are pseudo-Riemannian metrics of neutral signature defined on the cotangent bundle and associated in a canonical way to affine connections with vanishing torsion. The complete lift of the connection is defined to be the Levi-Civita one with respect to its Riemann extension. Complete and vertical lifts of different kind of tensors are also presented in [6]. Finally, horizontal lifts of connections are presented in [7].

In this paper, the complete lift of dual connections is defined and yields a pair of dual connections which have vanishing torsion if the original connections have. The strategy adopted is to lift the skewness tensor, here defined in a more general setting as a  $(0, 3)$ -tensor with cyclic symmetry. The procedure described in [6] is adapted to this case, effectively allowing to get a skewness tensor on  $TT^*M$ .

## 2. Statistical structures

In information geometry, dual connections are the basic objects defining the so-called statistical manifold structure [4]. In the sequel,  $M$  is a smooth  $n$ -dimensional manifold endowed with a Riemann metric  $g$ .

**Definition 1.** Let  $\nabla, \nabla^*$  be affine connections on  $TM$ . They are said to be dual if for any triple  $X, Y, Z$  of vector fields:

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z^* Y) \quad (2)$$

The torsion of a connection  $\nabla$  is the tensor  $T$  defined as:  $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ . The next well known proposition relates the torsion tensors of dual connections.

**Proposition 1.** Let  $\nabla, \nabla^*$  be dual connections. Let  $T$  (resp.  $T^*$ ) be the torsion tensor of  $\nabla$  (resp.  $\nabla^*$ ). Then,  $T = T^*$ .

**Proof.** For any triple  $(X, Y, Z)$  of vector fields:

$$\begin{aligned} g(T^*(X, Y), Z) &= g(\nabla_X^* Y, Z) - g(\nabla_Y^* X, Z) - g([X, Y], Z) \\ &= Xg(Y, Z) - g(\nabla_X Z, Y) - Yg(X, Z) + g(\nabla_Y Z, X) - g([X, Y], Z) \\ &= g(Z, \nabla_X Y) - g(Z, \nabla_Y X) - g([X, Y], Z) \\ &= g(T(X, Y), Z) \end{aligned}$$

□

As a particular, but important case, if the torsion of  $T$  vanishes, so does the torsion of  $T^*$ .

**Proposition 2.** Let  $\nabla, \nabla^*$  be dual connections. Then  $\nabla g = -\nabla^* g$

**Proof.** For any triple  $(X, Y, Z)$  of vector fields:

$$(\nabla_Z g)(X, Y) = Z(g(X, Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

and

$$(\nabla_Z^* g)(X, Y) = Z(g(X, Y)) - g(\nabla_Z^* X, Y) - g(X, \nabla_Z^* Y)$$

Using the relations:

$$Z(g(X, Y)) - g(\nabla_Z^* X, Y) = g(X, \nabla_Z Y)$$

and:

$$Z(g(X, Y)) - g(X, \nabla_Z^* Y) = g(\nabla_Z X, Y)$$

52 the claim follows.  $\square$

**Definition 2.** Let  $\nabla^1, \nabla^2$  be affine connections on  $TM$ . Their mutual torsion is the tensor:

$$D_{\nabla^1, \nabla^2}(X, Y) = \nabla_X^1 Y - \nabla_Y^2 X - [X, Y]$$

53 *Remark 1.* The divergence tensor is defined for dual connections  $\nabla, \nabla^*$  as  $D(X, Y) = \nabla_X Y - \nabla_X^* Y$ ,  
 54 which is related to  $D_{\nabla, \nabla^*}$  by the relation  $D_{\nabla, \nabla^*} = T(X, Y) + D(X, Y)$ . For torsion-less connections,  
 55 the two notions agree, i.e.  $D_{\nabla, \nabla^*} = D$ .

56 In the case of dual connections with vanishing torsion, the commutation defect of the divergence  
 57 is related to the mutual curvature of the connections.

**Definition 3.** Let  $(\nabla^1, \nabla^2)$  be a pair of connections. Their mutual curvature is the tensor  $(1, 3)$ -tensor:

$$R_{\nabla^1 \nabla^2}(X, Y, Z) = \nabla_X^1 \nabla_Y^2 Z - \nabla_Y^1 \nabla_X^2 Z - \nabla_{[X, Y]}^1 Z \quad (3)$$

As in the case of the curvature, it is often useful to introduce the  $(0, 4)$ -tensor:

$$\bar{R}_{\nabla^1 \nabla^2}(X, Y, Z, U) = g(R_{\nabla^1 \nabla^2}(X, Y, Z), U)$$

