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Combination of optimal control approaches for aircraft conflict avoidance via velocity regulation

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Abstract

This paper deals with optimal control applied to one of the most crucial and challenging problems in Air Traffic Management, that of aircraft conflict avoidance. We propose an optimal control model where aircraft separation is achieved by changing the speeds of aircraft, and the integral over a time window of their squared accelerations is minimized. Pairwise aircraft separation constraints constitute the main difficulties to be handled. We propose an original decomposition of the problem into three *zones*, in such a way that in two of them no conflict occurs. Then, using the Pontryagin maximum principle, two new formulations of the original optimal control problem are proposed and solved via direct shooting methods. Thanks to our decomposition, these numerical methods are applied on subproblems having reduced size with respect to the original one, thus improving the efficiency of the solution process. Thirty problem instances are numerically solved, showing the effectiveness of the proposed approaches.

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1 Introduction

The problem of aircraft conflict avoidance during en-route flights constitutes one of the most crucial and challenging problems in Air Traffic Management (ATM) and Control. It consists in ensuring that aircraft during their flight follow trajectories that respect minimal pairwise distances of separation, horizontally and vertically. When a loss of separation occurs, aircraft are said to be *in conflict*, and suitable separation maneuvers have to be issued to separate them. This is currently done by air traffic controllers, that continuously monitor some portions of the airspace, and, if a potential conflict is detected, give instruction to separate aircraft. This is crucial to ensure flight safety, and is more and more challenging due to the air traffic growth on the world scale. Increasing automation is therefore needed in ATM, that in turns requires the development of suitable mathematical models and efficient and reliable algorithms.

As one usually aims at solving the problem minimizing the deviation with respect to the original aircraft configuration and flight plans, aircraft conflict avoidance can naturally be modelled as an optimization or an optimal control problem. In this paper, we propose mathematical models and resolution methods based on optimal control.

Standard separation norms for commercial en-route aircraft are 5NM horizontally and 1000ft vertically (1NM (nautical mile) = 1852m; 1ft (feet) = 0.3048m). This defines a cylindric protection volume centered on every aircraft, which is then separated from the others if there is no other aircraft entering its protection volume. When aircraft are conflicting, they can be separated deviating for example their heading angle, their altitude or their speed. The first kind of maneuver is currently the most used in practice by air traffic controllers. Aircraft speed variations, on the other hand, are considered very promising, especially in the context of the so-called *subliminal speed control* [6], consisting in modifying the aircraft speeds within a very small range around their original speeds, without informing air traffic controllers. A review of conflict detection and resolution approaches up to the year 2000 is provided in [19]. A more recent discussion about modelling issues is provided in [8].

Local continuous optimization has been proposed in [21]. Mixed-integer optimization approaches date back to 2002, with linear models proposed in [20, 23]; more recently, nonlinear models are attracting a growing attention, see, *e.g.*, [1, 2, 9, 10].

Optimal control arises as a natural framework for the problem at hand, as well as for the close problem of trajectory planning, as it allows one to naturally model aircraft dynamics. Clements [13] proposes an optimal control for a minimum-time flight trajectory to a specified fly-to point. The same author in [14] presents a formulation where the control variable constraint is the maximum acceptable aircraft turn rate and the state variable constraint is defined in terms of the radius of the protected zone of conflicting aircraft. Bicchi and Pallottino [3] propose an optimal control model where aircraft velocities are considered constant and again the time for aircraft to go through given trajectory waypoints is minimized. The minimum-time aircraft trajectory problem is also studied in [24], where the optimal controls are found by either maximizing or minimizing a modified Hamiltonian.

In [16], turn and speed changes are considered as controls and some numerical results are obtained, that however show the limitation of the proposed approach in terms of numerical solutions.

Optimal control models in the considered framework are indeed rather difficult to solve. Shooting methods are usually applied (see [5] for a recent survey). Direct shooting methods, based on time discretization, discretize both control and states variables using numerical integrators (of, *e.g.*, Euler or Runge-Kutta type) to approximate the dynamic system. This leads to solve a finite dimensional nonlinear programming (NLP) problem. These methods yield possibly large-scale problems, and in addition suffer of approximation errors due to the time discretization. In contrast, methods based on the Pontryagin Maximum Principle (PMP) [22], map the variables at the initial time to their corresponding values at the final time, that leads to a finite dimensional problem with matching end-point conditions. In this case, there is no time-discretization and a high solution accuracy can be achieved. Nevertheless, these methods suffer of lack of *a priori* informations about the solution structure, and the initialization of the adjoint (costate) variables is in general difficult. To benefit of the advantages of the two classes of methods, a hybrid approach has been suggested (see, *e.g.* [7]), where the direct and indirect multiple shooting are combined in such a way that the numerical approximation computed by the direct method is used to initialize the adjoint variables in the indirect shooting method.

