Reducing Computational Cost in the Invariant Unscented Kalman Filtering For Attitude Estimation

Jean-Philippe Condomines, Gautier Hattenberger

Reducing Computational Cost in the Invariant Unscented Kalman Filtering For Attitude Estimation

Jean-Philippe Condomines, Gautier Hattenberger

***Abstract*—This article proposes a new formulation to derive the invariant unscented Kalman filter (IUKF) algorithm for attitude estimation problem, where both state and sigma-point are considered as a transformation group parametrization of the filter. The detailed IUKF equations are presented in this article. The filter equations relie on the same ideas as the usual Unscented Kalman Filter (UKF), but it uses a geometrically adapted correction term based on an invariant output error. The specific interest of the proposed formulation is that only the invariant state estimation errors between the predicted state and each sigma point must be known to determine the predicted outputs errors. As we have already computed the set of invariant state errors during the prediction step, the computation cost to find the covariance matrix of the invariant state estimation in the update step is greatly reduced.**

***Index Terms*—unscented Kalman filters (UKF), invariant filtering, attitude estimation, extended Kalman filter (EKF).**

I. Introduction

THE attitude estimation problem for nonlinear dynamic system is an important research topic and is a major concern in the aerospace engineering community ([1], [2], [3], [4], [5]). The unscented Kalman filter (UKF) is an efficient derivative free filtering algorithm for computing approximate solutions to discrete-time non-linear optimal filtering problems such as the estimation of Attitude and Heading reference System (AHRS) for autonomous systems. It has become prevalent as an alternative to the extended Kalman filter (EKF) that improves estimation and spares the pratictioner the computation of Jacobians. However, in its original form, the UKF cannot be directly applied to invariant filtering problems, where the linear estimation error and the linear predicted output error traditionaly used are defined as invariant state estimation error and invariant output error. Within this framework, the IUKF detailed nonlinear equations applied to attitude estimation problems can be analysed to reduce the computational cost.

This article focuses on a new formulation of the Invariant UKF-like estimator for Attitude estimation by analayzing the IUKF detailed nonlinear equations. Our main results, Theorems 1 and 2 allows to find a parametrization group reducing the computation complexity of the IUKF algorithm. A more detailed presentation of the contributions of this article is given in Section I-D.

A. Literature review

Various works has been conducted on nonlinear invariant observers over the past decade, most notably by N. Aghannan, P. Rouchon, S. Bonnabel, and E. Salaun ([6], [7], [8], [9], [10], [11]), who developed a constructive method based on a combination of differential geometry and group theory that can be used to create nonlinear filters for nonlinear system-state estimation problems. This last decade was characterized by the introduction of new techniques that question the classical definition of the estimation error traditionally used by methods of designing nonlinear observers. The first research based on a geometric approach was conducted by ([12], [13], [14], [15]). Their approach revolves around geometric Lie groups (quaternions, rotation matrices, etc.). Inspired by prior work by Krener and Isidori, it exploits the invariance of some nonlinear systems under certain geometric transformations (rotation, translation, etc.) to construct an observer that performs significantly better than observers defined locally around an equilibrium point. The various results established by S. Bonnabel ([7], [8], [16], [17], [18]) allowed a theory of invariant observers to be developed for a large class of systems with symmetries. *The linearization is no longer performed around an equilibrium point but around the identity element of a group. This enlarges the domain of convergence of the estimate and gives simplified expressions for the dynamics of the estimation error.* It requires however to tune an important number of setting parameters potentially when computing estimation gains, which can be cumbersome for complex system modeling. Thereafter, researchers have tried to develop more generic procedures which facilitate the design of invariant observers, by performing an automatic tuning of the correction gains which occurs in any filtering equation associated with nonlinear state observer. Regarding the state of the art, there exist two major techniques called Kalman-based invariant observers : the Invariant Kalman filter such as Invariant Extended Kalman Filter – IEKF or more recently the Unscented Kalman-based Invariant filter and the Invariant Particle Filter – IPF. The IEKF ([19], [17], [20], [21], [22]) is characterized by a larger convergence domain, due to the exploitation of systems' symmetries within the estimation algorithm (*i.e.,* within filter equations and gains computation), and present very good performances in practice. An important well known drawback in this method is that it requires to linearize the system of differential equations which govern the invariant state estimation error dynamics. Such an operation appears suitable for simple system modeling but for more complex system modeling, this linearization may be difficult to carry out. To overcome this problem, the UKF algorithm based on invariant framework has been recently proposed in ([23], [24], [25], [26], [27]). It has been proved in these bibliographical references that an Invariant UKF-like estimator could be designed by using either a compatibility condition or a specific case in order to make equation more explicit for the special case of the attitude estimation problem. In a similar spirit to

J.P Condomines and G. Hattenberger are with ENAC, Université de Toulouse, BP 54005, Toulouse Cedex 4, 31055, France. E-mail: (jean-philippe.condomines@enac.fr,gautier.hattenberger@enac.fr).

research from a few years ago on the IEKF (Invariant Extended Kalman Filter) algorithm, the correction gains of this estimator, which are specifically designed to be invariant, may be deduced by performing the same computational steps as UKF-type filtering (either in factorized or non-factorized form). However, before we can integrate the procedure for computing the correction gains (an algorithm borrowed from unscented Kalman filtering) with invariant observer theory, a series of methodological developments are required, as described in this article. Similarly, an extension of nonlinear invariant observers has been made for Rao-Blackwellized Particle Filters (PF) that can be used for nonlinear state estimation ([29], [30], [31]). Invariant PFs (IPF) rely on the notion of *conditional invariance* which corresponds to classical system invariance properties, but once some state variables are assumed to be known. It is those known states that will be sampled throughout the estimation process. It is noteworthy that, for the obtained IPF, the Kalman gains computed are identical for all particles which drastically reduces the computational effort usually needed to implement any PF.

B. Links with Attitude and Heading for Autonomous Systems

New applications in aerial robotics popularized invariant filtering. Many recent progresses in the miniaturization of sensors have led to the design of small and cheap integrated navigation system hardwares (complete IMU: Inertial Measurement Unit, GPS: Global Positioning System module, etc.), which have, for their part, contributed to boost significantly the market of mini-UAVs (Unmanned Air Vehicles) over the last decades, making them more accessible to everyone. Nevertheless, this accessibility is frequently inconsistent with good measurement performances. For instance, the GPS modules used commonly in the Paparazzi autopilot [1] deliver an absolute position with an average accuracy of 5m, up to 10m under certain flight conditions. Therefore, a need for multiple sensors data fusion arises to develop robust and powerful advanced control strategies for mini-UAVs that can be viewed as complex autonomous system. So much so that full state (or estimated state) feedback designs (cf. LQG/LTR syntheses) provide full authority to control efficiently in terms of stability and performances UAVs for accomplishing various missions. To this aim, *nonlinear estimation* offers several well-proven algorithmic techniques which permit to recover an acceptable level of accuracy on some key flight parameters (anemometric angles, orientation/attitude, linear and angular speeds, position, etc.) for mini-UAVs closed-loop handling qualities. Many bibliographical references [20], [16], [32] tackle this specific issue exploiting nonlinear invariant observers in the domain of autonomous systems in robotics for solving nonlinear Attitude and Heading Reference System (AHRS) estimation problems from both inertial/vision multisensors data fusion. Both properties and capabilities of this peculiar class of method make any invariant observer-based estimation scheme dedicated to dynamical system navigation appealing, especially when there exists, in addition, hardware redundancy. In that case, autonomous vehicles can reach an acceptable level of robustness w.r.t. degraded operating conditions for example in GPS-denied environments or multiple sensor faults. Using an invariant observer-based algorithm to merge an extended (and potentially redundant) set of measurements can still provide good performances and convergence properties in such situtations. Another interesting application of invariant observers theory can be found in [34]. It reformulates the standard Linear Quadratic Gaussian (LQG) controller synthesis into an Invariant LQG (ILQG) design by making use of an IEKF for the observer part. The resulting controller appears to be more robust and less sensitive to both estimated trajectory and misestimates.

[1]Paparazzi project at: https://blog.paparazziuav.org

C. About the standard IUKF computational Complexity

The computational complexity of the standard IUKF can be seen to be 20% higher than the computational complexity of the general UKF filter, when compared in terms of calculation time for both invariant state and measurement errors. When the state dimension is n, the UKF needs $2n$+1 evaluations of the state and measurement equations when IUKF needs $4n^2$+$2n$. In this paper, we reduce this computational complexity to $2n$.

D. Paper's Organizations and Contributions

Among methods in Section I-A, only a few tried to customize equations in order to make them more explicit for the special case of the Attitude and Heading Reference System (AHRS) ([27], [33], [35]). But none of them is able to reduce the computation complexity. This paper focuses on a new formulation of the Invariant UKF-like estimator for AHRS in order to reduce this complexity. The contributions of the paper include :

1) The presentation in Section II of theoretical prerequisities dealing with unscented Kalman filtering where both invariant state and output error are introduced. The invariant framework dedicated to unscented Kalman filter developed in this article is exceedingly convenient as filter equations can be specialized.
2) The invariant unscented Kalman filter equations are naturally specialized in Section IV, and the system of differential equations that described the proposed IUKF in the benchmarking case of an attitude estimation system is derived. Our focus is on finding parametization group that reduce the computational cost of the filter.
3) Finally the compuational results that appear in Theorem 2 are validated in Section V on the basis of simulated data generated by dynamic model simulations describing the free fall of a parafoil.

A comprehensive set of results validate the methodological principles presented in this article and compare the performances reached by the UKF, the standard IUKF and the developed IUKF dedicated to AHRS named $\mathrm{IUKF}_{\mathbf{x}}$. The invariance properties of the $\mathrm{IUKF}_{\mathbf{x}}$ algorithm are also visible on the state estimation errors which enlarge the convergence boundaries.

II. Some preliminaries

This section introduces the unscented Kalman filter this article is concerned with, as well as the invariant unscented Kalman filter our results apply to.

A. Unsented Kalman Filter

The standard UKF framework ([36], [37]) involves estimation of the state $\mathbf{x}_k \in \mathbb{R}^n$ of a discrete-time nonlinear dynamic system,

$$\begin{array}{rcl} \mathbf{x}_{k+1} & = & f(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k \\ \mathbf{y}_k & = & h(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{v}_k, \end{array} \tag{1}$$

where $\mathbf{y}_k \in \mathbb{R}^m$ is the output of the modeled system. $\mathbf{v}_k \in \mathbb{R}^m$ (resp. $\mathbf{w}_k \in \mathbb{R}^n$) refers to the discrete Gaussian process $\mathbf{w}_k \sim N(0, \mathbf{W}_k)$ (resp. observation $\mathbf{v}_k \sim N(0, \mathbf{V}_k)$).

The UKF estimation process starts with the calculation of the scaled Unscented Transform (UT), in order to pick a minimal set of sample points, also called *sigma points*, around the mean state vector denoted by $\boldsymbol{\mathcal{X}}$, s.t. $\boldsymbol{\mathcal{X}}^{(0)}_{k|k} = \hat{\mathbf{x}}_{k|k}$. This calculation provides a set of $(2n+1)$ sigma points and also two series of $(2n+1)$ scalar weighting factors, denoted by $\{W^{(i)}_{(m)}\}$ and $\{W^{(i)}_{(c)}\}$ ($i \in [\![0\,; 2n]\!]$). These sigma points are then propagated through the nonlinear state $f(\cdot)$ and output $h(\cdot)$ equations, providing a cloud of evolving points. The mean $\hat{\mathbf{x}}_{k|k}$ and covariance $\mathbf{P}^{\mathbf{x}}_{k|k}$ of the transformed points are the computed based on their statistics. The mean and covariance of the initial state $\mathbf{x}_0$ are denoted $\hat{\mathbf{x}}_0$ and $\mathbf{P}^{\mathbf{x}}_0$, respectively.

The unscented transform can be seen as a function (or functional) from $(f||h,\hat{\mathbf{x}}_k,\mathbf{P}^{\mathbf{x}}_k)$ to $(\hat{\mathbf{x}}_{k+1|k}||\hat{\mathbf{y}}_{k+1|k},\tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k}||\tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k})$ depending if the unscented transform is applied on the state or ($||$) output equation (See appendix B):

$$(\hat{\mathbf{x}}_{k+1|k}||\hat{\mathbf{y}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k}||\tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k}) = \mathrm{UT}(f||h, \hat{\mathbf{x}}_k, \mathbf{P}^{\mathbf{x}}_k). \tag{2}$$

In terms of the unscented transform $\mathrm{UT}(\cdot)$ the *unscented Kalman filter* prediction and update steps can be written as follows :

- *Prediction:* Compute the predicted state mean $\hat{\mathbf{x}}_{k+1|k}$ and the predicted covariance $\mathbf{P}^{\mathbf{x}}_{k+1|k}$:

$$\begin{array}{rcl} [\hat{\mathbf{x}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k}] & = & \mathrm{UT}(f, \hat{\mathbf{x}}_{k|k}, \mathbf{P}^{\mathbf{x}}_{k|k}) \\ \mathbf{P}^{\mathbf{x}}_{k+1|k} & = & \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k} + \mathbf{W}_k \end{array} \tag{3}$$

- *Update:* Compute the predicted mean $\hat{\mathbf{y}}_{k+1|k}$ and covariance of the measurement $\mathbf{P}^{\mathbf{y}}_{k+1|k}$, and the cross-corvariance of the state and measurement $\mathbf{P}^{\mathbf{xy}}_{k+1|k}$:

$$\begin{array}{rcl} [\hat{\mathbf{y}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k}] & = & \mathrm{UT}(h, \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}^{\mathbf{x}}_{k+1|k}) \\ \mathbf{P}^{\mathbf{y}}_{k+1|k} & = & \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k} + \mathbf{V}_k \\ \mathbf{P}^{\mathbf{xy}}_{k+1|k} & = & \displaystyle\sum_{i=0}^{2n} W^{(i)}_{(c)} (\hat{\mathbf{x}}^{(i)}_{k+1|k} - \hat{\mathbf{x}}_{k+1|k}) \\ & \times & (\hat{\mathbf{y}}_{k+1|k} - \hat{\mathbf{y}}^{(i)}_{k+1|k}) \end{array} \tag{4}$$

An estimation $\hat{\mathbf{x}}_{k+1|k+1}$ of $\mathbf{x}_{k+1}$ is then computed by the Kalman filtering equations :

$$\begin{array}{rcl} \hat{\mathbf{x}}_{k+1|k+1} & = & \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}) \\ \mathbf{P}^{\mathbf{x}}_{k+1|k+1} & = & \mathbf{P}^{\mathbf{x}}_{k+1|k} - \mathbf{K}_{k+1}\mathbf{P}^{\mathbf{y}}_{k+1|k}\mathbf{K}^T_{k+1}, \end{array} \tag{5}$$

where $\mathbf{K}_{k+1} = \mathbf{P}^{\mathbf{xy}}_{k+1|k} \cdot {\mathbf{P}^{\mathbf{y}}_{k+1|k}}^{-1}$ and $\mathbf{y}_{k+1}$ is the raw measurements.