58 The curvature and the mutual curvature of dual connections enjoy symmetry properties.

**Proposition 3.** Let  $(\nabla, \nabla^*)$  be a pair of dual connections. Then, for any vector fields  $X, Y, Z, U$ :

$$\begin{cases} \bar{R}(X, Y, Z, U) = \bar{R}^*(X, Y, U, Z) \\ \bar{R}_{\nabla^* \nabla}(X, Y, Z, U) = \bar{R}_{\nabla \nabla^*}(X, Y, U, Z) \end{cases} \quad (4)$$

**Proof.** The proof of the first property is found in, e.g. [4]. For the second, the definition of  $\bar{R}_{\nabla \nabla^*}$  is written as:

$$\bar{R}_{\nabla^* \nabla}(X, Y, Z, U) = g(\nabla_X^* \nabla_Y Z, U) - g(\nabla_Y^* \nabla_X Z, U) - g(\nabla_{[X, Y]}^* Z, U)$$

Using the duality property:

$$\begin{aligned} \bar{R}_{\nabla^* \nabla}(X, Y, Z, U) &= X(g(\nabla_Y Z, U)) - g(\nabla_Y Z, \nabla_X U) \\ &\quad - Y(g(\nabla_X Z, U)) + g(\nabla_X Z, \nabla_Y U) \\ &\quad - g(\nabla_{[X, Y]}^* Z, U) \end{aligned}$$

Using duality once again:

$$\begin{aligned}\bar{R}_{\nabla^*\nabla}(X, Y, Z, U) &= XY(g(Z, U)) - Xg(Z, \nabla_Y^*U) - Y(g(Z, \nabla_X^*U)) + g(Z, \nabla_Y^*\nabla_X U) \\ &\quad - YX(g(Z, U)) + Y(g(Z, \nabla_X^*U)) \\ &\quad + X(g(Z, \nabla_X^*U)) - g(Z, \nabla_X^*\nabla_Y U) \\ &\quad - [X, Y]g(Z, U) + g(Z, \nabla_{[X, Y]}U) = -\bar{R}_{\nabla\nabla^*}^*(Y, X, U, Z) = \bar{R}_{\nabla\nabla^*}^*(X, Y, U, Z)\end{aligned}$$

59  $\square$

60 In the case of dual connections without torsion, the definition of  $D(X, Y)$  simplifies to  $\nabla_X Y -$   
61  $\nabla^* X_Y$ . Letting  $D_X : Y \rightarrow D(X, Y)$ , the next proposition relates the commutation defect to the  
62 curvatures.

**Proposition 4.** For any vector fields  $X, Y, Z$ :

$$D_X D_Y Z - D_Y D_X Z = R(X, Y, Z) + R^*(X, Y, Z) - R_{\nabla\nabla^*}(X, Y, Z) - R_{\nabla^*\nabla}(X, Y, Z)$$

**Proof.** By simple computation:

$$\begin{aligned}D_X D_Y Z - D_Y D_X Z &= (\nabla_X - \nabla_X^*)(\nabla_Y Z - \nabla_Y^* Z) - (\nabla_Y - \nabla_Y^*)(\nabla_X Z - \nabla_X^* Z) \\ &= \nabla_X \nabla_Y Z - \nabla_X \nabla_Y^* Z - \nabla_X^* \nabla_Y Z + \nabla_X^* \nabla_Y^* Z \\ &\quad - \nabla_Y \nabla_X Z + \nabla_Y \nabla_X^* Z + \nabla_Y^* \nabla_X Z - \nabla_Y^* \nabla_X^* Z\end{aligned}$$

63 and the claims follows by identification of the terms.  $\square$

**Proposition 5.** Let  $\nabla, \nabla^*$  be dual affine connections on  $TM$ . Then, for any triple  $X, Y, Z$  of vector fields:

$$g(\nabla_X Y, Z) = g(\nabla_X^{lc} Y, Z) + \frac{1}{2} [g(D_{\nabla, \nabla^*}(Z, X), Y) - g(D_{\nabla, \nabla^*}(Y, Z), X) + g(D_{\nabla, \nabla^*}(X, Y), Z)] \quad (5)$$

64 where  $\nabla^{lc}$  is the Levi-Civita connection.

**Proof.** Since the two connections are dual:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z)$$

Using the definition of  $D_{\nabla, \nabla^*}$  it comes:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_Z X) - g(D_{\nabla, \nabla^*}(Z, X), Y) - g([Z, X], Y)$$

Using then an alternating sum over the cyclic permutations of  $(X, Y, Z)$  and the Koszul formula:

$$\begin{aligned}2g(\nabla_X^{lc} Y, Z) &= X(g(Y, Z)) - Z(g(X, Y)) + Y(g(Z, X)) \\ &\quad + g(Y, [Z, X]) - g(X, [Y, Z]) + g(Z, [X, Y])\end{aligned}$$

65 yields the result.  $\square$

66 **Remark 2.** Prop. 5 is the analogue of the Koszul formula for dual connections. It is a defining property  
67 given  $D_{\nabla\nabla^*}$ .