In this paper, we propose to use the direct shooting method on new formulations based on the PMP through an original decomposition of the addressed problem. We observe that in aircraft conflict avoidance, we can identify a sub-problem where aircraft are potentially conflicting, and so inequality constraints on state variables for pairwise separation have to be applied, and two other subproblems where this kind of constraints do not need to be verified, because aircraft are not yet in the conflict zone or have already passed such a zone (and so, they are separated by construction).

We propose an optimal control model for aircraft conflict avoidance based on speed regulation, where aircraft accelerations are the commands and where the cost function is the integral on the time window of the squared accelerations. Then, we propose a decomposition of the overall problem in subproblems, that we call *prezone*, *zone*, and *postzone*. Using the PMP on the *prezone* and on the *postzone*, we derive analytical solutions. This leads to two new formulations of the original optimal control problem, where the number of variables and constraints in the NLPs obtained by applying direct shooting methods is strongly reduced. We show that these two original resolution strategies allow us to obtain very promising results on problem instances containing up to 13 aircraft.

This paper is organized as follows. In Section 2, we present an optimal control model based on speed changes for aircraft conflict avoidance. In Section 3, we propose our original decomposition of the addressed problem. The main contribution of this paper is presented in Sections 4, 5 and 6, where we present the formulations of the three subproblems, their properties in terms of existence of solutions, and their numerical solution via the application of the PMP or the direct shooting method. In Section 7, we present numerical results validating the proposed approach on a set of 30 problem instances. Section 8 concludes the paper.

2 Optimal control model through acceleration command

In this section, we present an optimal control model to achieve aircraft separation by only modifying their speeds. This model uses the accelerations of the aircraft as commands of the dynamical system, keeping the trajectories unchanged in terms of headings.

Let us denote x_i and v_i the state variables representing respectively the position and the velocity of aircraft $i \in I := \{1, \dots, n\}$, where n is the number of aircraft. The acceleration u_i , for each aircraft i , is the command of the system. Aircraft are considered during their *en-route* cruise phase and expected to be at the same altitude (they are assigned the same flight level), thus we deal with a planar configuration. The velocity v_i and so the acceleration u_i are with values in \mathbb{R} , and the position x_i is with values in \mathbb{R}^2 with $\dot{x}_i(t) = v_i(t)d_i$ where $d_i \in \mathbb{R}^2$ is fixed. For each aircraft i , the velocity v_i and the acceleration u_i are bounded; *i.e.*, belonging to the interval $[\underline{v}_i, \bar{v}_i]$ and $[\underline{u}_i, \bar{u}_i]$ respectively.

We consider a *tactical* en-route phase of flights, that means that an air sector is monitored during a relatively short time window $[t_0, t_f]$, and aircraft separation maneuvers are issued in order to have aircraft separated over all this time window. The air sector and the time window can be easily considered in a bijection, *i.e.*, the

air sector is defined by the positions $x_i(t_0)$ and $x_j(t_f)$ of all aircraft i and j . Thus, the initial and the final times (t_0 and t_f) are fixed and identical for all aircraft. Let D be the minimum required horizontal separation distance between each pair of aircraft and d_i the direction (heading) of the i^{th} aircraft.

The position $x(t)$ of an aircraft at time t in a planar configuration is to be considered in cartesian coordinates $x(t) := (x^X(t), x^Y(t))^T, \forall t \in [t_0, t_f]$. Note that the controls u_i are searched in a set of square integrable functions over $[t_0, t_f]$. Moreover, bound constraints on the controls are directly added in the following problems.

The mathematical model is the following:

$$(\mathcal{P}) \left\{ \begin{array}{ll} \min_{u,v,x} & \sum_{i \in I} \int_{t_0}^{t_f} u_i^2(t) dt \\ \text{s.t.} & \\ & \dot{v}_i(t) = u_i(t) \quad \forall t \in [t_0, t_f], \forall i \in I \\ & \dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [t_0, t_f], \forall i \in I \\ & \underline{u}_i \leq u_i(t) \leq \bar{u}_i \quad \forall t \in [t_0, t_f], \forall i \in I \\ & \underline{v}_i \leq v_i(t) \leq \bar{v}_i \quad \forall t \in [t_0, t_f], \forall i \in I \\ & x_i(t_0) = x_i^0, \quad v_i(t_0) = v_i^0 \quad \forall i \in I \\ & x_i(t_f) \text{ free}, \quad v_i(t_f) = v_i^f \quad \forall i \in I \\ & \|x_i(t) - x_j(t)\|^2 \geq D^2 \quad \forall t \in [t_0, t_f], \\ & \quad \quad \quad \forall (i, j) \in I^2, i < j \end{array} \right.$$

We choose to minimize a quadratic energy cost function depending on speed variations. This criterion takes into account the speed changes of all aircraft by acceleration or deceleration. Note that problem (\mathcal{P}) is twice continuously differentiable, with a convex criterion (strictly convex if only u is considered as variable). The state equations, depending on the state variables v and x , are all linear. The main difficulties in the solution of (\mathcal{P}) are related to the tight bound constraints on the state variables $v_i(\cdot)$ (specially in the case of subliminal velocity control), and to the concave inequality constraints expressing the pairwise aircraft separation.