The linear correction $(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k})$ is weighted by the gain $\mathbf{K}_{k+1}$ in such a way as to minimize the covariance of the state estimation error $(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k})$. Numerical efficient square root versions of the UKF are presented in [38].

B. Invariant Unscented Kalman filtering

This subsection is an extention of first research dealing with IUKF [26]. The motivation is using the udapte equations of the IUKF algorithm we can specialize each step to make them more explicit in Section IV. If the dynamics of the observed system have invariance properties (symmetries) such as $f(\cdot)$ is G-invariant and $h(\cdot)$ is G-equivariant (see [39] for details), we cannot directly construct an estimator of the system state with analogous properties directly from the basic equations of the UKF algorithm. For convergence, it would be extremely desirable for any candidate estimator filters to satisfy the same invariance properties as the system itself, in the same spirit as the invariant observers of the IEKF algorithm. To achieve this, IUKF algorithm adapts the UKF algorithm so that it yields an invariant estimator. From the same principles and computation steps as the UKF algorithm, a *natural* reformulation of the equations aiming to adapt the method for estimation in an invariant setting can be obtained by redefining the error terms used of the standard algorithm. The linear state error $(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k})$, the linear predicted output error $(\hat{\mathbf{y}}_{k+1|k} - \hat{\mathbf{y}}^{(i)}_{k+1|k})$ used in $\mathbf{P}^{\mathbf{y}}_{k+1|k}$ and $(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k})$ conventionally used in Eq.(5) do not preserve any of the symmetries and invariance properties of the system. Instead, we consider in the IUKF algorithm the following invariant state error[2] and predicted output error on Lie group G such as $\forall \mathbf{g} \in G$,$\forall i \in [\![0\,; 2n]\!]$:

$$\begin{array}{rcl} \boldsymbol{\eta}(\mathbf{x}_{k+1}, \hat{\mathbf{x}}_{k+1|k}) & = & \mathbf{x}^{-1}_{k+1}\hat{\mathbf{x}}_{k+1|k} \\ \mathsf{E}(\hat{\mathbf{y}}_{k+1}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) & = & \rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k}) \end{array} \tag{6}$$

Where $\mathbf{x}^{-1}_{k+1}$ is deduced from Cartan moving frame method and local transformation $\rho_{\mathbf{g}}$ is defined as for a dynamical system preserving symmetries [7]. The unscented transfrom can be re-written in invariant form where the weighted sum of sigma point are written as equivalent invariant expressions.

Lemma 1. *(The invariant state error form of UT) : The unscented transform can be written with an invariant state error form as follow :*

$$\begin{array}{rcl} \boldsymbol{\mathcal{X}}_{k+1|k} & = & [\boldsymbol{\mathcal{X}}^{(0)}_{k+1|k} \;\; \boldsymbol{\mathcal{X}}^{(1)}_{k+1|k} \;\; \ldots \;\; \boldsymbol{\mathcal{X}}^{(2n)}_{k+1|k}] = f(\boldsymbol{\mathcal{X}}_{k|k}, \mathbf{u}_k) \\ \hat{\mathbf{x}}_{k+1|k} & = & \displaystyle\sum_{i=0}^{2n} W^{(i)}_{(m)} \boldsymbol{\mathcal{X}}^{(i)}_{k+1|k} \\ \mathbf{P}^{\mathbf{x}}_{k+1|k} & = & \displaystyle\sum_{i=0}^{2n} W^{(i)}_{(c)} ({\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}}^{-1} \cdot \hat{\mathbf{x}}_{k+1|k})({\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}}^{-1} \cdot \hat{\mathbf{x}}_{k+1|k})^T \end{array} \tag{7}$$

Lemma 2. *(The invariant output error form of UT) : The unscented transform can be written with an invariant output error form parametrized by the Lie group* $\mathbf{g}$ *as follow :*

$$\begin{array}{rcl} \hat{\mathbf{Y}}_{k+1|k} & = & [\hat{\mathbf{y}}^{(0)}_{k+1|k} \;\; \hat{\mathbf{y}}^{(1)}_{k+1|k} \;\; \ldots \;\; \hat{\mathbf{y}}^{(2n)}_{k+1|k}] = h(\boldsymbol{\mathcal{X}}_{k+1|k}, \mathbf{u}_k) \\ \hat{\mathbf{y}}_{k+1|k} & = & \displaystyle\sum_{i=0}^{2n} W^{(i)}_{(m)} \hat{\mathbf{y}}^{(i)}_{k+1|k} \\ \mathbf{P}^{\mathbf{y}}_{k+1|k} & = & \displaystyle\sum_{i=0}^{2n} W^{(i)}_{(c)} \mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) \mathsf{E}^T(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) \end{array} \tag{8}$$

[2] The group action conincides with left translations (resp. right translations), see [17] for details.

Proof. See Appendix C-(A) & C-(B) □

The prediction and update steps of the invariant unscented Kalman filter are the following [26]:

- *Prediction:* The covariance matrix of the state estimation error leads to a set of $(2n+1)$ invariant errors defined between each sigma point and the predicted state. Thus, the covariance has now been conceptually redefined in terms of the invariant state error instead of a linear error term. This is the most important nuance compared to classical unscented Kalman filtering. Other than this key modification, the prediction step is essentially identical to the original algorithm. Next, we need to modify the update step accordingly to compute the gain terms.

- *Update:* The update step of the standard UKF algorithm requires more extensive modifications. With our new definition of the covariance matrix, which is now associated with the invariant estimation errors, we need to make the following changes: the calculation of the predicted covariance of the output $\mathbf{y}$ now becomes: $\mathbf{P}^{\mathbf{y}}_{k+1|k} \propto \mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k})$, and the calculation of predicted cross-covariance between the estimation errors in the state $\mathbf{x}$ and the output $\mathbf{y}$ is now: $\mathbf{P}^{\mathbf{xy}}_{k+1|k} \propto \big(\eta(\boldsymbol{\mathcal{X}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}), \mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k})\big)$. An estimation $\hat{\mathbf{x}}_{k+1|k+1}$ of $\mathbf{x}_{k+1}$ is computed by the invariant Kalman filtering equations : $\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}\mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}, \mathbf{y}_{k+1}, \omega(\hat{\mathbf{x}}_{k+1|k}))$

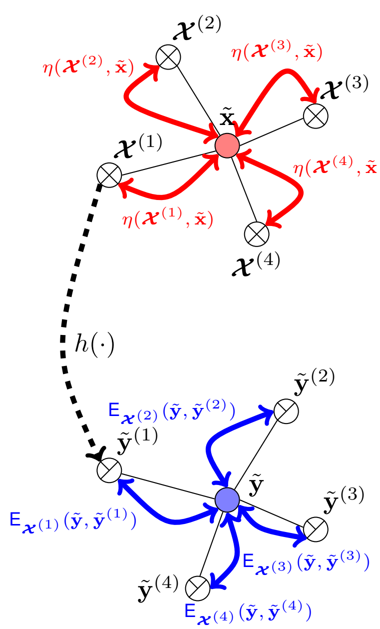


Fig. 1: *Principle of the standard IUKF approach*

By transitivity, it naturally follows that any correction gains for the state estimation computed using a UKF-type technique are functions of the invariant output errors $\mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}, \mathbf{y}_{k+1}, \omega(\hat{\mathbf{x}}_{k+1|k}))$. The invariant setting defined for the system also requires us to modify the correction equations for the predicted state. The additive correction term now includes: (a) a gain term that depends on the invariants of the estimation problem; (b) an invariant innovation term. This correction term is projected onto the invariant frame in such a way that the predicted state can be corrected component by component, i.e. along each of the n vectors in the standard basis of $\mathbb{R}^n$ formed by the invariant vector field $\mathcal{B}(\hat{\mathbf{x}}_{k+1|k}) = \{\omega_i(\hat{\mathbf{x}}_{k+1|k})\}_{i\in[\![1\,;\,n]\!]}$ (see [18] for more details). This is the most natural approach to adapting the UKF algorithm in order to construct an invariant estimator such that:

1) the correction terms are computed by a UKF-type scheme adapted to the invariant setting of the estimation problem which samples the state space using a classical unscented transform technique, like conventional sigma-points Kalman filtering;
2) the correction terms also preserve the specific symmetries of the system, since they are are constructed from an invariant innovation term and gain terms that are functions of the fundamental invariants and the invariant output error.

Algorithm 1. *(Invariant Unscented Kalman filter):*

In terms of the Unscented Transform UT($\cdot$) the *invariant unscented Kalman filter* prediction and update steps can be written by using **Lemma 1.** and **Lemma 2.** as follows :

- *Prediction:* Compute the predicted state mean $\hat{\mathbf{x}}_{k+1|k}$ and the predicted covariance $\mathbf{P}^{\mathbf{x}}_{k+1|k}$ as

$$\begin{aligned} [\hat{\mathbf{x}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k}] &= \mathrm{UT}(f, \hat{\mathbf{x}}_{k|k}, \mathbf{P}^{\mathbf{x}}_{k|k}, \boldsymbol{\eta}(\cdot)) \\ \mathbf{P}^{\mathbf{x}}_{k+1|k} &= \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k} + \mathbf{W}_k \end{aligned} \tag{9}$$

- *Update:* Compute the predicted mean $\hat{\mathbf{y}}_{k+1|k}$ and covariance of the measurement $\mathbf{P}^{\mathbf{y}}_{k+1|k}$, and the cross-corvariance of the state and measurement $\mathbf{P}^{\mathbf{xy}}_{k+1|k}$:

$$\begin{aligned} [\hat{\mathbf{y}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k}] &= \mathrm{UT}(h, \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}^{\mathbf{x}}_{k+1|k}, \mathsf{E}(\cdot)) \\ \mathbf{P}^{\mathbf{y}}_{k+1|k} &= \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k} + \mathbf{V}_k \\ \mathbf{P}^{\mathbf{xy}}_{k+1|k} &\propto \big(\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}), \cdots \\ &\qquad \mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k})\big) \end{aligned} \tag{10}$$

An estimation $\hat{\mathbf{x}}_{k+1|k+1}$ of $\mathbf{x}_{k+1}$ is then computed by the Kalman filtering equations :

$$\begin{aligned} \hat{\mathbf{x}}_{k+1|k+1} &= \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1} \dots \\ &\times \mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}, \mathbf{y}_{k+1}, \omega(\hat{\mathbf{x}}_{k+1|k})) \\ \mathbf{P}^{\mathbf{x}}_{k+1|k+1} &= \mathbf{P}^{\mathbf{x}}_{k+1|k} - \mathbf{K}_{k+1} \cdot \mathbf{P}^{\mathbf{y}}_{k+1|k} \cdot \mathbf{K}^T_{k+1}, \end{aligned} \tag{11}$$

where $\mathbf{K}_{k+1} = \mathbf{P}^{\mathbf{xy}}_{k+1|k} \cdot \mathbf{P}^{\mathbf{y}^{-1}}_{k+1|k}$ and $\mathbf{y}_{k+1}$ is the raw measurements.

The nonlinear correction $\mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}, \mathbf{y}_{k+1}, \omega(\hat{\mathbf{x}}_{k+1|k}))$ is weighted by the gain $\mathbf{K}_{k+1}$ in such a way as to minimize the covariance of the invariant state estimation error $\eta(\boldsymbol{\mathcal{X}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k})$.

The IUKF Matlab code will be available on the personal website of authors for the final manuscript (see http://recherche.enac.fr/~jean-philippe.condomines/wp/).