**Notation 1.** The  $(0, 3)$ -tensor:

$$U_{\nabla_1, \nabla_2}(X, Y, Z) = g(D_{\nabla_1, \nabla_2}(Z, X), Y) - g(D_{\nabla_1, \nabla_2}(Y, Z), X) + g(D_{\nabla_1, \nabla_2}(X, Y), Z) \quad (6)$$

68 is the skewness tensor associated the connections  $\nabla_1, \nabla_2$ . When no confusion is possible in the case of  
 69 dual connections, the subscripts will be dropped so that  $U(X, Y, Z)$  stands for  $U_{\nabla, \nabla^*}(X, Y, Z)$

*Remark 3.* The formula of prop. 5 can be rewritten to give the expression of  $\nabla^*$ :

$$g(\nabla_X^* Y, Z) = g(\nabla_X^{lc} Y, Z) - \frac{1}{2}U(Y, X, Z)$$

**Proposition 6.** For any triple  $(X, Y, Z)$ :

$$U(X, Y, Z) = U(Y, X, Z) + 2g(T(X, Y), Z) \quad (7)$$

70 where  $T$  is the torsion of  $\nabla$ .

**Proof.** Using the definition:

$$\nabla_X Y = \nabla_Y X + [X, Y] + T(X, Y)$$

and the fact that the Levi-Civita has vanishing torsion:

$$g(\nabla_X Y, Z) = g(\nabla_X^{lc} Y, Z) + \frac{1}{2}U(X, Y, Z)$$

thus:

$$\begin{aligned} g(\nabla_Y X, Z) &= g(\nabla_Y^{lc} X, Z) - g(T(X, Y), Z) + \frac{1}{2}U(X, Y, Z) \\ &= g(\nabla_Y^{lc} X, Z) + \frac{1}{2}U(Y, X, Z) \end{aligned}$$

and so:

$$U(X, Y, Z) = U(Y, X, Z) + 2g(T(X, Y), Z)$$

71  $\square$

**Proposition 7.** The tensor  $U$  has the cyclic symmetry property, that is for any triple  $(X, Y, Z)$  of vector fields:

$$U(X, Y, Z) = U(Z, X, Y) \quad (8)$$

**Proof.** Using the symmetry of the Riemann metric, the same derivation as in prop. 5 but applied to the terms  $X(g(Z, Y), Y(g(X, Z), Z(g(Y, X)$  yields:

$$\begin{aligned} 2g(\nabla_X^{lc} Z, Y) &= 2g(\nabla_X Z, Y) \\ &\quad - g(Z, D(Y, X)) + g(X, D(Z, Y)) - g(Y, D(X, Z)) \end{aligned} \quad (9)$$

By identification it comes:

$$U(X, Z, Y) = U(Y, X, Z) \quad (10)$$

72  $\square$

**Proposition 8.** Let  $U$  be a tensor with cyclic symmetry, then the connections defined by:

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\nabla_X^{lc} Y, Z) + \frac{1}{2}U(X, Y, Z) \\ g(\nabla_X^* Y, Z) &= g(\nabla_X^{lc} Y, Z) - \frac{1}{2}U(Y, X, Z) \end{aligned} \quad (11)$$

73 are dual.

**Proof.** For any triple  $(X, Y, Z)$  of vector fields:

$$X(g(Y, Z)) = g(\nabla_X^{\text{lc}} Y, Z) + g(Y, \nabla_X^{\text{lc}} Z)$$

Under the assumption of eq. 10, it comes:

$$\begin{aligned} X(g(Y, Z)) &= g(\nabla_X Y, Z) + \frac{1}{2}U(X, Y, Z) \\ &\quad + g((Y, \nabla_X^* Z) - \frac{1}{2}U(Z, X, Y)) \end{aligned}$$

and since  $U$  has cyclic symmetry:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g((Y, \nabla_X^* Z)$$

74  $\square$

**Proposition 9.** Let  $\nabla_1, \nabla_2$  be a pair of affine connections. For any triple  $(X, Y, Z)$  of vector fields:

$$g(Y, D_{\nabla_1, \nabla_2}(Z, X)) = \frac{1}{2} [U_{\nabla_1, \nabla_2}(X, Y, Z) + U_{\nabla_1, \nabla_2}(Z, X, Y)] \quad (12)$$

75 **Proof.** Direct computation from the definition of  $U$ .  $\square$

76 *Remark 4.* Prop. 9 shows that the mutual torsion of a pair of dual connections is uniquely defined by a  
77 cyclic symmetric tensor. Conversely, for a pair  $\nabla_1, \nabla_2$  of connections, the cyclic symmetry defect of  
78 the tensor  $U_{\nabla_1, \nabla_2}$ , namely  $A(X, Y, Z) = U_{\nabla_1, \nabla_2}(X, Y, Z) - U_{\nabla_1, \nabla_2}(Z, X, Y)$  is the obstruction of being  
79 dual. Please note also that the torsion for a pair of dual connections can be seen as the obstruction for  
80 the tensor  $U$  to be totally symmetric.