To solve an optimal control problem, one can derive necessary optimality conditions through the PMP. The solution can then be obtained numerically by applying an indirect shooting technique. However, problem (\mathcal{P}) owns constraints involving state variables that makes it difficult to apply such an indirect method. As an alternative to solve problem (\mathcal{P}) , one can apply direct shooting methods. They are based on the discretization of the time window, yielding a large-scale nonlinear (continuous) optimization problem (NLP), that can be solved by standard local optimization solvers. Note that discretizing problem (\mathcal{P}) , the number of control

variables amounts to $n \times p$, where n is the number of aircraft and p is the number of discretization points, and the number of separation constraints amounts to $\frac{n(n-1)p}{2}$ respectively. As an example, let us consider a conflict avoidance problem with only three aircraft, flying during 1h discretized by using steps of 15". This yields a non-linear problem with 720 ($n = 3$ and $p = 240$) control variables, 720 separation inequality constraints (plus 1440 inequality constraints on the velocity bounds).

In the following, we show that problem (\mathcal{P}) can be decomposed into three subproblems in such a way that the PMP can be relatively easily applied to two of them. This implies that the direct shooting method is applied to the remaining subproblem. In this way, the size of the NLP to be solved is drastically reduced, improving the efficiency of the solution process. In the next sections, we present our decomposition method of problem (\mathcal{P}) into three parts: *prezone*, *zone* and *postzone*, such that no conflict can occur in the *prezone* and in the *postzone*, and we propose a solution approach for each of these subproblems.

3 Decomposing the problem into *zones*

Given the aircraft predicted planar (straight-line) trajectories, one can check intersections of these trajectories so as to identify spatial regions where the separation constraints must be actually checked. The corresponding time window where the separation constraints have to be checked can then be identified as well. This is the basis of the original strategy of resolution of the problem at hand, proposed in this paper. In the following, we discuss the decomposition of the problem.

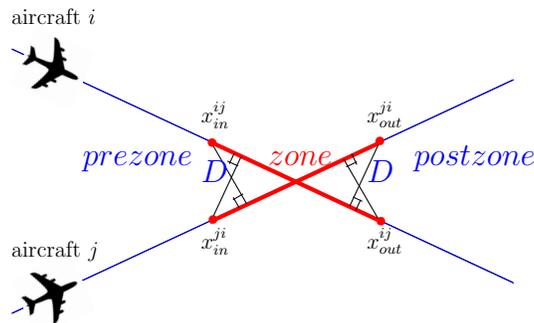


Figure 1: Spatial decomposition for two aircraft.

Let S_i define the line segment corresponding to all the positions $x_i(t)$ of aircraft i , for all time t in the time window $[t_0, t_f]$. Two aircraft i and j are in conflict when some points on S_i have a distance less than or equal to D to another point on S_j , and reciprocally (see Fig. 1). We can thus easily identify a planar region where i

and j are in conflict. For each pair of aircraft, we will denote this planar region by *zone*. The *zone* is characterized by the fact that no conflict occurs outside it, while inside it conflicts may occur, and so separation constraints have to be imposed. We define *prezone* the region preceding the *zone*, and respectively *postzone* the remaining region. This is illustrated in Fig. 1.

In the remaining of this section, we present the way a *zone* is identified in practice. Let us consider aircraft i and j . We first have to determine their entrance and exit points x_{in}^{ij} and x_{out}^{ij} , if they exist. This means that we have to check if S_i and S_j own points at a distance less than or equal to D . Let us denote by $(D_i) : a_i x + b_i y + c_i = 0$ the equation of the line supporting S_i ; this defines the equation of the trajectory of aircraft i and so $a_i = d_i^Y$, $b_i = -d_i^X$, and $c_i = -a_i x_i^X(t_0) - b_i x_i^Y(t_0)$. For i and j following trajectories (D_i) and (D_j) respectively, to determine the points x_{in}^{ij} and x_{out}^{ij} (if they exist), we use the fact that x_{in}^{ij} and x_{out}^{ij} must belong to (D_i) , for aircraft i , and have distance from (D_j) equal to D (reciprocally for aircraft j). This yields the following two equations:

$$\begin{aligned} a_i x + b_i y + c_i &= 0 \\ \frac{|a_j x + b_j y + c_j|}{\sqrt{a_j^2 + b_j^2}} &= D \end{aligned}$$

that give two distinct but similar linear systems:

$$\begin{cases} a_i x + b_i y = -c_i \\ a_j x + b_j y = D\sqrt{a_j^2 + b_j^2} - c_j \end{cases} \quad \begin{cases} a_i x + b_i y = -c_i \\ a_j x + b_j y = -D\sqrt{a_j^2 + b_j^2} - c_j \end{cases}$$