III. Problem setting

A. Considered Discret-Time model

We consider an Attitude and Heading Reference Systems (AHRS) in discrete-time [2] with step dt, characterized by its quaternion $\mathbf{q}_k$ with the quaternion product $*$. Eq (1) now becomes :

$$\begin{aligned}
\mathbf{q}_{k+1} &= \mathbf{q}_k + 0.5.\mathbf{q}_k * (\boldsymbol{\omega}_{m_k} - \boldsymbol{\omega}_{b_k}).dt + \mathbf{w}_{\mathbf{q_k}} * \mathbf{q}_k \\
\boldsymbol{\omega}_{b_{k+1}} &= \boldsymbol{\omega}_{b_k} + {\mathbf{q}_k}^{-1} * dt.\mathbf{w}_{\mathbf{w_k}} * \mathbf{q}_k \\
a_{s_{k+1}} &= a_{s_k} + dt.\mathbf{w}_{a_k} \\
b_{s_{k+1}} &= b_{s_k} + dt.\mathbf{w}_{b_k} \\
\begin{pmatrix} \mathbf{y_A}_k \\ \mathbf{y_B}_k \end{pmatrix} &= \begin{pmatrix} a_{s_k}.\mathbf{q}_k^{-1} * \mathbf{A} * \mathbf{q}_k + \mathbf{v}_{\mathbf{A_k}} \\ b_{s_k}.\mathbf{q}_k^{-1} * \mathbf{B} * \mathbf{q}_k + \mathbf{v}_{\mathbf{B_k}} \end{pmatrix},
\end{aligned} \tag{12}$$

Where $\mathbf{w}_{\mathbf{q_k}}, \mathbf{w}_{\mathbf{w_k}}, \mathbf{w}_{a_k}, \mathbf{w}_{b_k}$ (resp. $\mathbf{v}_{\mathbf{A_k}}, \mathbf{v}_{\mathbf{B_k}}$) refer to the process (resp. measurement) Gaussian white noise covariance matrices. The AHRS is endowed with 3 triaxial sensors: 3 magnetometers measuring Earth's magnetic field, which is known constant and expressed in the body-fixed frame s.t. vector $\mathbf{y}_{\mathbf{B}_{k+1}} = \mathbf{q}_{k+1}^{-1} * \mathbf{B} * \mathbf{q}_{k+1}$ (where $\mathbf{B} = (B_x\ B_y\ B_z)^T$) can be considered as an output of the observation equations; 3 gyroscopes produce the measurements associated with the instantaneous angular velocities gathered in $\boldsymbol{\omega}_{m_k} \in \mathbb{R}^3$; and 3 accelerometers provide the measured output signals corresponding to the acceleration. Constant $\mathbf{A} = (0\ 0\ g)^T$ refers to the local Earth's gravity vector. Moreover, we add constant bias vector $\boldsymbol{\omega}_{b_k}$ on the angular velocities vector measurement $\boldsymbol{\omega}_{m_k}$ and constant scaling factor, denoted by a_{s_k} and b_{s_k}, which adjust and correct the predicted outputs $\mathbf{y_A}_k$ and $\mathbf{y_B}_k$.

B. Comments on modeling imperfections of inertial sensors

Taking into account the maximum number of sensors' imperfections (such as low frequency disturbances) within the estimation process requires the introduction of several additive state variables. A 1^{st}-order observability analysis (see [40] for more calculation details) shows that up to 6 additional unknown constants can be estimated without introducing inobservability. Thus, the choice of considering an additive constant bias vector $\boldsymbol{\omega}_b$ on the angular rates vector measurement $\boldsymbol{\omega}_m$ has been made. Then, only 2 (of possible 3) additional parameters have been introduced. Doing so allows to rely not too much on the possibly perturbated magnetic field within the estimation process of $\mathbf{y_A}$. These 2 additive variables correspond to positive constant scaling factors, denoted by a_s and b_s, which adjust and correct the predicted outputs $\mathbf{y_A}$ and $\mathbf{y_B}$ respectively. All these sensor imperfections are modeled as pseudo-Gaussian random walks which can be physically interpreted as slowly varying parameters. Note that the noise $\mathbf{w}_{\mathbf{w_k}}$ is defined as a Langevin noise i.e, this noise is said isotropic and enter into the system in an invariant way (see Definition 1 in [33] for more details).

C. Invariance properties of the considered model

By taking advantage of the Galilean invariance properties of the problem, the model equations can be expressed equivalently in both aircraft coordinates and ground coordinates. Given Eq 12 and the Lie group $G = \mathbb{H}_1 \times \mathbb{R}^5$ (where $\mathbb{H}_1$ is the Lie algebra of quaternions of norm one) acting on the entire state space, the dynamics of the system is indeed G-equivariant. we have that:

Lemma 3. *Let G a Lie-group, $\forall \mathbf{g}_0 = (\mathbf{q}_0^T\ \boldsymbol{\omega}_0^T\ a_0\ b_0)^T \in G$, the following output transformation proves that system modeling is G-equivariant [8] : $\rho_{\mathbf{g}_0}(\mathbf{y}_{k+1}) = ((a_0.\mathbf{q}_0^{-1} * \mathbf{y}_{\mathbf{A}_{k+1}} * \mathbf{q}_0)^T\ (b_0.\mathbf{q}_0^{-1} * \mathbf{y}_{\mathbf{B}_{k+1}} * \mathbf{q}_0)^T)^T$.*

Moreover, the invariant state estimation error vector $\boldsymbol{\eta}(\mathbf{x}_{k+1}, \hat{\mathbf{x}}_{k+1|k})$, which is a transposition of the linear error to the multiplicative group may be defined by the following expression:

Lemma 4. *Consider $(2n+1)$ sigma points $\boldsymbol{\mathcal{X}}$, s.t. $\boldsymbol{\mathcal{X}}_{k|k}^{(0)} = \hat{\mathbf{x}}_{k|k} = (\hat{\mathbf{q}}_{k|k}^T\ \hat{\boldsymbol{\omega}}_{b_{k|k}}^T\ \hat{a}_{s_{k|k}}\ \hat{b}_{s_{k|k}})$. An invariant state estimation error $\eta(\boldsymbol{\mathcal{X}}_{k+1|k}^{(i)}, \hat{\mathbf{x}}_{k+1|k}) = {\boldsymbol{\mathcal{X}}_{k+1|k}^{(i)}}^{-1} \cdot \hat{\mathbf{x}}_{k+1|k} / i \in [\![\, 0\,; 2n\,]\!]$ can be expressed by [8]*

$$\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \hat{\mathbf{x}}_{k+1|k}) = \begin{pmatrix} \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}}^{-1} * \hat{\mathbf{q}}_{k+1|k} \\ \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * (\boldsymbol{\omega}_{b,\boldsymbol{\mathcal{X}}^{(i)}} - \hat{\boldsymbol{\omega}}_{b,k+1|k}) * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}}^{-1} \\ \hat{a}_{s_{k+1|k}} / a_{s,\boldsymbol{\mathcal{X}}^{(i)}} \\ \hat{b}_{s_{k+1|k}} / b_{s,\boldsymbol{\mathcal{X}}^{(i)}} \end{pmatrix}. \tag{13}$$

Where ${\boldsymbol{\mathcal{X}}^{(i)}}^{-1} = (\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}}^{-1}\ \ \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \boldsymbol{\omega}_{b,\boldsymbol{\mathcal{X}}^{(i)}} * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}}^{-1}\ (a_{s,\boldsymbol{\mathcal{X}}^{(i)}})^{-1}\ (b_{s,\boldsymbol{\mathcal{X}}^{(i)}})^{-1})^T$.

The invariance properties of the IUKF applied on AHRS are closely intertwined with the invariant state estimation error. Along the line of the Theorem 2 presented in [33], we consider the variable $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \hat{\mathbf{x}}_{k+1|k})$ as Markov processes,and is independent of the inputs $\boldsymbol{\omega}_{mk}$.[3] The most important consequence of this property is that the invariant filter gain(s) calculation can be addressed *ad hoc* by choosing gain value(s) which will meet some predifined requirements in terms of : -convergence (guarantee and domain); -decoupling purposed. The convergence propertie of filters will be highlighted in Section V.

D. Toward an invariant unscented Kalman filter for AHRS

At this point, we should note that the algorithm presented in Section II uses a multiple parametrization of the transformation group obtained by successively defining the inverse of each sigma point as a parameter of the composite mapping $\phi_{\mathbf{g}} = (\mathbf{x}_{k+1}^{-1}, \rho_{\mathbf{g}})$. This is ultimately equivalent to defining a set of $(2n+1)$ n-dimensional moving frames in the state space, sending each sigma point to the identity element $\mathbf{e}$ via the local mapping $\mathbf{x}_{k+1}^{-1}$. The algorithm proposed here is generic in the sense that it does not assume any specific form for the equations of the observation model nor the relations which define the group transformation $\rho_{\mathbf{g}}$. Nevertheless, it can sometimes be useful to extend and specialize the computations in each of the steps listed above in order to make them more explicit for AHRS.
The problem is to find a parametrization group $\mathbf{g}$, reducing the computation cost of the IUKF algorithm from Eq.(12) and the

[3] When the consante biais vector $\boldsymbol{\omega}_{b_k}$ is correctly estimated that is the case in practice.

invariant predicted output error of unscented transform (See Lemma 2) such that : $\mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) = \rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k})$. We are now in a position to study the modification of the considered parametrization in the invariant predicted output error of the IUKF algorithm.

IV. A PARAMETRIC FORMULATION STUDY FOR ATTITUDE ESTIMATION

This section contains the main theoretical result of this article. This result, Theorem 2., consists of a formulation avoiding to find $4n^2+2n$ invariant state errors, when the state dimension is n, between the sigma points. The main idea of the paper is to consider either $\mathbf{g_0} = {\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}}^{-1}$ or $\mathbf{g_0} = \hat{\mathbf{x}}^{-1}_{k+1|k}$. We futher make the following assumption :

Assumption 1. *Without loss of generality and emphasize the role of the parametrization, we assume evolution and observation noises in Eq.(12) equal to zero (i.e., $\mathbf{w}_k = \mathbf{v}_k = 0$).*

A. The considered parametrization

1) Sigma-point as parametrization ($\mathbf{g_0} = {\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}}^{-1}$): For AHRS, there is a straightforward way to express the $(2n+1)$ invariant output errors in terms of the constant vector $(\mathbf{A}^T\ \mathbf{B}^T)^T$, the transformation group $\rho_{\mathbf{g_0}}$, and a set of invariant state estimation errors satisfying the following relation :

Theorem 1. *Let $(\mathbf{A}^T\ \mathbf{B}^T)^T$, two constant vectors. For any sigma points $\boldsymbol{\mathcal{X}}^{(i)}$, $\forall i \in [\![\,0\,;2n\,]\!]$ the predicted invariant output error $\mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(i)}, \hat{\mathbf{y}}^{(i)}_{k+1|k})$, denoted $\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}}$ for convenience can be expressed such as*

$$\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}} = \sum_{\substack{j\,=\,0\\ j\,\neq\, i}}^{2n} W^{(j)}_{(m)} \left[\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \rho_{\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)},\boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \right]. \tag{14}$$

Proof. See Appendix D-(A) □

Eq.(14) shows that the invariant prediction output errors can be written as a weighted sum of the (invariant) distances between the constant vector $(\mathbf{A}^T\ \mathbf{B}^T)^T$ and its image under the mapping $\rho_{\mathbf{g}}$ over the Lie group parametrized by the elements $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \boldsymbol{\mathcal{X}}^{(j)})$, where i ranges over $[\![\,0\,;2n\,]\!]$ and $\forall j \in [\![\,0\,;2n\,]\!]$. This expression requires us to compute the values of the $(2n+1)$ invariant error vectors between the sigmas points $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \boldsymbol{\mathcal{X}}^{(j)})$. As a special case, whenever $i \in [\![\,0\,;2n\,]\!]$, then $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \boldsymbol{\mathcal{X}}^{(i)}) = \vec{0}$.

The algorithm therefore requires to find $(2n+1) \times (2n+1) - (2n+1) = 4n^2 + 2n$ terms $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \boldsymbol{\mathcal{X}}^{(j)})$ after these trivial cases are eliminated. In the simple case of an AHRS with $n = 9$, the IUKF approach therefore requires to compute 342 invariant error vectors between the sigma points at each iteration (i.e, prediction and correction step). For the AHRS, given a set of $(2n+1)$ sigma points, we implicitly need to compute a potentially large number of invariant estimation errors between the sigma points in order to compute the invariant output errors.

2) Predicted state as parametrization ($\mathbf{g_0} = \hat{\mathbf{x}}^{-1}_{k+1|k}$) : The parameter $\mathbf{g_0}$ of the group transformation $\rho_{\mathbf{g_0}}$can also be chosen to be constant and equal to $\hat{\mathbf{x}}_{k+1|k}$, without changing the performance of the estimation algorithm (see section V). In the case where $\mathbf{g_0} = \hat{\mathbf{x}}_{k+1|k}$, invariant output predicted errors can be expressed as following:

Theorem 2. *Solution to problem : Let $(\mathbf{A}^T\ \mathbf{B}^T)^T$, two constant vectors. For any predicted state $\hat{\mathbf{x}}_{k+1|k}, \forall i \in [\![\,0\,;2n\,]\!]$ the predicted invariant output error $\mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}, \hat{\mathbf{y}}^{(i)}_{k+1|k})$, denoted $\hat{\mathsf{E}}_{\hat{\mathbf{x}}}$ for convenience, can be expressed such as*

$$\hat{\mathsf{E}}_{\hat{\mathbf{x}}} = \rho_{\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)},\hat{\mathbf{x}}_{k+1|k})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(j)},\hat{\mathbf{x}}_{k+1|k})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \tag{15}$$

Proof. See Appendix D-(B) □

The significance of this result is that each elementary error term takes an invariant of the estimation problem as an argument, namely the constant vector $(\mathbf{A}^T\ \mathbf{B}^T)^T$, and that the parametrization of the Lie group ranges over the index j of the weighted sum, depending on the sigma point considered in each elementary calculation. Unlike the earlier case, we only need to know $2n = 18$ invariant state estimation errors between the predicted state and each sigma point in order to compute the invariant output errors.