81 *Remark 5.* A statistical manifold may be defined as a quadruple  $(M, g, \nabla, U)$  with  $M$  a smooth  
82 manifold,  $g$  a Riemannian metric,  $\nabla$  an affine connection and  $U$  a tensor with cyclic symmetry. It  
83 slightly more general than the usual definition since  $U$  is not required to be totally symmetric, thus  
84 allowing connections with torsion.

### 85 3. Dual connections lifts

86 Let  $U$  be a coordinate neighborhood in  $M$  and let  $\pi: T^*M \rightarrow M$  be the canonical projection.  
87  $\phi^{-1}(U)$  is a coordinate neighborhood in  $T^*M$  with coordinates denoted as  $(x^1, \dots, x^n, p_1, \dots, p_n)$ .

The lift of connections on the cotangent bundle has been studied in [6,7] using the Riemann extension defined in [10]. Another kind of lift is introduced in [11] along with a metric on  $T^*M$ . Let  $(M, g)$  be a smooth Riemannian manifold and let  $\nabla$  be an affine connection. The kernel of  $d\pi: TT^*M \rightarrow T^*M$  defines an integrable distribution, called the vertical distribution, hereafter denoted by  $VT^*M$ . It is spanned by the vectors:

$$e_{j+n} = \delta^j = \frac{\partial}{\partial p_j}, j = 1 \dots n \quad (13)$$

Complementary to it, there is an horizontal distribution spanned by the vectors:

$$e_j = \partial_j + \Gamma_{ji}^k p_k \delta^i, j = 1 \dots n \quad (14)$$

with:

$$\partial_j = \frac{\partial}{\partial x^j}$$

These basis vectors are conveniently put into a matrix form, following the convention of [11]:

$$L = \begin{pmatrix} Id & 0 \\ \Gamma & Id \end{pmatrix} \quad (15)$$

where  $\Gamma$  is the matrix with entries:

$$\Gamma_{ji} = \Gamma_{ji}^k p_k \quad (16)$$

**Definition 4.** The Riemannian extension of a torsion-free affine connection  $\nabla$  on  $TM$  is the symmetric  $(0,2)$ -tensor with component matrix:

$$\nabla^R = \begin{pmatrix} -2\Gamma & Id \\ Id & 0 \end{pmatrix}$$

where  $\Gamma$  is the matrix defined in 16.

**Proposition 10.** Let  $\nabla$  be a torsion-free affine connection on  $M$  and let  $(e_j)_{1,\dots,2n}$  be its adapted frame in  $TT^*M$ . With respect to it, the component matrix of the Riemannian extension is:

$$\begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix}$$

**Proof.** In the adapted frame, the expression of the component matrix of the Riemannian extension is:

$$L^t \begin{pmatrix} -2\Gamma & Id \\ Id & 0 \end{pmatrix} L$$

which is equal to:

$$\begin{pmatrix} -2\Gamma + \Gamma + \Gamma^t & Id \\ Id & 0 \end{pmatrix}$$

using the assumption that  $\nabla$  is torsion-free,  $\Gamma^t = \Gamma$  and the claim follows.  $\square$

**Definition 5.** The Levi-Civita connection with respect to Riemannian extension, denoted by  $\nabla^c$ , is called the complete lift of the connection  $\nabla$ .

**Proposition 11.** The Christoffel symbols of the complete lift  $\nabla^c$  are given by:

$${}^c\Gamma_{ji}^k = \Gamma_{ji}^k, {}^c\Gamma_{ji}^{k+n} = p_l R_{kij}^l, {}^c\Gamma_{j(i+n)}^{k+n} = -\Gamma_{jk}^i, i, j, k = 1, \dots, n$$

When  $\nabla = \nabla^{lc}$ , the torsion-free assumption is automatically satisfied, so that in an adapted frame the Riemannian extension reduces to the one of prop. 10

**Proposition 12.** Let  $(\nabla, \nabla^*)$  be a pair of dual affine connections on  $TM$ . Then, with respect to the Riemannian extension  $\nabla^R$  of  $\nabla^{lc}$ , the following relations hold:

$$L^t \nabla^R L^* = L^{*t} \nabla^R L = \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix} \quad (17)$$

$$L^t \nabla^R L = \begin{pmatrix} \frac{1}{2} (\tilde{D} + \tilde{D}^t) & Id \\ Id & 0 \end{pmatrix} \quad (18)$$

$$L^{*t} \nabla^R L^* = \begin{pmatrix} -\frac{1}{2} (\tilde{D} + \tilde{D}^t) & Id \\ Id & 0 \end{pmatrix} \quad (19)$$

where  $\tilde{D}$  is the matrix with entries:

$$\tilde{D}_{ji} = p_k D_{ji}^k$$

94 and  $L$  (resp.  $L^*$ ) is the component matrix of the adapted frame to  $\nabla$  (resp.  $\nabla^*$ ).