Denoting by $(K_i, K_j)^T$ the second right hand side of the first system (for the second one, we just have to change a sign in K_j), we obtain:

$$x = \frac{b_j K_i - b_i K_j}{a_i b_j - a_j b_i} \quad \text{and} \quad y = \frac{a_j K_i - a_i K_j}{a_j b_i - a_i b_j}.$$

Depending on the values of K_j , these yield two distinct points which are x_{in}^{ij} and x_{out}^{ij} . We just have to check which point is *in* and which one is *out*, and also if they exist and belong to S_i . Note that this computation, while being easy, has to be done twice for each pair of aircraft. This therefore yields $n \times (n - 1)$ computations.

Once entry and exit points x_{in}^{ij} and x_{out}^{ij} are known, we can determine the *time zone* for an aircraft i , i.e., the time window corresponding to the *zone*. We denote it by $[t_{in}^i, t_{out}^i]$ (see Fig. 2). We have to solve the following equations: $\dot{x}_i(t) = \bar{v}_i d_i$ and $\dot{x}_i(t) = \underline{v}_i d_i$ to obtain t_{in}^i and t_{out}^i respectively. Considering $\dot{x}_i(t) = \bar{v}_i d_i$, one

has $x_i(t) = \bar{v}_i d_i t + K_1$, where $K_1 := x_i(t_0) - \bar{v}_i d_i t_0$ and thus

$$t_{in}^{ij} := \begin{cases} \frac{(x_{in}^{ij})^X - K_1}{\bar{v}_i d_i^X}, & \text{if } d_i^X \neq 0 \\ \frac{(x_{in}^{ij})^Y - K_1}{\bar{v}_i d_i^Y}, & \text{else.} \end{cases}$$

Considering now the second equation, we obtain similarly:

$$t_{out}^{ij} := \begin{cases} \frac{(x_{out}^{ij})^X - K_2}{\underline{v}_i d_i^X}, & \text{if } d_i^X \neq 0 \\ \frac{(x_{out}^{ij})^Y - K_2}{\underline{v}_i d_i^Y}, & \text{else.} \end{cases}$$

where $K_2 := x_i(t_0) - \underline{v}_i d_i t_0$.

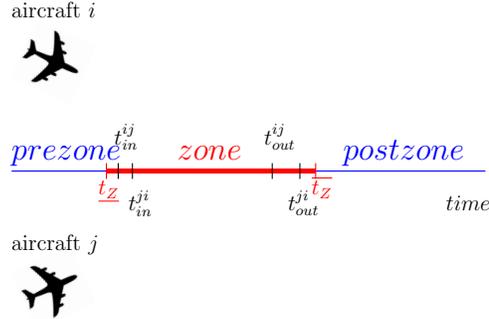


Figure 2: Decomposition in *time zones* for all aircraft. Time t_{in} and t_{out} are shown for aircraft i and j .

Remark that for i and j , if $t_{in}^{ij} \neq t_{in}^{ji}$, we take the maximum value, and similarly for $t_{out}^{ij} \neq t_{out}^{ji}$. Finally, we can define the *general time zone* $[\underline{t}_Z, \bar{t}_Z]$ as:

$$\underline{t}_Z := \min_{i \in I} \min_{j \in I, j > i} \max\{t_{in}^{ij}, t_{in}^{ji}\} \quad \text{and} \quad \bar{t}_Z := \max_{i \in I} \max_{j \in I, j > i} \min\{t_{out}^{ij}, t_{out}^{ji}\} \quad (1)$$

that so requires a computational complexity of $O(n^2)$. The *general time zone*, illustrated in Fig. 2, corresponds to a unique time window for all the n aircraft and it is the only window where aircraft separation constraints have to be satisfied.

4 Applying the PMP on the *postzone*

In this section, we focus on the *postzone* to derive necessary optimality conditions using the PMP.

From a temporal point of view, the *postzone* corresponds to $[\bar{t}_Z, t_f]$, i.e., from the end of the *zone* until the final time. As conflicts occur only in the *zone*, starting from \bar{t}_Z , there is no more conflict between the aircraft. Thus, the aircraft separation constraints can be discarded yielding the following problem:

$$(\mathcal{P}_{\text{postzone}}) \left\{ \begin{array}{l} \min_{u,v,x} \quad \sum_{i \in I} \int_{\bar{t}_Z}^{t_f} u_i^2(t) dt \\ \text{s.t.} \\ \dot{v}_i(t) = u_i(t) \quad \forall t \in [\bar{t}_Z, t_f], \forall i \in I \\ \dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [\bar{t}_Z, t_f], \forall i \in I \\ \underline{u}_i \leq u_i(t) \leq \bar{u}_i \quad \forall t \in [\bar{t}_Z, t_f], \forall i \in I \\ \underline{v}_i \leq v_i(t) \leq \bar{v}_i \quad \forall t \in [\bar{t}_Z, t_f], \forall i \in I \\ x_i(\bar{t}_Z) = x_i^{\bar{t}_Z}, \quad v_i(\bar{t}_Z) = v_i^{\bar{t}_Z} \quad \forall i \in I \\ x_i(t_f) \text{ free}, \quad v_i(t_f) = v_i^f \quad \forall i \in I \end{array} \right.$$