B. Invariant unscented Kalman filter equations for AHRS

Algorithm 2. *(Invariant Unscented Kalman filter for AHRS):*

In terms of the unscented transform UT($\cdot$) the *invariant unscented Kalman filter for AHRS* prediction and update steps can be written by using **Lemma 4.** and **Theorem 2.** as follows :

- *Prediction:* Compute the predicted state mean $\hat{\mathbf{x}}_{k+1|k}$ and the predicted covariance $\mathbf{P}^{\mathbf{x}}_{k+1|k}$ as

$$\begin{array}{rcl}[\hat{\mathbf{x}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k}] & = & \mathrm{UT}(f, \hat{\mathbf{x}}_{k|k}, \mathbf{P}^{\mathbf{x}}_{k|k}, \mathbf{x}^{-1}_k \hat{\mathbf{x}}_{k|k}) \\ \mathbf{P}^{\mathbf{x}}_{k+1|k} & = & \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k} + \mathbf{W}_k \end{array} \tag{16}$$

- *Update:* Compute the predicted mean $\hat{\mathbf{y}}_{k+1|k}$ and covariance of the measurement $\mathbf{P}^{\mathbf{y}}_{k+1|k}$, and the cross-corvariance of the state and measurement $\mathbf{P}^{\mathbf{xy}}_{k+1|k}$:

$$\begin{array}{rcl}[\hat{\mathbf{y}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k}] & = & \mathrm{UT}(h, \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}^{\mathbf{x}}_{k+1|k}, \hat{\mathsf{E}}_{\hat{\mathbf{x}}}) \\ \mathbf{P}^{\mathbf{y}}_{k+1|k} & = & \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k} + \mathbf{V}_k \\ \mathbf{P}^{\mathbf{xy}}_{k+1|k} & \propto & \left(\mathbf{x}^{-1}_{k+1}\hat{\mathbf{x}}_{k+1|k}, \hat{\mathsf{E}}_{\hat{\mathbf{x}}}\right) \end{array} \tag{17}$$

An estimation $\hat{\mathbf{x}}_{k+1|k+1}$ of $\mathbf{x}_{k+1}$ is then computed by the Kalman filtering equations using Eq.(15) :

$$\begin{aligned}\hat{\mathbf{x}}_{k+1|k+1} = \sum_{j=0}^{2n} W^{(j)}_{(m)} \Bigg[& \boldsymbol{\mathcal{X}}^{(j)}_{k+1|k} \cdots \\ & + \sum_{i=1}^{n} \mathbf{K}^{(i)}_{k+1} \Bigg(\rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\mathbf{y}_{k+1}) - \rho_{\boldsymbol{\eta}(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \Bigg) \\ & \omega_i(\hat{\mathbf{x}}_{k+1|k}) \Bigg] \end{aligned} \tag{18}$$

Proof. See Appendix D-(C) □

At each time step, the estimated state is therefore computed in the form of a correction of the prediction derived from the left or right-invariant dynamics of the system (i.e., $\omega_i(\hat{\mathbf{x}}_{k+1|k})$, see [39] for more details) expressed as a weighted sum of invariant innovation terms over the Lie group parametrized directly or indirectly by $\hat{\mathbf{x}}_{k+1|k}$. After combining all of these resuults, Eq. (18) can be rewritten as follows in the special case of an AHRS :

$$\begin{aligned}
\hat{q}_{k+1|k+1} &= \hat{q}_{k+1|k} + \sum_{i=1}^{4} \mathbf{K}_{k+1}^{(i)} \cdots \\
&\times \begin{pmatrix} \hat{a}_{s,k+1|k}^{-1} \hat{q}_{k+1|k} * \left[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\right] \\ \hat{b}_{s,k+1|k}^{-1} \hat{q}_{k+1|k} * \left[\mathbf{y}_{k+1}^{4:6} - \hat{\mathbf{y}}_{k+1|k}^{4:6}\right] \end{pmatrix}
\end{aligned} \tag{19}$$

$$\begin{aligned}
\hat{\omega}_{b,k+1|k+1} &= \hat{\omega}_{b,k+1|k} + \hat{q}_{k+1|k}^{-1} * \sum_{i=5}^{7} \mathbf{K}_{k+1}^{(i)} \cdots \\
&\times \begin{pmatrix} \hat{a}_{s,k+1|k}^{-1} \hat{q}_{k+1|k} * \cdots \\ \hat{b}_{s,k+1|k}^{-1} \hat{q}_{k+1|k} * \cdots \\ \left[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\right] * \hat{q}_{k+1|k}^{-1} \\ \left[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\right] * \hat{q}_{k+1|k}^{-1} \end{pmatrix} * \hat{q}_{k+1|k}
\end{aligned} \tag{20}$$

$$\begin{aligned}
\hat{a}_{s,k+1|k+1} &= \hat{a}_{s,k+1|k} + \mathbf{K}_{k+1}^{(8)} \\
&\times \begin{pmatrix} \hat{q}_{k+1|k} * \cdots \\ \hat{a}_{s,k+1|k} \hat{b}_{s,k+1|k}^{-1} \hat{q}_{k+1|k} * \cdots \\ \left[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\right] * \hat{q}_{k+1|k}^{-1} \\ \left[\mathbf{y}_{k+1}^{4:6} - \hat{\mathbf{y}}_{k+1|k}^{4:6}\right] * \hat{q}_{k+1|k}^{-1} \end{pmatrix}
\end{aligned} \tag{21}$$

$$\begin{aligned}
\hat{b}_{s,k+1|k+1} &= \hat{b}_{s,k+1|k} + \mathbf{K}_{k+1}^{(9)} \\
&\times \begin{pmatrix} \hat{b}_{s,k+1|k} \hat{a}_{s,k+1|k}^{-1} \hat{q}_{k+1|k} * \cdots \\ \hat{q}_{k+1|k} * \cdots \\ \left[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\right] * \hat{q}_{k+1|k}^{-1} \\ \left[\mathbf{y}_{k+1}^{4:6} - \hat{\mathbf{y}}_{k+1|k}^{4:6}\right] * \hat{q}_{k+1|k}^{-1} \end{pmatrix}
\end{aligned} \tag{22}$$

Proof. See Appendix D-(D) □

Figures 3 and Figures 4 in appendix A, illustrates the two theorems for the AHRS, and compares them with the more general approach presented in Section II-B, which can be applied to any type of invariant model.

C. Features of the IUKF$_{\mathbf{x}}$ for AHRS

*1) Sound geometric stucture for the quaternion estimation (**Feature #1**):* by constuction Eq.(19) preserves the unit norm of the estimated quaternion.

*2) Symmetry-preserving structure (**Feature #2**):* in the context of attitude estimation for mini-UAV applications, we can give physical interpretations of the invariance properties of the kinematic relations. Each invariance correspond to a symmetry of the dynamics of the motion of the aircraft that is independent of the reference system (aircraft or earth coordinates) in which it is expressed. Thus, the estimation error does not depend on the trajectory of the system.

*3) reducing computation cost expected (**Feature #3**):* we have already computed the set $\{\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \hat{\mathbf{x}}_{k+1|k})/i \in [\![\, 0\,; 2n \,]\!]\}$ during the prediction step to find the covariance matrix of the invariant state estimation errors. This formulation therefore avoids the need to find the $4n^2 + 2n$ invariant state errors between the sigma points.

V. Simulation results

Fig. 2: Pictorial overview of SPQR concept.

The motivation for this simulation is led by the development of Small Payload Quick-Return (SPQR) study intended to routinely deliver small payloads from International Space Station (ISS) on-demand [4]. The SPQR concept, originating from NASA Ames Research Center at Moffett Field, CA, relies on a three-stage method of returning payloads, after being stored until needed and then loaded while on-board the ISS (cf. Figure 2): a) Deorbit, by means of a passive deployable drag system; b) Atmospheric reentry, via the deployment of a passively self-stabilizing reentry body; c) Terminal descent of the temperature-controlled paylaod canister beneath an autonomous guided parafoil. To mature this final phase of the SPQR concept, an autonomous parafoil system which satisfies the demands of landing precision requirements must be developed by using a small AHRS payload. We illustrate the behavior of the UKF and the IUKF dedicated to AHRS with both parametrizations (IUKF$_{\mathbf{x}}$,IUKF$_{\boldsymbol{\mathcal{X}}}$) by simulations and compare this one with experiment data. A set of results for the AHRS estimation problem generated from simulated noisy data are then presented to demonstrate the well-foundedness of the IUKF algorithm using state as parametrization (IUKF$_{\mathbf{x}}$), as well as its potential benefits in both theoretical and SPQR contexts.

A. Simuation Setting

The reference input data used for our evaluation of an IUKF-type approach were generated by dynamic model simulations describing the free fall of a parafoil. These simulated data provide a straightforward way to validate the methodological principles presented in this article, configure the parameters of each method, and establish conclusions regarding the analysis on the computational complexity. We consider the data both with and without added noise (Figure 4). The reference simulation that we used to validate our algorithms had a duration of slightly over 100 seconds. The simulated parachute system exhibits relatively strong dynamics. The roll, heading, and pitch angles vary by up to several dozen degrees. The

[4]https://www.nasa.gov/mission_pages/station/research/experiments/2543.html

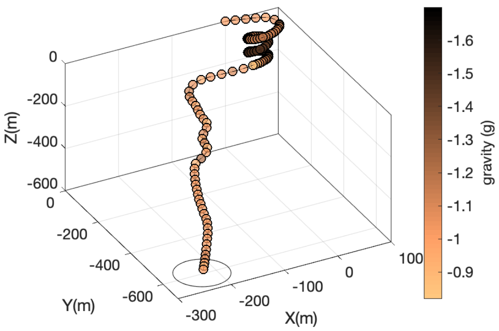
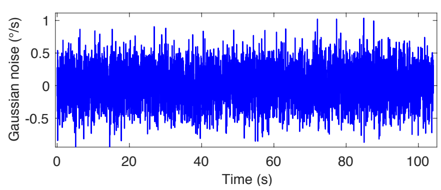
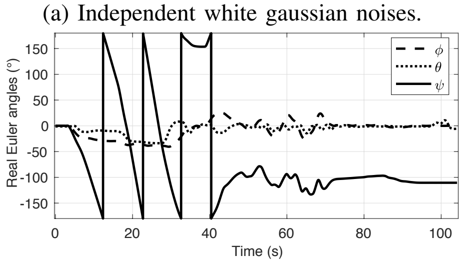


Fig. 3: 3D trajectory of the parafoil in terminal descent starting from the position (0,0,0). The simulated parachute system exhibits relatively strong dynamics. The linear acceleration $\dot{V}$ is non-zero from (0,0,0) to (0,-200,-200).

UAV also experienced significant variations in the velocity, partially invalidating one of the hypotheses of the model in Eq.(12), namely the assumption that the linear acceleration is negligible i.e, $\dot{V} = 0$ (Figure 3). It would therefore be interesting to investigate the effects of the error introduced into the estimation process by assuming that $\dot{V} = 0$. Analysis of the simulated data shows that $\dot{V}$ was non-zero throughout the period $t \in [5\,;40]$.



(a) Independent white gaussian noises.



(b) Typical attitude anlges (ϕ, θ, ψ) in the ideal noise-free case.

Fig. 4: We illustrate a typical realization of the gaussian white noise processes with zero mean which perturbate the nonlinear state estimation problem to solve. Then, attitudes angles are plotted on the same subfigure. The roll, heading, and pitch angles vary by up to several dozen degrees.

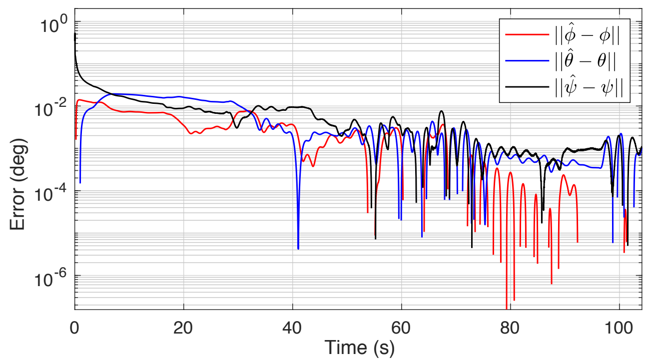
B. Noise-Free Simulations

1) Attitude estimation: To emphase the effect of the parametrization (**Feature #1**) on algorithms performance, we first assume that the sensors are perfect (see **Assumption 1.**), i.e., without noise. Figure 5 shows the estimated attitude errors computed by the UKF, IUKF$_{\boldsymbol{\mathcal{X}}}$ and IUKF$_{\mathbf{x}}$ algorithms. Each estimate is compared against the pseudo-measurements reconstructed from the components of the reference quaternion state vector. The estimated angles match the reference values almost perfectly. The attitude of the parafoil is correctly reconstruct with respect to all three axes. Note that the error introduced into the initial state of the simulation was corrected very rapidly, after only a few computation steps (characteristic time < 0.5 sec.). However, the estimation errors (with a log scale along the vertical axis) show that the IUKF estimator converges more closely and quickly to the true values of the flight parameters. The IUKF$_{\mathbf{x}}$ achieves smaller estimation errors than the UKF and IUKF$_{\boldsymbol{\mathcal{X}}}$. Additionally, the comparison of these error plots suggests that the residuals of the state estimate constructed by IUKF$_{\mathbf{x}}$ appear to be more stable over time; a slight albeit slow decrease of these residuals may be observed in the results generated by the UKF algorithm due to $\dot{V} \neq 0$ throughout the period $t \in [5\,;40]$. The estimates of the new IUKF parametrization therefore outperforms the standard variant of the UKF algorithm.