**Proof.** In the case of dual connections, eq. 12 yields:

$$g(D(X, Y), Z) = U(X, Y, Z)$$

and so:

$$\nabla = \nabla^{lc} + \frac{1}{2}D \quad (20)$$

$$\nabla^* = \nabla^{lc} - \frac{1}{2}D^t \quad (21)$$

where  $D^t(X, Y) = D(Y, X)$ . From 20 (resp. 21), it comes:

$$\Gamma = \Gamma^{lc} + \frac{1}{2}\tilde{D} \quad (22)$$

$$\Gamma^* = \Gamma^{lc} - \frac{1}{2}\tilde{D}^t \quad (23)$$

$$(24)$$

When then have:

$$\nabla^R L = \begin{pmatrix} -\Gamma^{lc} + \frac{\tilde{D}}{2} & Id \\ Id & 0 \end{pmatrix}$$

and:

$$L^{*t} \nabla^R L = \begin{pmatrix} -\frac{\tilde{D}}{2} + \frac{\tilde{D}}{2} & Id \\ Id & 0 \end{pmatrix} = \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix}$$

95 The other equations are proved the same way.  $\square$

The above relations show that the horizontal subspaces of  $\nabla$  and  $\nabla^*$  are related by the Riemannian extension in a very simple way. Let  $X, Y$  be a vector in  $T_{x,p}T^*M$  with decomposition  $X = X_V + X_H$  (resp.  $Y = Y_{V^*} + Y_{H^*}$ ) according to the horizontal subspace of  $\nabla$  (resp.  $\nabla^*$ ), then:

$$\nabla^R(Y, X) = \langle Y_{V^*}, X_H \rangle + \langle X_V, Y_{H^*} \rangle$$

96 with  $\langle \cdot, \cdot \rangle$  the euclidean inner product.

97 Another interesting fact is that with respect to the adapted frames of  $\nabla$  (resp.  $\nabla^*$ ), the Riemannian  
98 extension becomes a modified Riemannian extension in the sense of [12]. To a given modified  
99 Riemannian extension, it is thus possible to associate a pair of dual connections with a given torsion  
100 (this last restriction comes from the fact that only the symmetric part of the tensor  $D$  enters the  
101 expression).

102 Since duality is related to metric, it is not so obvious how to lift a pair of mutually dual connections  
103 in a canonical way since the complete lifts of  $\nabla$  and  $\nabla^*$  involve different Riemannian extensions.  
104 The preferred approach will be thus to lift the mutual torsion  $D$  to a  $(0,3)$ -tensor, what can be done  
105 extending the approach of [6], and to exploit the fact that it has the cyclic symmetry property.

In the sequel, the symmetric (resp. anti-symmetric) part with respect to the contravariant indices of the  $(1,2)$ -tensor  $D$  will be denoted by  ${}^s D$  (resp.  ${}^a D$ ), i.e.:

$${}^s D_{ij}^k = \frac{1}{2} (D_{ij}^k + D_{ji}^k)$$

$${}^a D_{ij}^k = \frac{1}{2} (D_{ij}^k - D_{ji}^k)$$



**Proposition 13.** *The expression:*

$$\sigma = \frac{1}{2} p_k^a D_{ij}^k dx^i \wedge dx^j$$

*defines a 2-form on  $TT^*M$ . Its exterior derivative  $d\sigma$  is given by:*

$$d\sigma = \frac{1}{2} p_l \frac{\partial^a D_{ij}^l}{\partial x^k} dx^k \wedge dx^i \wedge dx^j + \frac{1}{2} {}^a D_{ij}^k dp_k \wedge dx^i \wedge dx^j$$

Rearranging the terms, the form  $d\sigma$  can be rewritten as:

$$\begin{aligned} 6d\sigma = & p_l \left( \frac{\partial^a D_{ij}^l}{\partial x^k} + \frac{\partial^a D_{ki}^l}{\partial x^j} + \frac{\partial^a D_{jk}^l}{\partial x^i} \right) dx^i \wedge dx^j \wedge dx^k \\ & + {}^a D_{ij}^k dp_k \wedge dx^i \wedge dx^j + {}^a D_{jk}^i dx^k \wedge dp_i \wedge dx^j + {}^a D_{ki}^j dx^k \wedge dx^i \wedge dp_j \end{aligned} \quad (25)$$

106 It turns out that the above tensor has cyclic symmetry since it is  $(0,3)$  and skew-symmetric. This  
 107 can made more explicit by first noticing that the first line in the right hand side has obviously this  
 108 property. In the second line, considering as an example the first term  ${}^a D_{ij}^k dp_k \wedge dx^i \wedge dx^j$ , a cyclic  
 109 permutation of the arguments yields  ${}^a D_{jk}^i dx^k \wedge dp_i \wedge dx^j$ . Now, the indices change  $j \rightarrow k, k \rightarrow i, i \rightarrow j$   
 110 gives  ${}^a D_{jk}^i dx^k \wedge dp_i \wedge dx^j$ , which is exactly the original second term. The remaining terms can be  
 111 worked the same way.