Remark 1 *The fixed values of the velocity at times \bar{t}_Z and t_f , respectively $v_i^{\bar{t}_Z}$ and v_i^f , have to be in the interval $[\underline{v}_i, \bar{v}_i]$, otherwise it would not be any point satisfying the constraints of problem $(\mathcal{P}_{\text{postzone}})$. In the following, we suppose that the values of $v_i^{\bar{t}_Z}$ and v_i^f are in the interval.*

The following two propositions provide some properties about the solution of problem $(\mathcal{P}_{\text{postzone}})$.

Proposition 1 (Conditions on the existence of a realizable solution) *A realizable solution of problem $(\mathcal{P}_{\text{postzone}})$ exists if and only if*

$$\underline{u}_i \leq \frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z} \leq \bar{u}_i, \quad \forall i \in I.$$

Proof 1 *Let \tilde{u} be a realizable control, that implies $\underline{u}_i \leq \tilde{u}_i(t) \leq \bar{u}_i \quad \forall t \in [\bar{t}_Z, t_f], \forall i \in I$.*

By using the monotonicity property of the integral, one has:

$$\int_{\bar{t}_Z}^{t_f} \underline{u}_i \leq \int_{\bar{t}_Z}^{t_f} \tilde{u}_i(t) dt \leq \int_{\bar{t}_Z}^{t_f} \bar{u}_i \quad \forall i \in I.$$

Hence, denoting by \tilde{v}_i the velocity corresponding to the acceleration \tilde{u}_i , one obtains

$$(t_f - \bar{t}_Z) \times \underline{u}_i \leq \tilde{v}_i(t_f) - \tilde{v}_i(\bar{t}_Z) \leq (t_f - \bar{t}_Z) \times \bar{u}_i \quad \forall t \in [\bar{t}_Z, t_f], \forall i \in I.$$

The result follows from $\tilde{v}_i(\bar{t}_Z) = v_i^{\bar{t}_Z}$ and $\tilde{v}_i(t_f) = v_i^f$.

Reciprocally, suppose that

$$\underline{u}_i \leq \frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z} \leq \bar{u}_i, \quad \forall i \in I.$$

By considering a constant control $\tilde{u}_i(t) := \frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z}$, $\forall t \in [\bar{t}_Z, t_f], \forall i \in I$, we obtain the corresponding velocity:

$$\tilde{v}_i(t) = \int_{\bar{t}_Z}^t \tilde{u}_i(\tau) d\tau + \tilde{v}_i(\bar{t}_Z) = \int_{\bar{t}_Z}^t \frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z} d\tau + \tilde{v}_i(\bar{t}_Z) = (t - \bar{t}_Z) \times \frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z} + \tilde{v}_i(\bar{t}_Z).$$

Thus, $\tilde{v}_i(t_f) = (t_f - \bar{t}_Z) \times \frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z} + \tilde{v}_i(\bar{t}_Z) = v_i^f - v_i^{\bar{t}_Z} + \tilde{v}_i(\bar{t}_Z)$.

Hence, setting $\tilde{v}_i(\bar{t}_Z) := v_i^{\bar{t}_Z}$, one has $\tilde{v}_i(t_f) = v_i^f$. Moreover, as \tilde{v}_i is affine with respect to t , and as $v_i^{\bar{t}_Z}$ and v_i^f are in $[v_i, \bar{v}_i]$, one has that $\tilde{v}_i(t) \in [v_i, \bar{v}_i]$ for all $t \in [\bar{t}_Z, t_f]$.

Therefore, this solution (\tilde{u}, \tilde{v}) is realizable for problem $(\mathcal{P}_{\text{postzone}})$. Note that there is no constraint on the final position and so, \tilde{x} is just defined from \tilde{v} by fixing $\tilde{x}(\bar{t}_Z) = x^{\bar{t}_Z}$.