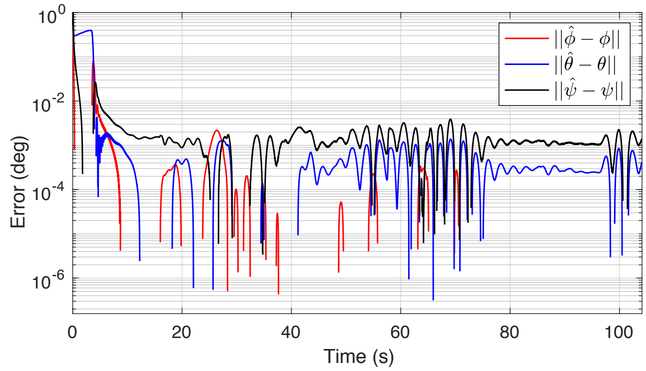
2) Invariance properties: Although we have established that IUKF$_{\mathbf{x}}$ behaves well in terms of reconstructing the state, we still need to verify that it has the same invariance properties as the IEKF algorithm (**Feature #2**, see section III-C for details). We therefore need to check the symmetry-preserving framework and hence that the estimated quaternion $\hat{q}(t) = (q_0(t), q_1(t), q_2(t), q_3(t))$ are now independent of the trajectory, despite the presence of non-linear dynamics. The initial state $\hat{\mathbf{x}}_0$ was varied through $\alpha \in (3; 6; 12)$ for the three algorithms, while keeping the norm of the initial state estimation error constant at all times. This gives the following quaternions:

	True system	Initial state
q_0	0.99	$\cos(\pi/\alpha)$
q_1	0	$\sin(\pi/\alpha)/\sqrt{3}$
q_2	-0.0103	-$\sin(\pi/\alpha)/\sqrt{3}$
q_3	0	$\sin(\pi/\alpha)/\sqrt{3}$

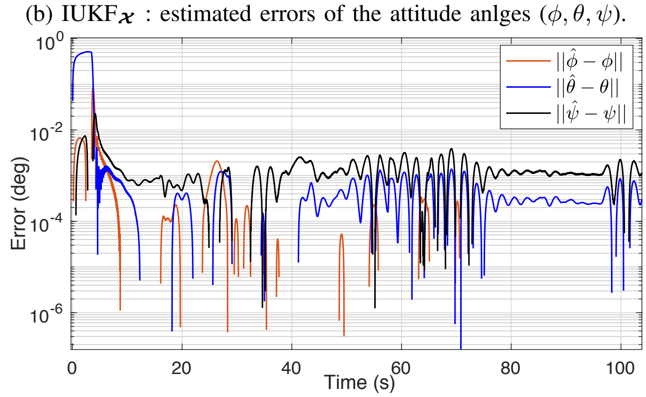
In these calculations, the initial state $\hat{\mathbf{x}}_0$ was varied for both algorithms UKF and IUKF$_{\mathbf{x}}$, while keeping the norm of the initial state estimation error constant at all times. Figures 6 plot the norm of the estimation errors over time for each of the states q, ω_b, a_s, and b_s. The results reveal the invariance properties of the IUKF$_{\mathbf{x}}$ algorithm; its behavior with regard to estimation errors is globally the same regardless of the estimate chosen for the initial condition, unlike the classical UKF algorithm. On every trajectory generated by varying the initial conditions, the dynamics of the convergence of the IUKF$_{\mathbf{x}}$ estimate to the true state were identical. We then run 500 Monte-Carlo simulations for different initial state on the pitch and roll in order to demonstrate the convergence boundarie (i.e limit up to instability case) for both algorithms. Figure 7 shows that the convergence boundarie of the IUKF$_{\mathbf{x}}$ is larger than the UKF which is extremely useful for SPQR context. We illustrate in Figure 8 convergence results obtained by applying each of two UKF algorithms to the AHRS for the same initial condition. Considering all of the results established above, it seems reasonable to conclude that the invariant unscented Kalman filter IUKF$_{\mathbf{x}}$ proposed in this article is capable of accurately estimate the state, in a similar spirit to the various other invariant filters that can already be found in the literature. The relatively constant nature of the invariant state error makes it an extremely valuable source of information that could be exploited by various other problems (robust controls, fault detection,...).



(a) UKF : estimated errors of the attitude anlges (ϕ, θ, ψ).



(b) IUKF$_{\boldsymbol{x}}$: estimated errors of the attitude anlges (ϕ, θ, ψ).

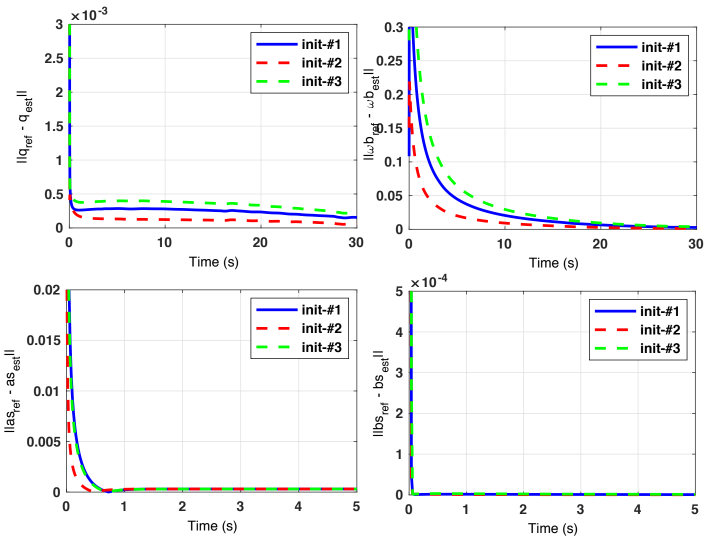


(c) IUKF$_{\mathbf{x}}$: estimated errors of the attitude anlges (ϕ, θ, ψ).

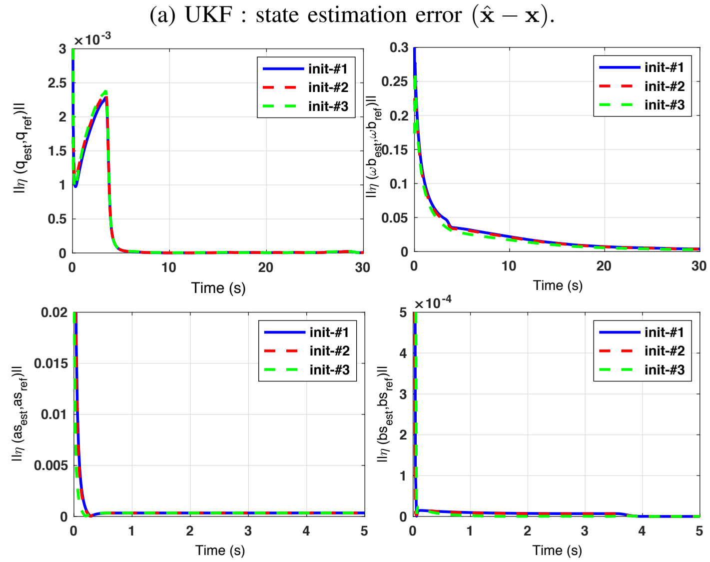
Fig. 5: Attitude estimation errors with wrong initial angles and $\dot{V} \neq 0$ throughout the period $t \in [5\,;40]$. The plots show that the IUKF$_{\mathbf{x}}$ estimator converge more closely and quikly to the true values of the flight parameters.

C. Measurement Noise

We now study the impact of the measurement noise on the algorithms performances. Here, a series of additive noise terms were incorporated into the reference simulation as perturbations of the measurements $\boldsymbol{\omega}_m$, $\boldsymbol{y_{Am}}$, and $\boldsymbol{y_{Bm}}$. The experimental data are sampled with a frequency equal to 50Hz which characterized the inertial measurement unit and the magnetometers. The measurements are corrupted by gaussian white noises whose standard deviations are set to : $\sigma_{\text{gyro}} = 0.2\ ^\circ/s$, $\sigma_{\text{accelero}} = 0.2\text{g}$ and $\sigma_{\text{magneto}} = 500\text{nT}$. In order to validate our filters, we have also introduced a biases vector on $\boldsymbol{\omega}_m$ s.t. $\boldsymbol{\omega}_b = [0.1 \text{ rad/s } 0.05 \text{ rad/s } 0.02 \text{ rad/s}]^T$. The two positive scalar factor values are set to $a_s = 1.2$ and $b_s = 0.9$ re-



(a) UKF : state estimation error $(\hat{\mathbf{x}} - \mathbf{x})$.



(b) IUKF$_{\mathbf{x}}$: state estimation error $\boldsymbol{\eta}(\hat{\mathbf{x}}, \mathbf{x})$. The only difference relative to (a) is the choice of initial conditions.

Fig. 6: Linear state estimation error computed by UKF, as well as the invariant state estimation computed by IUKF$_{\mathbf{x}}$.

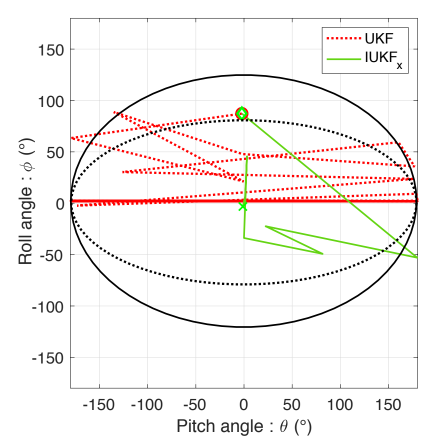


Fig. 7: Convergence boundaries on Roll and Pitch for UKF (black dashed dot line) and IUKF$_{\mathbf{x}}$ (black line). The convergence boundarie of the IUKF$_{\mathbf{x}}$ is larger than the UKF.

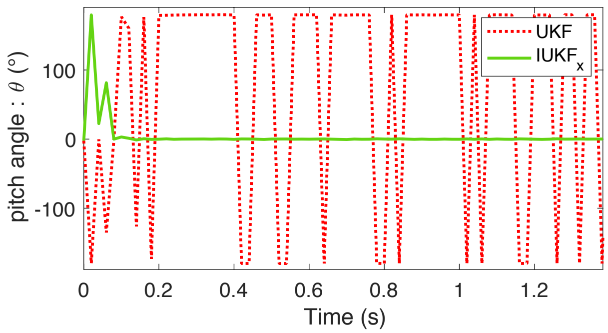
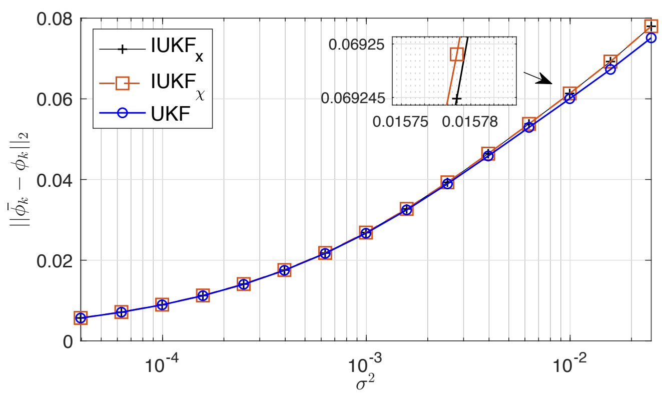
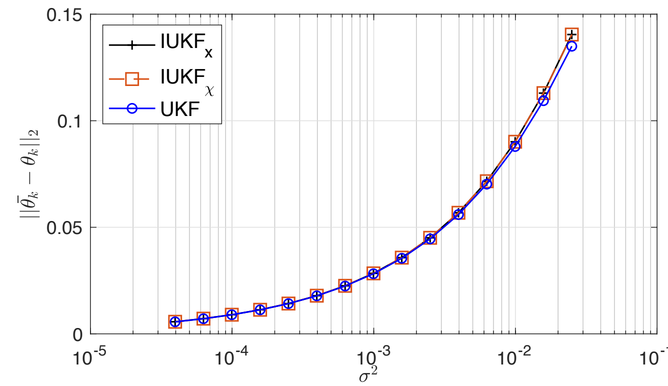


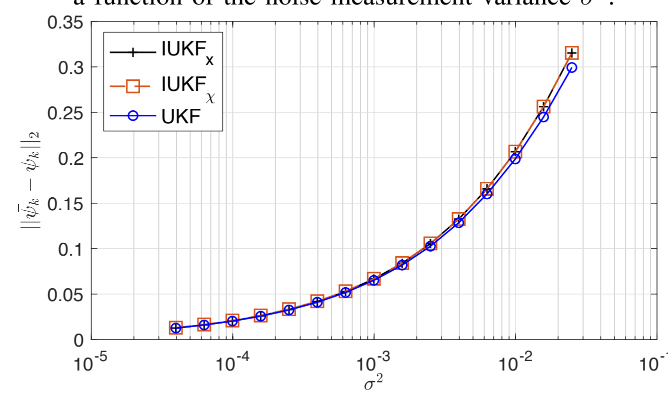
Fig. 8: Attitude estimation on roll angle with initial state outside the UKF convergence boundarie. The IUKF$_{\mathbf{x}}$ is stable, the UKF is unstable.



(a) Monte-Carlo average of the Root Mean Square Error on $(\phi_k)_{1\leqslant k\leqslant N}$ over the whole trajectory, as a function of the noise measurement variance σ^2.



(b) Monte-Carlo average of the Root Mean Square Error on $(\theta_k)_{1\leqslant k\leqslant N}$ over the whole trajectory, as a function of the noise measurement variance σ^2.



(c) Monte-Carlo average of the Root Mean Square Error on $(\psi_k)_{1\leqslant k\leqslant N}$ over the whole trajectory, as a function of the noise measurement variance σ^2.

Fig. 9: Noisy case - comparison of the average Root Mean Square Error of IUKF$_{\mathbf{x}}$,IUKF$_{\boldsymbol{\chi}}$ and UKF. The IUKF$_{\mathbf{x}}$ is sightly better than IUKF$_{\boldsymbol{\chi}}$ algorithm but the UKF is sightly better than IUKF when noise is moderate.

spectively. Initially, we set for the filters with incorrect angles $\bar{\phi} \sim \mathcal{N}\Big(0, (\pi/4)^2\Big), \bar{\theta} \sim \mathcal{N}\Big(0, (\pi/4)^2\Big), \bar{\psi} \sim \mathcal{N}\Big(0, (\pi/2)^2\Big)$. We then run 500 Monte-Carlo simulations for different levels of measurement noise $\sigma^2_{\text{accelero}} = [4.10^{-4}, 3.10^{-2}]$ and compare the (average) Root Mean Square Errors w.r.t the reference values over the whole trajectory. Figures 9 compare the results produced by the IUKF$_{\mathbf{x}}$,IUKF$_{\boldsymbol{\chi}}$ and UKF algorithms. For this trajectory, we see that IUKF$_{\mathbf{x}}$ is sightly better than IUKF$_{\boldsymbol{\chi}}$. The UKF is sightly better than IUKF when noise is moderate.

D. Claimed reduction of computational burned of the IUKF

In case of a real time embedded application, another interesting filter characteristic is the computational effort (**Feature #3**). In section IV, we claimed the reduction of the computational complexity of the IUKF parametrize by the state ($\mathbf{x}_k$). A computational analysis consist in computing the average time calculation using 500 Monte-Carlo simulation using an Intel Core I7 2.Ghz. We thus see in Figure 10 that the computation time of the IUKF is higher ($\sim$19.5%) than UKF algorithm due to additional operation on invariant state error (See section I-D). But results clearly reveal that the IUKF parametrized by the state ($\mathbf{x}_k$) is more computation time efficient ($\sim$7%) on *Update step* and ($\sim$13%) on *Output errors* (See **Algorithm 2**) than IUKF parametrized by sigma-point.