Considering now the symmetric part of  $D$ , a similar procedure can applied to obtain a fully symmetric  $(0,3)$ -tensor. Let us denote by  $\odot$  the symmetric tensor product, that is:

$$x \odot y = (x \otimes y + y \otimes x)/2$$

. From  ${}^s D$ , a symmetric tensor on  $TT^*M$  can be defined as:

$$\theta = \frac{1}{2} p_k {}^s D_{ij}^k dx^i \odot dx^j$$

112 Following the construction of 13 and the formula of [13], a fully symmetric lift can be defined.

**Definition 6.** The symmetric lift of  ${}^s D$  is the  $(0,3)$ -tensor with components:

$$\begin{aligned} \frac{1}{6} \left( p_l \left( \frac{\partial^s D_{ij}^l}{\partial x^k} + \frac{\partial^s D_{ki}^l}{\partial x^j} + \frac{\partial^s D_{jk}^l}{\partial x^i} \right) dx^i \odot dx^j \odot dx^k \right. \\ \left. + {}^s D_{ij}^k dp_k \odot dx^i \odot dx^j + {}^s D_{jk}^i dx^k \odot dp_i \odot dx^j + {}^s D_{ki}^j dx^k \odot dx^i \odot dp_j \right) \end{aligned} \quad (26)$$

113 Gathering things together, both the symmetric and the anti-symmetric part of  $D$  can be lifted to a  
 114 cyclic symmetric  $(0,3)$ -tensor. In the sequel, the notation of [6] is adopted: Latin letters  $i, j, \dots$  refer to  
 115  $x$  components, overlined letters  $\bar{i}, \bar{j}, \dots$  refers to  $p$  components and capital letters can be used for both.  
 116 As an example,  $dx^{\bar{i}} = dp_i, \delta^i = \partial_{\bar{i}}$ .

**Definition 7.** The cyclic symmetric complete lift of the  $(1,2)$ -tensor  $D$ , denoted  $U^c$ , is the  $(0,3)$ -tensor with components  $u_{ABC}^c dx^A \otimes dx^B \otimes dx^C$ :

$$\begin{cases} u_{ijk}^c = p_l \left( \frac{\partial D_{ij}^l}{\partial x^k} + \frac{\partial D_{ki}^l}{\partial x^j} + \frac{\partial D_{jk}^l}{\partial x^i} \right) \\ u_{ijk}^c = D_{jk}^i, u_{ijk}^c = D_{ki}^j, u_{ijk}^c = D_{ij}^k \\ u_{i\bar{j}A}^c = u_{iA\bar{j}}^c = u_{A\bar{i}j}^c = 0 \end{cases}$$

From  $U^c$ , the complete lift of  $D$  can be defined as the  $(1,2)$ -tensor  $D^c$  such that for any triple of vector fields:

$$\nabla^R (X, D^c(Y, Z)) = U^c(X, Y, Z) \quad (27)$$

Given the matrix form of the Riemannian extension:

$$\nabla^R = \begin{pmatrix} -2\Gamma & Id \\ Id & 0 \end{pmatrix}$$

its inverse is readily obtained as:

$$\Delta = \begin{pmatrix} 0 & Id \\ Id & 2\Gamma \end{pmatrix}$$

The components of  $D^c$  in coordinates can be obtained by composing the matrix  $A$ , yielding:

$$D_{AB}^{cC} = \Delta^{CD} u_{DAB}$$

$$\left\{ \begin{array}{l} D_{jk}^{ci} = D_{jk}^i \\ D_{jk}^{c\bar{i}} = p_l \left( \frac{\partial D_{ij}^l}{\partial x^k} + \frac{\partial D_{ki}^l}{\partial x^j} + \frac{\partial D_{jk}^l}{\partial x^i} \right) + 2\Gamma_l^{\bar{i}} D_{jk}^l \\ D_{jk}^{c\bar{j}} = D_{ki}^j \\ D_{jk}^{c\bar{k}} = D_{ij}^k \\ D_{jk}^{ci} = D_{jk}^{c\bar{i}} = D_{jk}^{c\bar{j}} = D_{jk}^{c\bar{k}} = 0 \end{array} \right. \quad (28)$$

117 with the notation  $\Gamma_l^{\bar{i}} = \Gamma_{il}$ . Please note that the above relations are different from the one given in [6]  
 118 for the complete lift of a skew-symmetric  $(1,2)$ -tensor since here the Riemann extension is used in  
 119 place of the canonical  $\epsilon$   $(1,1)$ -tensor and only the cyclic symmetry is assumed. This last fact can be  
 120 noticed in the third and forth lines of eq. (28).