Proposition 2 (Solution of problem $(\mathcal{P}_{\text{postzone}})$) *The unique solution of problem $(\mathcal{P}_{\text{postzone}})$, if a realizable solution exists, is:*

$$u_i^*(t) := \frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z}, \quad \forall i \in I.$$

This optimal control provides the following optimal state variables:

$$v_i^*(t) := \frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z} t + \frac{v_i^{\bar{t}_Z} \times t_f - v_i^f \times \bar{t}_Z}{t_f - \bar{t}_Z}$$

and

$$x_i^*(t) := d_i \times \left(\frac{v_i^f - v_i^{\bar{t}_Z}}{2(t_f - \bar{t}_Z)} t^2 + \frac{v_i^{\bar{t}_Z} \times t_f - v_i^f \times \bar{t}_Z}{t_f - \bar{t}_Z} t \right) + X^0$$

where $X^0 = x^{\bar{t}_Z} - d_i \times \left(\frac{v_i^f - v_i^{\bar{t}_Z}}{2(t_f - \bar{t}_Z)} \bar{t}_Z^2 + \frac{v_i^{\bar{t}_Z} \times t_f - v_i^f \times \bar{t}_Z}{t_f - \bar{t}_Z} \bar{t}_Z \right)$, and the following optimal cost

$$J^* := J(u^*) = \sum_{i \in I} \frac{(v_i^f - v_i^{\bar{t}_Z})^2}{t_f - \bar{t}_Z}.$$

Proof 2 Let us first relax problem $(\mathcal{P}_{\text{postzone}})$ by discarding bound constraints on u and v : $\underline{u}_i \leq u_i(t) \leq \bar{u}_i$ and $\underline{v}_i \leq v_i(t) \leq \bar{v}_i$, $\forall t \in [\bar{t}_Z, t_f] \forall i \in I$.

Note that problem $(\mathcal{P}_{\text{postzone}})$ corresponds to the sum of n independent sub-problems, one for each $i \in I$, as follows:

$$(\mathcal{P}_{\text{postzone}}^i) \left\{ \begin{array}{l} \min_{u_i, v_i, x_i} \int_{\bar{t}_Z}^{t_f} u_i^2(t) dt \\ \text{s.t.} \\ \dot{v}_i(t) = u_i(t) \quad \forall t \in [\bar{t}_Z, t_f], \\ \dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [\bar{t}_Z, t_f], \\ x_i(\bar{t}_Z) = x_i^{\bar{t}_Z}, \quad v_i(\bar{t}_Z) = v_i^{\bar{t}_Z} \\ x_i(t_f) \text{ free}, \quad v_i(t_f) = v_i^f \end{array} \right.$$

The Hamiltonian of problem $(\mathcal{P}_{\text{postzone}}^i)$ is:

$$\begin{aligned} H_i(x_i^X(t), x_i^Y(t), v_i(t), z_{1,i}(t), z_{2,i}(t), z_{3,i}(t), z_{0,i}, u_i(t)) \\ = z_{1,i}(t) v_i(t) d_i^X + z_{2,i}(t) v_i(t) d_i^Y + z_{3,i}(t) u_i(t) + z_{0,i} u_i^2(t), \end{aligned} \quad (2)$$

where the real scalar $z_{0,i}$ is associated to the cost function and $z_{1,i}(t), z_{2,i}(t), z_{3,i}(t)$ are the costate variables; $z_{1,i}(t), z_{2,i}(t), z_{3,i}(t)$ are associated to $x_i^X(t), x_i^Y(t)$ and $v_i(t)$ respectively.

The associated system of differential equations, for all $t \in [\bar{t}_Z, t_f]$, is:

$$\begin{aligned}
\dot{x}_i^X(t) &= \frac{\partial H_i(\cdot)}{\partial z_{1,i}} = v_i(t)d_i^X, \\
\dot{x}_i^Y(t) &= \frac{\partial H_i(\cdot)}{\partial z_{2,i}} = v_i(t)d_i^Y, \\
\dot{v}_i(t) &= \frac{\partial H_i(\cdot)}{\partial z_{3,i}} = u_i(t), \\
\dot{z}_{1,i}(t) &= -\frac{\partial H_i(\cdot)}{\partial x_i^X} = 0, \\
\dot{z}_{2,i}(t) &= -\frac{\partial H_i(\cdot)}{\partial x_i^Y} = 0, \\
\dot{z}_{3,i}(t) &= -\frac{\partial H_i(\cdot)}{\partial v_i} = -z_{1,i}(t)d_i^X - z_{2,i}(t)d_i^Y.
\end{aligned}$$

Maximizing the Hamiltonian with respect to u_i , and then fixing the value $z_{0,i}$ to -1 without loss of generality (just the normal extremals are studied here), we obtain:

$$u_i^* = \operatorname{argmax}_{u_i} H_i(\cdot) = \frac{z_{3,i}(t)}{2}$$

This result is obtained by deriving H_i with respect to u_i ($\frac{\partial H_i}{\partial u_i} = z_{3,i} - 2u_i$), and by solving $\frac{\partial H_i}{\partial u_i} = 0$, thus providing the necessary optimality conditions at the first order. By deriving two times with respect to u_i , we obtain $\frac{\partial^2 H_i}{\partial u_i^2} = -2 < 0$, which proves that u_i^* is indeed the unique maximum of H_i .