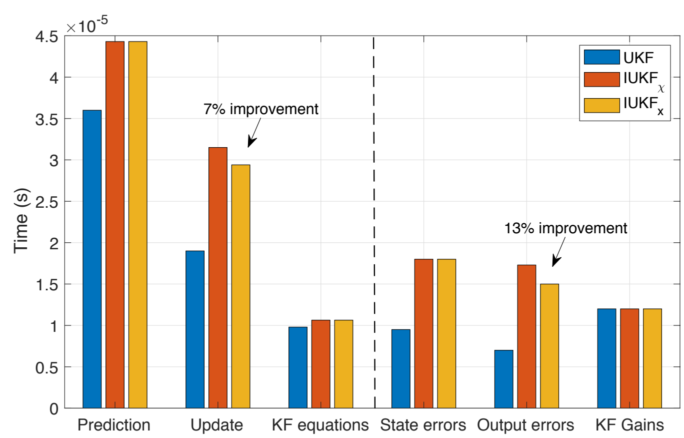


Fig. 10: Comparison of the Computation effort for UKF, IUKF$_{\boldsymbol{\chi}}$ and IUKF$_{\mathbf{x}}$. The IUKF$_{\mathbf{x}}$ is more computation time efficient ($\sim$7%) on *Update step* and ($\sim$13%) on *Output errors*

VI. Conclusion and Discussion

We presented in this article a new formulation of invariant unscented Kalman filtering for attitude estimation. These latter, named $\text{IUKF}_{\mathbf{x}}$, combines both invariant observers and the nonlinear unscented Kalman filtering theories while reducing the computational cost of the standard IUKF. Its methodological foundation, which forms the main contribution of this article, consists in adapting the computational steps of the IUKF technique to the attitude estimation problem by i) investigate the IUKF detailed nonlinear equation, ii) define an invariant state estimation error to update through time all covariance matices and iii) study transformation group parametrization on output errors.

In comparison with the state-of-the-art, our proposed $\text{IUKF}_{\mathbf{x}}$ nonlinear state estimation algorithm presents one main advantage when considering computational aspects, avoiding to find $4n^2 + 2n$ invariant state errors in the predicted step of the IUKF algorithm. The simulation results presented in Section V have demonstrated that this specific formulation can reduce the computation complexity without compromising the stability and precision of the filter. At this stage some interesting (but still preliminary) conclusions on the proposed formulation for AHRS are :

- As discussed in Section II, using an invariant state errors, covariances are left unchanged by dynamical systems'symmetries. This confers to gains some properties of invariance which leads, by transitivity, to design an IUKF symmetry-preserving state observer in the same way that [33]. Stability of estimated standard deviation from these covariances, which characterize estimated state trajectory uncertainties, could facilitate new control strategies design with less conservatism;
- A large convergence boundarie must be highlighted since this proof of stability could be useful for the SPQR context, in particular during the step of Atmospheric reentry where angles of the parafoil could vary by up to hundred degrees;
- The proposed formulation is very simple to implement and induces a very limited computational burden compared to the standard IUKF.

Finally, we have shown an equivalent capability of our proposed $\text{IUKF}_{\mathbf{x}}$ formulation in comparison with a standard IUKF method for invariant nonlinear state estimation. Future work will include performance test on embedded micro-controllers for drone navigation and SLAM.

ACKNOWLEDGMENT

The authors would like to thank Dr. Miquel from ENAC, the french aviation school, ENAC Lab and Dr. Seren from ONERA, the French Aerospace Lab, Information Processing and Systems Department, Toulouse, France for their evaluable helps about parametric formulation.

APPENDIX A
ILLUSTRATIONS

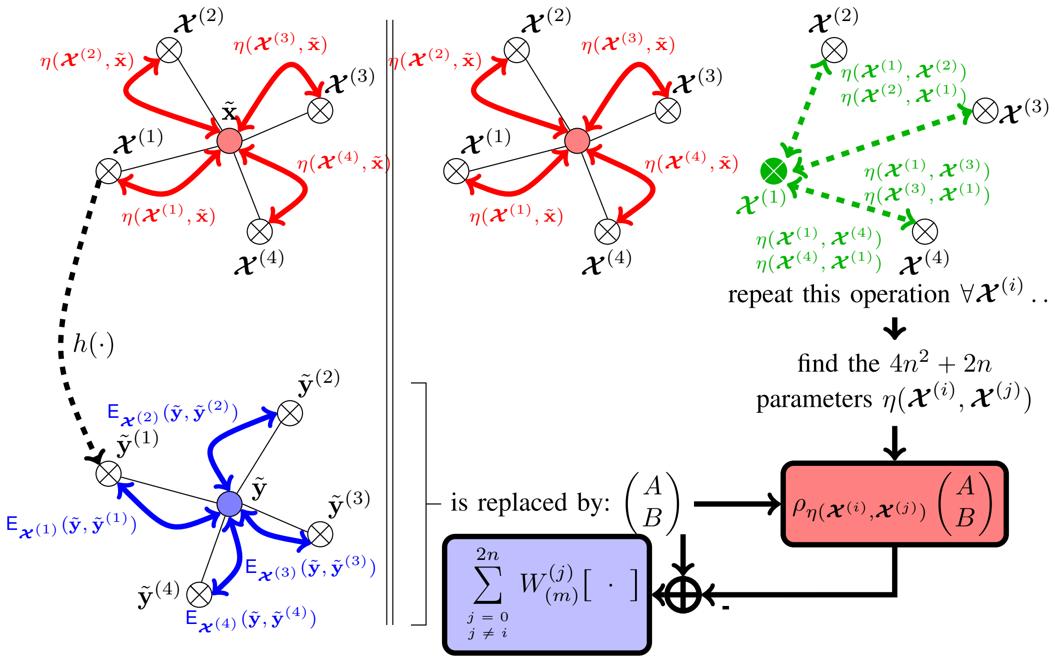


Fig. 11: *Comparison of standard IUKF and an applied version of IUKF specifically adapted to the AHRS, parametrized by the sigma points* $\{\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(i)})/i \in [\![\, 0\,; 2n \,]\!]\}$

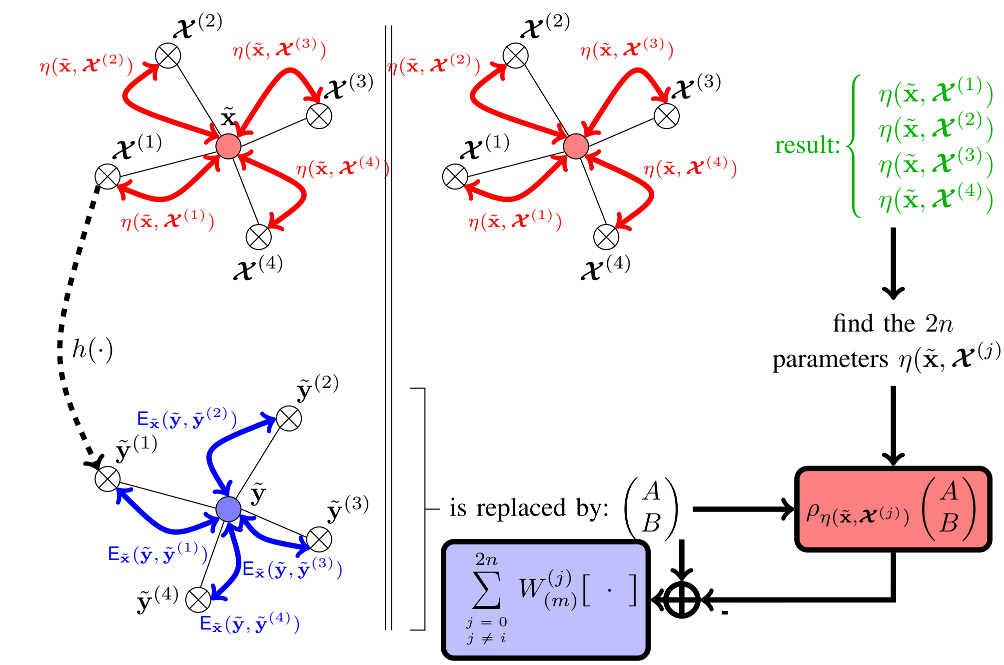


Fig. 12: *Comparison of standard IUKF and an applied variant of IUKF specifically adapted to the AHRS, parametrized by the predicted state* $\hat{\mathbf{x}}_{k+1|k} = \sum_{j \in [\![\, 0\,; 2n \,]\!]} W_{(m)}^{(j)} \, \boldsymbol{\mathcal{X}}_{k+1|k}^{(j)}$

APPENDIX B
UNSCENTED TRANSFROM

The *unscented transform* [41] can be used for forming a Gaussian approximation to the joint distribution of random variable $\mathbf{x}_{k|k}$ and $\mathbf{y}_{k|k}$, when the random variable $\mathbf{y}_{k|k}$ is obtained by a non-linear transformation of the Gaussian random variable $\mathbf{x}_{k|k}$ as follow :

$$\begin{cases}\mathbf{x}_{k|k} \sim N(\hat{\mathbf{x}}_{k|k}, \mathbf{P}_{k|k}) \\ \mathbf{y}_{k|k} = \gamma(\mathbf{x}_{k|k}, k)\end{cases} \tag{23}$$

The aims of the basic UT is to form a fixed number of deterministically chosen sigma-points $\boldsymbol{\mathcal{X}}_{k|k}$, which capture the "true" mean $\hat{\mathbf{x}}_{k|k}$ and covariance $\mathbf{P}_{k|k}$ of the original distribution $\mathbf{x}_{k|k}$. This set of points must represent accurately the first and second order moments. These sigma-points are then propagated through the nonlinear functions Eq.(23) providing a cloud of evolving points. The mean $\hat{\mathbf{x}}_{k|k+1}$ and estimated covariance matrix $\mathbf{P}_{k|k+1}$ of the transformed points are then computed based on their statistics. The unscented transform can be used for forming

$$\begin{pmatrix}\mathbf{x}_{k|k} \\ \mathbf{y}_{k|k}\end{pmatrix} \sim N\left(\begin{pmatrix}\hat{\mathbf{x}}_{k|k} \\ \hat{\mathbf{y}}_{k+1|k}\end{pmatrix}, \begin{pmatrix}\mathbf{P}^{\mathbf{x}}_{k|k} & \mathbf{P}^{\mathbf{xy}}_{k|k} \\ \mathbf{P}^{\mathbf{yx}}_{k|k} & \mathbf{P}^{\mathbf{y}}_{k|k}\end{pmatrix}\right) \tag{24}$$

to the joint probability density of $\mathbf{x}_{k|k} \in \mathbb{R}^n$ and $\mathbf{y}_{k|k} \in \mathbb{R}^m$. The unscented transform is the following :

1) Form the set of $2n+1$ sigma points from the coloms of the $n \times n$ matrix $(\sqrt{n+\lambda})P_{k|k}$ as follows:

$$\begin{cases}\boldsymbol{\mathcal{X}}^{(0)}_{k|k} = \mathbf{x}_{k|k} \\ \boldsymbol{\mathcal{X}}^{(i)}_{k|k} = \mathbf{x}_{k|k} + \left[\sqrt{(n+\lambda)\mathbf{P}^{\mathbf{x}}_{k|k}}\right], i = 1, \cdots, n \\ \boldsymbol{\mathcal{X}}^{(i)}_{k|k} = \mathbf{x}_{k|k} - \left[\sqrt{(n+\lambda)\mathbf{P}^{\mathbf{x}}_{k|k}}\right], i = n+1, \cdots, 2n\end{cases} \tag{25}$$

and compute the associated weights $W^{(0)}_m$,$W^{(O)}_c$,$W^{(i)}_m$,$W^{(i)}_c$.[5]

2) Transform each of sigma as

$$\mathbf{y}^{(i)}_{k+1|k} = \gamma(\boldsymbol{\mathcal{X}}^{(i)}_{k|k}), i = 0, \cdots, 2n. \tag{26}$$

3) Mean and covariance estimates for $\mathbf{y}_{k+1|k}$ can be computed as

$$\hat{\mathbf{y}}_{k+1|k} \approx \sum_{i=0}^{2n} W^{(i)}_m \hat{\mathbf{y}}^{(i)}_{k+1|k} \tag{27}$$

$$\mathbf{P}^{\mathbf{y}}_{k+1|k} \approx \sum_{i=0}^{2n} W^{(i)}_c (\hat{\mathbf{y}}^{(i)}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k})(\hat{\mathbf{y}}^{(i)}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k})^T \tag{28}$$

[5] $W^{(0)}_m = \lambda/(n+\lambda), W^{(O)}_c = \lambda/(n+\lambda) + (1-\alpha^2+\beta), W^{(i)}_m = 1/\{2(n+\lambda)\}, i = 1, \cdots, 2n, W^{(i)}_c = 1/\{2(n+\lambda)\}, i = 1, \cdots, 2n$. The parameter λ is a scaling parameter defined as $\lambda = \alpha^2(n+\kappa)-n$. The positive constants α, β and κ are used as parameters of the method. We set uncented transform parameters to $\kappa = 0$ and $\beta = 2$. α keeps a free-parameter chosen by the practitioner, which must be small ($\alpha = 10^{-3}$ in our applications).