121 The next definitions are recalled for the sake of completeness.

**Definition 8.** Let  $\omega = \omega_i dx^i$  be a degree 1 differential form. Its vertical lift to  $TT^*M$  is the vector field:

$$\omega^V = \omega_i \delta^i$$

122 Vector fields admit both a vertical and a complete lift. Only the later will be used here.

**Definition 9.** Let  $X = X^i \partial_i$  be a vector field on  $M$ . Its complete lift to  $TT^*M$  is the vector field:

$$X^c = X^i \partial_i - p_l \frac{\partial X^l}{\partial x^k} \delta^k$$

123 Finally  $(1,1)$ -tensors can be lifted in a quite obvious way:

**Definition 10.** Let  $F$  be a  $(1,1)$ -tensor field. Its vertical lift to  $TT^*M$  is the vector field:

$$F^V = p_l F_k^l \delta^k$$

124 The action of  $D^c$  on vertical and complete lift can now be obtained.

**Proposition 14.** Let  $X$  be a vector field and  $\omega, \theta$  be 1-forms. Then:

$$\left\{ \begin{array}{l} D^c(\omega^V, \theta^V) = 0 \\ D^c(\omega^V, X^c) = (\omega D_X)^V, D^c(X^c, \omega^V) = (\omega D_X)^V \end{array} \right. \quad (29)$$

125 where  $D_X$  (resp.  $D^X$ ) is the  $(1, 1)$ -tensor defined by:  $D_X(Y) = D(X, Y)$  (resp.  $D^X(Y) = D(Y, X)$ ).

**Proof.** Let  $\omega = \omega_i dx^i, \theta = \theta_j dx^j$ . Then  $D(\omega^V, \theta^V) = \omega^i \theta^j D_{ij}^{cA} = 0$ . Let  $X$  be vector field and  $X^c$  its complete lift. By linearity:

$$D^{cA}(\omega^V, X^c) = \omega^i X^j D_{ij}^{cA} - p_l \frac{\partial X^l}{\partial x^k} D_{ik}^{cA}$$

Since  $D_{ik}^{cA} = 0$ , the second term in the right hand side vanishes. For the first one, only  $D_{ij}^{c\bar{k}} = D_{jk}^i$  is non-zero, so that:

$$D^c(\omega^V, X^c) = \omega_i X^j D_{jk}^i \delta^k$$

126 The tensor  $D_X$  has expression  $D_X(Y) = D_{ij}^k X^i Y^j \partial_k$ , so that  $\omega D_X$  is the form  $\omega D_X = \omega_k X^i D_{ij}^k dx^j$ , whose  
127 vertical lift is  $\omega_k X^i D_{ij}^k \partial^k$ .  $\square$

128 Please note while the expression obtained is similar to the one of [6], the sign is opposite.

The case of the action on two complete lifts is a little bit more complicated. First of all, given two vector fields  $X = X^i \partial_i, Y = Y^j \partial_j$ , a simple computation yields:

$$\begin{aligned} D^c(X^c, Y^c) &= X^i Y^j D_{ij}^k \partial_k + X^i Y^j p_l \left( \frac{\partial D_{ij}^l}{\partial x^k} + \frac{\partial D_{ki}^l}{\partial x^j} + \frac{\partial D_{jk}^l}{\partial x^i} \right) \delta^k \\ &\quad + 2p_l \Gamma_{kl} D_{ij}^l X^i Y^j \delta^k - X^i p_l \frac{\partial Y^l}{\partial x^j} D_{ki}^j \delta^k - Y^j p_l \frac{\partial X^l}{\partial x^i} D_{jk}^i \delta^k \end{aligned} \quad (30)$$

After rewriting, eq. (30) becomes:

$$\begin{aligned} D^c(X^c, Y^c) &= X^i Y^j D_{ij}^k \partial_k + X^i p_l \left( Y^j \frac{\partial D_{ki}^l}{\partial x^j} - \frac{\partial Y^l}{\partial x^j} D_{ki}^j \right) \delta^k \\ &\quad + Y^j p_l \left( X^i \frac{\partial D_{jk}^l}{\partial x^i} - \frac{\partial X^l}{\partial x^i} D_{jk}^i \right) \delta^k + X^i Y^j p_l \frac{\partial D_{ij}^l}{\partial x^k} \delta^k + 2p_l \Gamma_{kl} D_{ij}^l X^i Y^j \delta^k \end{aligned} \quad (31)$$