By replacing $u_i(t)$ in the above differential equations by $\frac{z_{3,i}(t)}{2}$, this yields a system with 6 equations, 6 unknowns, 3 initial conditions and just 1 final condition. The transversality conditions provide 2 new final conditions on the costate variables: $z_{1,i}(t_f) = 0$ and $z_{2,i}(t_f) = 0$. Considering the two differential equations $\dot{z}_{1,i}(t) = 0$ and $\dot{z}_{2,i}(t) = 0$, this implies that $z_{1,i}(t) = 0$ and $z_{2,i}(t) = 0$, for all $t \in [\bar{t}_Z, t_f]$. Therefore, the last differential equation $\dot{z}_{3,i}(t) = -z_{1,i}(t)d_i^X - z_{2,i}(t)d_i^Y = 0$, and then $z_{3,i}$ is constant for all $t \in [\bar{t}_Z, t_f]$. By denoting A this constant, one has $u_i^*(t) = \frac{A}{2}$ and hence (considering the third differential equation of our system), $v_i^*(t) = \frac{A}{2}t + B$, where B is another real constant. By using the initial and final conditions on v_i , one obtains the values of the two constants A and B ; i.e., $v_i^*(\bar{t}_Z) = \frac{A}{2}\bar{t}_Z + B = v_i^{\bar{t}_Z}$ and $v_i^*(t_f) = \frac{A}{2}t_f + B = v_i^f$, hence $A = 2\frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z}$ and $B = \frac{t_f \times v_i^{\bar{t}_Z} - \bar{t}_Z \times v_i^f}{t_f - \bar{t}_Z}$. Substituting the values of these two constants, we obtain the expression of the optimal solution u^*, v^*, x^* and J^* .

Let us now discuss about the relaxed bound constraints on u and v .

First consider the bound constraints on the velocity v : $\underline{v}_i \leq v_i(t) \leq \bar{v}_i$. From the values of A and B obtained by solving the relaxed problem, one has that $v_i^*(t) = \frac{A}{2}t + B$ and then $v_i^*(t)$ is affine. Let us suppose that $v_i^f \leq v_i^{\bar{t}_Z}$, then one has $A \leq 0$ and then $v_i^*(t) \leq v_i^*(\bar{t}_Z) = v_i^{\bar{t}_Z}$ for all $t \in [\bar{t}_Z, t_f]$. Moreover, as $A \leq 0$, one has $v_i^*(t) \geq v_i^*(t_f) = v_i^f$ for all $t \in [\bar{t}_Z, t_f]$. Hence, $\underline{v}_i \leq v_i^f \leq v_i^*(t) \leq v_i^{\bar{t}_Z} \leq \bar{v}_i$, for all $t \in [\bar{t}_Z, t_f]$. Let us now suppose that $v_i^f \geq v_i^{\bar{t}_Z}$, then one has $A \geq 0$ and then $v_i^*(t) \geq v_i^*(\bar{t}_Z) = v_i^{\bar{t}_Z}$ for all $t \in [\bar{t}_Z, t_f]$. Moreover, as $A \geq 0$, one has $v_i^*(t) \leq v_i^*(t_f) = v_i^f$ for all $t \in [\bar{t}_Z, t_f]$. Thus, one has $\underline{v}_i \leq v_i^{\bar{t}_Z} \leq v_i^*(t) \leq v_i^f \leq \bar{v}_i$, for all $t \in [\bar{t}_Z, t_f]$. In conclusion, if the optimal control u^* exists then the bound constraints on the velocity v are satisfied.

Let us now consider the bound constraints on u : $\underline{u}_i \leq u_i(t) \leq \bar{u}_i$ for all $i \in I$ and $t \in [\bar{t}_Z, t_f]$. As $u_i^*(t)$ is constant and equal to $\frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z}$, if $\exists i \in I$ such that $\frac{v_i^f - v_i^{\bar{t}_Z}}{t_f - \bar{t}_Z} \notin [\underline{u}_i, \bar{u}_i]$ from Proposition 1, one can conclude that there is no realizable control for problem $(\mathcal{P}_{\text{postzone}})$. In conclusion, if a control u^* exists, the bound constraints on u are satisfied.

Therefore, we proved that from the relaxed problem $(\mathcal{P}_{\text{postzone}})$, the optimal control u^* can be provided using the PMP; the velocity, the position and the final cost can then be derived. Moreover, such an optimal constant solution u^* exists if and only if the bound constraints on u are satisfied.

Remark 2 In the addressed application, the bounds on the control are in practice quite large, and so easily satisfied. It follows that in general, we have a solution of problem $(\mathcal{P}_{\text{postzone}})$.

By decomposing the time interval into two parts $[t_0, \bar{t}_Z]$ and $[\bar{t}_Z, t_f]$, and using Propositions 1 and 2, one can reformulate problem (\mathcal{P}) . The following theorem presents this new formulation of problem (\mathcal{P}) , involving a lower number of variables with respect to the initial one.