APPENDIX C
DERIVATIONS

A. Derivation of the invariant state error form of UT

Let's consider a group action, full-rank and transitive (i.e. $\mathsf{dim}(G) = \mathsf{dim}(\mathcal{X}) = n$). G can be identified with the state space $\mathcal{X} = \mathbb{R}^n$ in such a way that the local transformation on the state $\varphi_{\mathbf{g}}$ is viewed as the left or right-multiplication mapping $\varphi_{\mathbf{g}}(\mathbf{x}) = \mathbf{g} \cdot \mathbf{x}$. Solving the normalization equations to obtain $\varphi_{\mathbf{g}}(\mathbf{x}) = \mathbf{g} \cdot \mathbf{x} = e$ from Cartan moving frame method, where e is the identity element of the group G, gives us the moving frame $\gamma(\mathbf{x}) = \mathbf{x}^{-1}$ as a solution [42]. If we define the invariant state error on Lie group G such as

$$\boldsymbol{\eta}(\mathbf{x}_{k+1}, \hat{\mathbf{x}}_{k+1|k}) = \mathbf{x}^{-1}_{k+1}.\hat{\mathbf{x}}_{k+1|k} \tag{29}$$

then the unscented transform in Eq.(3) can be written in form of the last equation in Eq.(7).

$$\begin{aligned}
\boldsymbol{\mathcal{X}}_{k+1|k} &= [\boldsymbol{\mathcal{X}}^{(0)}_{k+1|k} \ \boldsymbol{\mathcal{X}}^{(1)}_{k+1|k} \ \dots \ \boldsymbol{\mathcal{X}}^{(2n)}_{k+1|k}] = f(\boldsymbol{\mathcal{X}}_{k|k}, \mathbf{u}_k) \\
\hat{\mathbf{x}}_{k+1|k} &\approx \sum_{i=0}^{2n} W^{(i)}_{(m)} \boldsymbol{\mathcal{X}}^{(i)}_{k+1|k} \\
\mathbf{P}^{\mathbf{x}}_{k+1|k} &\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} (\varphi_{\boldsymbol{\mathcal{X}}^{(i)-1}_{k+1|k}}(\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}) - \varphi_{\boldsymbol{\mathcal{X}}^{(i)-1}_{k+1|k}}(\hat{\mathbf{x}}_{k+1|k})) \\
&\times \ (\varphi_{\boldsymbol{\mathcal{X}}^{(i)-1}_{k+1|k}}(\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}) - \varphi_{\boldsymbol{\mathcal{X}}^{(i)-1}_{k+1|k}}(\hat{\mathbf{x}}_{k+1|k}))^T \\
&\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} \eta(\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}) \eta^T(\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}) \\
&\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} (\boldsymbol{\mathcal{X}}^{(i)^{-1}}_{k+1|k} \cdot \hat{\mathbf{x}}_{k+1|k})(\boldsymbol{\mathcal{X}}^{(i)^{-1}}_{k+1|k} \cdot \hat{\mathbf{x}}_{k+1|k})^T
\end{aligned}$$

which leads to last equation in Lemma 1.

B. Derivation of the invariant output error form of UT

If we define the invariant output error using the Lie group $\forall \mathbf{g} \in G$ such as

$$\mathsf{E}(\hat{\mathbf{y}}_{k+1}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) = \rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k}) \tag{30}$$

then the unscented transform in Eq.(4) can be written in form of the last equation in Eq.(8).

$$\begin{aligned}
\hat{\mathbf{Y}}_{k+1|k} &= [\hat{\mathbf{y}}^{(0)}_{k+1|k} \ \hat{\mathbf{y}}^{(1)}_{k+1|k} \ \dots \ \hat{\mathbf{y}}^{(2n)}_{k+1|k}] = h(\boldsymbol{\mathcal{X}}_{k+1|k}, \mathbf{u}_k) \\
\hat{\mathbf{y}}_{k+1|k} &\approx \sum_{i=0}^{2n} W^{(i)}_{(m)} \hat{\mathbf{y}}^{(i)}_{k+1|k} \\
\mathbf{P}^{\mathbf{y}}_{k+1|k} &\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} (\rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k})) \\
&\times \ (\rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k}))^T \\
&\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} \mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \mathbf{y}^{(i)}_{k+1|k}) \\
&\times \ \mathsf{E}^T(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \mathbf{y}^{(i)}_{k+1|k})
\end{aligned}$$

which leads to last equation in Lemma 2.

Appendix D
Proofs of the results of section V

A. Proof of Theorem 1

Let $\mathbf{g_0} = {\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}}^{-1}$, we have :

$$\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}} = \rho_{\boldsymbol{\mathcal{X}}^{(i)-1}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\boldsymbol{\mathcal{X}}^{(i)-1}}(\hat{\mathbf{y}}^{(i)}_{k+1|k})$$
$$= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \begin{pmatrix} a^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}}\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \hat{\mathbf{y}}_{\mathbf{A}_{k+1|k}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \\ b^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}}\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \hat{\mathbf{y}}_{\mathbf{B}_{k+1|k}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \cdots$$
$$- \begin{pmatrix} a^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}}\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \left(\sum_{j=0}^{2n} W^{(j)}_{(m)} h_{\mathbf{A}}(\boldsymbol{\mathcal{X}}^{(j)})\right) * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \\ b^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}}\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \left(\sum_{j=0}^{2n} W^{(j)}_{(m)} h_{\mathbf{B}}(\boldsymbol{\mathcal{X}}^{(j)})\right) * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \end{pmatrix}$$

Where $h_{\mathbf{A}}(\cdot)$ and $h_{\mathbf{B}}(\cdot)$ denotes the restriction of the observation model to the outputs associated with the acceleration and the magnetic field respectively.

$$\begin{aligned}
\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}} &= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)} \cdots \\
&\times \begin{pmatrix} a^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}}\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \left(a_{s,\boldsymbol{\mathcal{X}}^{(j)}}\mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{A} * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}}\right) * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \\ b^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}}\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \left(b_{s,\boldsymbol{\mathcal{X}}^{(j)}}\mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{B} * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}}\right) * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \end{pmatrix} \\
&= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)} \cdots && (31)\\
&\times \begin{pmatrix} a^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} a_{s,\boldsymbol{\mathcal{X}}^{(j)}} \cdot (\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}})^{-1} * \ldots \\ b^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} b_{s,\boldsymbol{\mathcal{X}}^{(j)}} \cdot (\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}})^{-1} * \ldots \end{matrix} && (32)\\
&\qquad \begin{matrix} \mathbf{A} * (\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}}) \\ \mathbf{B} * (\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}}) \end{pmatrix} && (33)\\
&= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\eta(\boldsymbol{\mathcal{X}}^{(i)},\boldsymbol{\mathcal{X}}^{(j)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}
\end{aligned}$$

We have

$$\sum_{j=0}^{2n} W^{(j)}_{(m)} = 1.$$

Hence

$$\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}} = \sum_{j=0}^{2n} W^{(j)}_{(m)} \left[\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \rho_{\eta(\boldsymbol{\mathcal{X}}^{(i)},\boldsymbol{\mathcal{X}}^{(j)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\right]. \qquad (34)$$

Moreover

$$\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \rho_{\eta(\boldsymbol{\mathcal{X}}^{(i)},\boldsymbol{\mathcal{X}}^{(i)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \rho_{\vec{0}}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} = \vec{0}$$

Equation (34) can be expressed in the case where $j \neq i$, which concludes the proof with $\forall i \in [\![0 ; 2n]\!]$:

$$\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}} = \sum_{\substack{j=0\\ j\neq i}}^{2n} W^{(j)}_{(m)} \left[\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \rho_{\eta(\boldsymbol{\mathcal{X}}^{(i)},\boldsymbol{\mathcal{X}}^{(j)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\right].$$

B. Proof of Theorem 2

Let $\mathbf{g_0} = \hat{\mathbf{x}}^{-1}_{k+1|k}$. We have

$$\begin{aligned}
\hat{\mathsf{E}}_{\hat{\mathbf{x}}} &= \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\hat{\mathbf{y}}^{(i)}_{k+1|k}) \\
&= \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(h_{\mathbf{A},\mathbf{B}}(\boldsymbol{\mathcal{X}}^{(i)})) - \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\hat{\mathbf{y}}_{k+1|k}) \\
&= \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(h_{\mathbf{A},\mathbf{B}}(\boldsymbol{\mathcal{X}}^{(i)})) - \cdots && (35)\\
&\qquad \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \\
&= h_{\mathbf{A},\mathbf{B}}(\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(i)})) - \cdots && (36)\\
&\qquad \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}
\end{aligned}$$

With $h_{\mathbf{A},\mathbf{B}}(\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(i)}))$, we have :

$$\underbrace{\begin{pmatrix} a_{s,\boldsymbol{\mathcal{X}}^{(i)}} \cdot \hat{a}^{-1}_{s,k+1|k}\hat{\mathbf{q}}_{k+1|k} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} * \mathbf{A} * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \hat{\mathbf{q}}^{-1}_{k+1|k} \\ b_{s,\boldsymbol{\mathcal{X}}^{(i)}} \hat{b}^{-1}_{s,k+1|k} \cdot \hat{\mathbf{q}}_{k+1|k} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} * \mathbf{B} * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \hat{\mathbf{q}}^{-1}_{k+1|k} \end{pmatrix}}_{= \quad \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(i)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}}. \qquad (37)$$

The claimed predicted invariant output error can be expressed as a weighted sum of invariant output errors such as :

$$\hat{\mathsf{E}}_{\hat{\mathbf{x}}} = \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(i)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}.$$

C. Derivation of the IUKF for the AHRS

As shown in Section V, predicted state parametrization $\hat{\mathsf{E}}_{\mathbf{x}}$ avoids the need to find $4n^2 + 2n$ invariant state errors. The invariant unscented Kalman filter equations of the AHRS problem can be derived as follows.

$$\begin{aligned}
\hat{\mathbf{x}}_{k+1|k+1} &= \hat{\mathbf{x}}_{k+1|k} + \sum_{i=1}^{n} \mathbf{K}^{(i)}_{k+1} \cdots \\
&\times\ \mathsf{E}(\mathbf{y}_{k+1}, \hat{\mathbf{x}}_{k+1|k}, \hat{\mathbf{y}}_{k+1|k}) \cdot w_i(\hat{\mathbf{x}}_{k+1|k}) \\
&= \hat{\mathbf{x}}_{k+1|k} + \sum_{i=1}^{n} \mathbf{K}^{(i)}_{k+1}\big(\rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\mathbf{y}_{k+1}) \cdots \\
&\quad -\rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\hat{\mathbf{y}}_{k+1|k})\big) \cdot w_i(\hat{\mathbf{x}}_{k+1|k}) \\
&= \hat{\mathbf{x}}_{k+1|k} + \sum_{i=1}^{n} \mathbf{K}^{(i)}_{k+1}\Bigg(\rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\mathbf{y}_{k+1}) \cdots \\
&\quad - \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\Bigg) w_i(\hat{\mathbf{x}}_{k+1|k}) \\
&= \hat{\mathbf{x}}_{k+1|k} + \sum_{i=1}^{n} \mathbf{K}^{(i)}_{k+1} \sum_{j=0}^{2n} W^{(j)}_{(m)}\Bigg(\rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\mathbf{y}_{k+1}) \\
&\quad -\rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\Bigg) w_i(\hat{\mathbf{x}}_{k+1|k})
\end{aligned}$$

At each time step, the estimated state is therefore computed in the form of a correction of the prediction derived from the left or right-invariant dynamics of the system expressed as a weighted sum of invariant innovation terms over the Lie group parametrized directly or indirectly by $\hat{\mathbf{x}}_{k+1|k}$.

$$
\begin{aligned}
\hat{\mathbf{x}}_{k+1|k+1} = {} & \hat{\mathbf{x}}_{k+1|k} + \sum_{j=0}^{2n} W_{(m)}^{(j)} \sum_{i=1}^{n} \mathbf{K}_{k+1}^{(i)} \Bigg(\rho_{\hat{\mathbf{x}}_{k+1|k}^{-1}}(\mathbf{y}_{k+1}) \cdots \\
& -\rho_{\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix} A \\ B \end{pmatrix} \Bigg) \cdot w_i(\hat{\mathbf{x}}_{k+1|k}) \\
= {} & \sum_{j=0}^{2n} W_{(m)}^{(j)} \Bigg[\boldsymbol{\mathcal{X}}_{k+1|k}^{(j)} + \sum_{i=1}^{n} \mathbf{K}_{k+1}^{(i)} \Bigg(\rho_{\hat{\mathbf{x}}_{k+1|k}^{-1}}(\mathbf{y}_{k+1}) \\
& -\rho_{\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix} A \\ B \end{pmatrix} \Bigg) \cdot w_i(\hat{\mathbf{x}}_{k+1|k}) \Bigg]
\end{aligned}
$$

D. Derivation of the detailed IUKF equation for the AHRS

The discrete-time invariant unscented Kalman filter equations can be derived for the special case of an AHRS.