Let us consider, for  $X, Y$  fixed vector fields, the  $(1, 1)$ -tensor  $\nabla^{lc} D(X, Y)$ :

$$Z \mapsto \nabla_Z^{lc}(D(X, Y)) = Z^k \frac{\partial D_{ij}^l X^i Y^j}{\partial x_k} + \Gamma_{km}^l D_{ij}^m X^i Y^j Z^k$$

Its vertical lift is then:

$$\left( \nabla^{lc} D(X, Y) \right)^V = p_l \frac{\partial D_{ij}^l X^i Y^j}{\partial x^k} \delta^k + p_l \Gamma_{km}^l D_{ij}^m X^i Y^j \delta^k \quad (32)$$

On the other hand, the complete lift of the vector field  $D(X, Y)$  is:

$$(D(X, Y))^C = D_{ij}^k X^i Y^j \partial_k - p_l \frac{\partial D_{ij}^l X^i Y^j}{\partial x^k} \delta^k \quad (33)$$

Combining ed. (32) and eq. (33) yields:

$$2p_l \Gamma_{kl} D_{ij}^l X^i Y^j \delta^k + X^i Y^j D_{ij}^k \partial_k = 2 \left( \nabla^{lc} D(X, Y) \right)^V + (D(X, Y))^C - p_l \frac{\partial D_{ij}^l X^i Y^j}{\partial x^k} \delta^k \quad (34)$$

Putting the expression in eq. (31) yields:

$$\begin{aligned} D^c(X^c, Y^c) = & 2 \left( \nabla^{lc} D(X, Y) \right)^V + (D(X, Y))^C + X^i p_l \left( Y^j \frac{\partial D_{ki}^l}{\partial x^j} - \frac{\partial Y^l}{\partial x^j} D_{ki}^j \right) \delta^k \\ & + Y^j p_l \left( X^i \frac{\partial D_{jk}^l}{\partial x^i} - \frac{\partial X^i}{\partial x^i} D_{jk}^l \right) \delta^k - p_l D_{ij}^l \frac{\partial X^l}{\partial x^k} Y^j \delta^k - p_l D_{ij}^l X^i \frac{\partial Y_j}{\partial x^k} \delta^k \end{aligned} \quad (35)$$

Let  $K$  be a  $(1, 1)$  tensor  $K$ . Its Lie derivative can be written [14, p. 32, prop. 35.]:

$$\mathcal{L}_X K(Y) = [X, K(Y)] - K([X, Y])$$

It thus comes:

$$\mathcal{L}_Y D_X(Z) = [Y, D_X(Y)] - D_X([Y, Z]) \quad (36)$$

Which can be written in coordinates:

$$\begin{aligned} \mathcal{L}_Y D_X(Z)^l = & Y^j \frac{\partial D_{ik}^l X^i Z^k}{\partial x^j} - D_{ik}^j X^i \frac{\partial Y^l}{\partial x^j} Z^k - Y^j \frac{\partial Z^k}{\partial x^j} X^i D_{ik}^l + Z^j \frac{\partial Y^k}{\partial x^j} X^i D_{ik}^l \\ = & X^i \left( Y^j \frac{\partial D_{ik}^l}{\partial x_j} - D_{ik}^j \frac{\partial Y^l}{\partial x^j} \right) Z^k + \frac{\partial Y^k}{\partial x^j} X^i D_{ik}^l Z^j \end{aligned} \quad (37)$$

Plugging it into eq. (35) finally gives the reduced expression:

$$\begin{aligned} D^c(X^c, Y^c) = & 2 \left( \nabla^{lc} D(X, Y) \right)^V + (D(X, Y))^C \\ & + (\mathcal{L}_Y D^X + \mathcal{L}_X D_Y)^V \\ & + 2((\nabla^0 D)(X, Y) - \nabla^0(D(X, Y)))^V \end{aligned} \quad (38)$$

129 with  $\nabla^0$  the trivial connection with 0 Christoffel symbols. The equation eq. (38) completely defines the  
130 tensor  $D^c$ .

From the complete lift  $D^c$ , dual connections with respect to the Riemannian extension can be obtained:

$$\begin{cases} \tilde{\nabla} = \nabla^c + \frac{1}{2} D^c \\ \tilde{\nabla}^* = \nabla^c - \frac{1}{2} D^{c^t} \end{cases} \quad (39)$$

131 The pair  $(\tilde{\nabla}, \tilde{\nabla}^*)$  defines the complete lift of the original statistical structure to the pseudo-Riemannian  
132 manifold  $(T^*M, \nabla^R)$ . When  $\nabla$  is without torsion, then  $D$  is symmetric. Using eq. (38) and the fact that  
133 in such a case  $D_X = D^X$  show that  $D^c$  is itself symmetric, proving that  $\tilde{\nabla}$  has vanishing torsion.

134 **Conflicts of Interest:** “The author declare no conflict of interest

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