$$
\begin{aligned}
\hat{q}_{k+1|k+1} \quad = {} & \quad \tilde{q} + \sum_{j=0}^{2n} W_{(m)}^{(j)} \sum_{i=1}^{4} \mathbf{K}_{k+1}^{(i)} \cdots \\
\times {} & \quad \left(\rho_{\hat{\mathbf{x}}_{k+1|k}^{-1}}(\mathbf{y}_{k+1}) - \rho_{\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix} A \\ B \end{pmatrix} \right) * \tilde{q} \\
= {} & \quad \tilde{q} + \sum_{j=0}^{2n} W_{(m)}^{(j)} \sum_{i=1}^{4} \mathbf{K}_{k+1}^{(i)} \cdots \\
\times {} & \quad \begin{pmatrix} \big[\tilde{a}_s^{-1}\tilde{q} * \mathbf{y}_{k+1}^{1:3} * \tilde{q}^{-1} - \ldots \\ \big[\tilde{b}_s^{-1}\tilde{q} * \mathbf{y}_{k+1}^{4:6} * \tilde{q}^{-1} - \ldots \\ a_{s,\boldsymbol{\mathcal{X}}^{(j)}}\tilde{a}_s^{-1}\tilde{q} * q_{\boldsymbol{\mathcal{X}}^{(j)}}^{-1} * A * q_{\boldsymbol{\mathcal{X}}^{(j)}} * \tilde{q}^{-1}\big] * \tilde{q} \\ b_{s,\boldsymbol{\mathcal{X}}^{(j)}}\tilde{b}_s^{-1}\tilde{q} * q_{\boldsymbol{\mathcal{X}}^{(j)}}^{-1} * B * q_{\boldsymbol{\mathcal{X}}^{(j)}} * \tilde{q}^{-1}\big] * \tilde{q} \end{pmatrix} \\
= {} & \quad \tilde{q} + \sum_{j=0}^{2n} W_{(m)}^{(j)} \sum_{i=1}^{4} \mathbf{K}_{k+1}^{(i)} \cdots \\
\times {} & \quad \begin{pmatrix} \tilde{a}_s^{-1}\tilde{q} * \big[\mathbf{y}_{k+1}^{1:3} - a_{s,\boldsymbol{\mathcal{X}}^{(j)}} q_{\boldsymbol{\mathcal{X}}^{(j)}}^{-1} * A * q_{\boldsymbol{\mathcal{X}}^{(j)}}\big] \\ \tilde{b}_s^{-1}\tilde{q} * \big[\mathbf{y}_{k+1}^{4:6} - b_{s,\boldsymbol{\mathcal{X}}^{(j)}} q_{\boldsymbol{\mathcal{X}}^{(j)}}^{-1} * B * q_{\boldsymbol{\mathcal{X}}^{(j)}}\big] \end{pmatrix} \\
= {} & \quad \tilde{q} + \sum_{j=0}^{2n} W_{(m)}^{(j)} \sum_{i=1}^{4} \mathbf{K}_{k+1}^{(i)} \cdots \\
\times {} & \quad \begin{pmatrix} \tilde{a}_s^{-1}\tilde{q} * \big[\mathbf{y}_{k+1}^{1:3} - h^{1:3}(\boldsymbol{\mathcal{X}}^{(j)}, \mathbf{u}_k)\big] \\ \tilde{b}_s^{-1}\tilde{q} * \big[\mathbf{y}_{k+1}^{4:6} - h^{4:6}(\boldsymbol{\mathcal{X}}^{(j)}, \mathbf{u}_k)\big] \end{pmatrix} \\
= {} & \quad \tilde{q} + \sum_{i=1}^{4} \mathbf{K}_{k+1}^{(i)} \cdots \\
\times {} & \quad \begin{pmatrix} \tilde{a}_s^{-1}\tilde{q} * \sum_{j=0}^{2n} W_{(m)}^{(j)} \big[\mathbf{y}_{k+1}^{1:3} - h^{1:3}(\boldsymbol{\mathcal{X}}^{(j)}, \mathbf{u}_k)\big] \\ \tilde{b}_s^{-1}\tilde{q} * \sum_{j=0}^{2n} W_{(m)}^{(j)} \big[\mathbf{y}_{k+1}^{4:6} - h^{4:6}(\boldsymbol{\mathcal{X}}^{(j)}, \mathbf{u}_k)\big] \end{pmatrix} \\
\hat{q}_{k+1|k+1} \quad = {} & \quad \hat{q}_{k+1|k} + \sum_{i=1}^{4} \mathbf{K}_{k+1}^{(i)} \cdots \\
\times {} & \quad \begin{pmatrix} \hat{a}_{s,k+1|k}^{-1}\hat{q}_{k+1|k} * \big[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\big] \\ \hat{b}_{s,k+1|k}^{-1}\hat{q}_{k+1|k} * \big[\mathbf{y}_{k+1}^{4:6} - \hat{\mathbf{y}}_{k+1|k}^{4:6}\big] \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\omega}_{b,k+1|k+1} \;\; = {} & \;\; \hat{\omega}_{b,k+1|k} + \hat{q}_{k+1|k}^{-1} * \sum_{i=5}^{7} \mathbf{K}_{k+1}^{(i)} \cdots \\
& \times \begin{pmatrix} \hat{a}_{s,k+1|k}^{-1}\hat{q}_{k+1|k} * \cdots \; \big[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\big] * \hat{q}_{k+1|k}^{-1} \\ \hat{b}_{s,k+1|k}^{-1}\hat{q}_{k+1|k} * \cdots \; \big[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\big] * \hat{q}_{k+1|k}^{-1} \end{pmatrix} * \hat{q}_{k+1|k}
\end{aligned}
$$

$$
\begin{aligned}
\hat{a}_{s,k+1|k+1} = {} & \hat{a}_{s,k+1|k} + \mathbf{K}_{k+1}^{(8)} \cdots \\
& \times \begin{pmatrix} \hat{q}_{k+1|k} * \cdots \; \big[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\big] * \hat{q}_{k+1|k}^{-1} \\ \hat{a}_{s,k+1|k}\hat{b}_{s,k+1|k}^{-1}\hat{q}_{k+1|k} * \cdots \; \big[\mathbf{y}_{k+1}^{4:6} - \hat{\mathbf{y}}_{k+1|k}^{4:6}\big] * \hat{q}_{k+1|k}^{-1} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\hat{b}_{s,k+1|k+1} = {} & \hat{b}_{s,k+1|k} + \mathbf{K}_{k+1}^{(9)} \cdots \\
& \times \begin{pmatrix} \hat{b}_{s,k+1|k}\hat{a}_{s,k+1|k}^{-1}\hat{q}_{k+1|k} * \cdots \; \big[\mathbf{y}_{k+1}^{1:3} - \hat{\mathbf{y}}_{k+1|k}^{1:3}\big] * \hat{q}_{k+1|k}^{-1} \\ \hat{q}_{k+1|k} * \cdots \; \big[\mathbf{y}_{k+1}^{4:6} - \hat{\mathbf{y}}_{k+1|k}^{4:6}\big] * \hat{q}_{k+1|k}^{-1} \end{pmatrix}
\end{aligned}
$$

REFERENCES

[1] Crassidis, J. L., & Junkins, J. L. (2004). *Optimal Estimation of Dynamic Systems*. Chapman & Hall/CRC Press LLC.

[2] Crassidis, J. L., Markley, F. L., & Cheng, Y. (2007). Survey of Nonlinear Attitude Estimation Methods. *Journal of Guidance, Control and Dynamics*, *30* (1), 12–28.

[3] Vasconcelos, J and Silvestre, C and Oliveira, (2001). A nonlinear observer for rigid body attitude estimation using vector observations. In *17th World Congres. The International of Automatic Control*,p. 8599-8604, 2008.

[4] Bonnabel, S et al. (2015). An intinsic CramérRao bound on SO(3) for (dynamic) attitude filtering In *arXiv:1503.04701v2*, 2015.

[5] Bernal-Polo, P.; Martnez-Barber, H. (2019) Kalman Filtering for Attitude Estimation with Quaternions and Concepts from Manifold Theory. In Sensors, 19, 149.

[6] Aghannan, N., Rouchon P.(2003). ”An intrinsic observer for a class of Lagrangian systems.”*IEEE Trans. Autom. Control,* vol. 48, no.6, pp.936-945, 2003

[7] Bonnabel, S., Martin, P., & Rouchon, P. (2008). Symmetry-preserving observers. *IEEE Tansaction on Automatic Control,* 53 (11), 2514–2526.

[8] Bonnabel, S., Martin, P., & Rouchon, P. (2009). Non-linear symmetry-preserving observers on Lie groups. *IEEE Tansaction on Automatic Control,* 54 (7), 1709–1713.

[9] Martin,P., & Rouchon,P., and J. Rudolph (2004). Invariant tracking. *ESAIM: Control, Optimisation and Calculus of Variations* , 10:1–13, 2004.

[10] Martin, P., & Salaün, E. (2008). An invariant observer for earth-velocity-aided attitude heading reference systems. In *Proceedings of the 17th IFAC World Congress*, Seoul, Korea, 6–11 July 2008 (pp. 9857–9864).

[11] Martin, P., & Salaün, E. (2007). Invariant observers for attitude and heading estimation from low-cost inertial and magnetic sensors. In *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans (LA), USA, 12–14 December 2007 (pp. 1039–1045).

[12] Maithripala, D and Dayawansa, W P and Berg, J M (2005) Intrinsic observer-based stabilization for simple mechanical systems on Lie Groups. In *SIAM Journal on Control and Optimization*,44,(pp1691–1711), 2005.

[13] Mahony, R and Hamel, T and Pflimlin, J-M(2008) Nonlinear complementary filters on the special orthogonal group. *IEEE Transaction Automatic Control*,53,p1203-1218,2008.

[14] Lagemann, C J T and Mahony, R (2008). Observer design for invariant systems with homogeneous observations. In *arXiv:0810.0748*,2008.

[15] Aghannan, N., & Rouchon, P. (2002). On invariant asymptotic observers. In *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas (NV), USA, 10–13 December 2002 (pp. 1479–1484).

[16] Bonnabel, S. (2007a). Invariant asymptotic observers: theory and examples. *PhD thesis*, Mines ParisTech.

[17] Bonnabel, S., Martin, P., & Salaün, E. (2009). Invariant Extended Kalman Filter: Theory and Application to a Velocity-Aided Attitude Estimation Problem. In *Proceedings of the 48th IEEE Conference on Decision and Control*, Shanghai, China, 16–18 December 2009 (pp. 1297–1304).

[18] Bonnabel, S. (2007b). Left-invariant extended Kalman filter and attitude estimation. In *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans (LA), USA, 12–14 December 2007 (pp. 1027–1032).

[19] Barrau A. and Bonnabel S. (2017) "The Invariant Extended Kalman Filter as a Stable Observer," in *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1797-1812, April 2017.

[20] Barczyk, M., Bonnabel, S., & Deschaud, J-E. (2015). Invariant EKF Design for Scan Matching-Aided Localization. *IEEE Tansaction on Control Systems Technology*, *23* (6), 2440–2448.

[21] Bourmaud, G., Mégret,R., Giremus, A., Berthoumieu, Y. (2014). Continuous-Discrete Extended Kalman Filter on Matrix Lie Groups Using Concentrated Gaussian Distributions. *Journal Math Imaging Vis*, DOI 10.1007/s10851-014-0517-0.

[22] Bourmaud, G., Mégret,R., Giremus, A., Berthoumieu, Y. (2013). Discrete extended Kalman filter on Lie groups. *Signal Processing Conference (EUSIPCO), Proceedings of the 21st European*.

[23] Condomines, J-P., Seren, C., & Hattenberger, G. (2014). Pi-Invariant Unscented Kalman Filter for Sensor Fusion. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, Los Angeles (CA), USA, 15–17 December 2014 (pp. 1035–1040).

[24] J-P. Condomines, C. Seren and G. Hattenberger, "Nonlinear state estimation using an invariant unscented Kalman filter," In *Proc. of the AIAA Guidance, Navigation and Control Conf.*, pp. 1-15, 2013.

[25] J-P. Condomines, C. Seren and G. Hattenberger, *Optimal invariant observers theory for nonlinear state estimation.* Multisensor attitude estimation: fundamental concepts and applications, CRC Press, Taylor & Francis, pp. 391-408, 2017.

[26] J-P. Condomines, C. Seren and G. Hattenberger, "Invariant Unscented Kalman Filter with application to attitude estimation," In *Proc. of the 56th IEEE Conf. on Decision and Control*, hal-01509884v1, 2017.

[27] M. Brossard, S. Bonnabel and J-P. Condomines, "Unscented Kalman Filtering on Lie Groups," *IROS, IEEE/RSJ International Conference on Intelligent Robots and Systems*, Sep 2017, Vancouver, Canada, hal-01489204.

[28] Hauberg, S., Lauze, F., Pedersen, K.S. (2013) Unscented Kalman filter on Riemannian manifolds. *J. Math. Imaging Vis*,p 1-18.

[29] Barrau, A., & Bonnabel, S. (2014). Invariant particle filtering with application to localization. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, Los Angeles (CA), USA, 15–17 December 2014 (pp. 5599–5605).

[30] Snoussi, H., Mohammad-Djafari, A. (2006). Particle filtering on Riemannian manifold, *AIP Conference, Issue 1*, *872* (1), p219.

[31] Tompkins, F., Wolfe, PJ. (2007). Bayesian filtering on the Stiefel manifold. *IEEE international Workshop on Computational Advances in Multi-Sensor Adaptative Processing*.

[32] Li, D., Li, Q., Tang, L., Yang, S., Cheng, N., ... Song, J. (2015). Invariant Observer-Based State Estimation for Micro-Aerial Vehicles in GPS-Denied Indoor Environments Using an RGB-D Camera and MEMS Inertial Sensors. *Micromachines*, *6*, 487–522.

[33] Barrau, A., & Bonnabel, S. (2015). Intrinsic filtering on Lie groups with applications to attitude estimation. *IEEE Tansaction on Automatic Control*, *60* (2), 436–449.

[34] Diemer, S., & Bonnabel, S. (2015). An invariant linear quadratic Gaussian controller for a simplified car. In *Proceedings of the 2015 IEEE Conference on Robotics and Automation*, Seattle (WA), USA, 26–30 May 2015 (pp. 448–453).

[35] Khosravian, A., Trumpf, J., Mahony, R., & Lageman, C. (2015). Observers for invariant systems on Lie groups with biased input measurements and homogeneous outputs. *Automatica*, *55* (Issue C), 19–26.

[36] Julier, S. J., & Uhlmann, J. K. (2004). Unscented filtering and nonlinear estimation. In *Proceedings of the IEEE* (Invited Paper), *92* (3), (pp. 401–422).

[37] Sarkka, S. (2007). On unscented Kalman filtering for state estimation of continous-time nonlinear systems. *IEEE Transaction on Automatic Control*, *52* (9), 1631–1641.

[38] van der Merwe, R., & Wan, E. A. (2001). The square-root unscented Kalman filter for state and parameter estimation. In *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing*, Salt Lake City (UT), USA, 7–11 May 2001 (pp. 3461–3464).

[39] Barczyk, M., & Lynch, A. F. (2013). Invariant Observer Design for Helicopter UAV Aided Inertial Navigation System. *IEEE Transaction on Control Systems Technology*, *21* (3), 791–806.

[40] Salaün, E. (2009). Filtering algorithms and avionics systems for unmanned aerial vehicles. *PhD thesis*, Mines ParisTech.

[41] Julier, S. J. (2002). The scaled unscented transformation. In *Proceedings of the 2002 American Control Conference*, Anchorage (AK), USA, 8–10 May 2002 (pp. 4555–4559).

[42] Olver, P. J. (1999). *Classical Invariant Theory*. Cambridge University Press.