Theorem 1 (Reformulation of problem (\mathcal{P})) A reformulation of problem (\mathcal{P})

is:

$$(\mathcal{P}_1) \left\{ \begin{array}{ll} \min_{u,v,x} & \sum_{i \in I} \int_{t_0}^{\bar{t}_Z} u_i^2(t) dt + \sum_{i \in I} \frac{(v_i^f - v_i(\bar{t}_Z))^2}{t_f - \bar{t}_Z} \\ \text{s.t.} & \\ & \dot{v}_i(t) = u_i(t) \quad \forall t \in [t_0, \bar{t}_Z], \forall i \in I \\ & \dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [t_0, \bar{t}_Z], \forall i \in I \\ & \underline{u}_i \leq u_i(t) \leq \bar{u}_i \quad \forall t \in [t_0, \bar{t}_Z], \forall i \in I \\ & \underline{v}_i \leq v_i(t) \leq \bar{v}_i \quad \forall t \in [t_0, \bar{t}_Z], \forall i \in I \\ & \underline{u}_i \leq \frac{v_i^f - v_i(\bar{t}_Z)}{t_f - \bar{t}_Z} \leq \bar{u}_i \quad \forall i \in I \\ & x_i(t_0) = x_i^0, \quad v_i(t_0) = v_i^0 \quad \forall i \in I \\ & x_i(t_f) \text{ free}, \quad v_i(t_f) = v_i^f \quad \forall i \in I \\ & \|x_i(t) - x_j(t)\|^2 \geq D^2 \quad \forall t \in [t_0, \bar{t}_Z], \\ & \quad \quad \quad \forall (i, j) \in I^2, i < j \end{array} \right.$$

Proof 3 By decomposing the time window $[t_0, t_f]$ into two parts $[t_0, \bar{t}_Z]$ and $[\bar{t}_Z, t_f]$, problem (\mathcal{P}) becomes almost separable. The only link occurs at time \bar{t}_Z . Indeed, the two subproblems can provide disconnected solutions of u , v and x , but only at time \bar{t}_Z .

The control u can be discontinuous because it is possible for an aircraft to change roughly its acceleration. So u can be discontinuous at time \bar{t}_Z ; indeed, u has just to be integrable over $[t_0, \bar{t}_Z]$ and $[\bar{t}_Z, t_f]$. For the velocity v and the position x , one must keep the continuity of the solution at time \bar{t}_Z . Suppose that we solve the first subproblem on $[t_0, \bar{t}_Z]$, computing an optimal control u^1 , an optimal velocity v^1 and position x^1 . Using the values of $v^1(\bar{t}_Z)$ and $x^1(\bar{t}_Z)$ used as initial conditions of the second subproblem for t in $[\bar{t}_Z, t_f]$, and from Propositions 1 and 2, a solution of the second subproblem, if it exists, can be easily computed. The existence of a solution for the second subproblem provides a new constraint on $v^1(\bar{t}_Z)$: $\underline{u}_i \leq \frac{v_i^f - v_i^1(\bar{t}_Z)}{t_f - \bar{t}_Z} \leq \bar{u}_i$ for all $i \in I$. Note that if this constraint is not satisfied, the second subproblem (for time in $[\bar{t}_Z, t_f]$) has no realizable point (see Proposition 1). Therefore this constraint has to be added to problem (\mathcal{P}_{Z1}) and the result follows.

Remark 3 The new formulation of Theorem 1 only provides solutions for all $t \in [t_0, \bar{t}_Z]$, if such an optimal solution exists. All the remaining solutions, for all time t in $[\bar{t}_Z, t_f]$, have to be derived from Proposition 2.

5 Applying the PMP on the *prezone*

In this section, we focus on the *prezone* to derive necessary optimality conditions using the PMP. This provides a new formulation of problem (\mathcal{P}).

This section follows the same outline as the previous one dedicated to the study of the *postzone*, however the case treated here is much more difficult and major differences occur. For the sake of simplification and without loss of generality, we fix $t_0 = 0$ in this section.

From a temporal point of view, the *prezone* corresponds to $[0, \underline{t_Z}]$, i.e., from the initial time until the beginning of the *zone*. As conflicts occur only in the *zone*, from the initial time to the beginning of the *zone*, $\underline{t_Z}$, there is no conflict between aircraft. Thus, the aircraft separation constraints can be discarded yielding the following problem:

$$(\mathcal{P}_{\text{prezone}}) \left\{ \begin{array}{l} \min_{u,v,x} \sum_{i \in I} \int_0^{\underline{t_Z}} u_i^2(t) dt \\ \text{s.t.} \\ \dot{v}_i(t) = u_i(t) \quad \forall t \in [0, \underline{t_Z}], \forall i \in I \\ \dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [0, \underline{t_Z}], \forall i \in I \\ \underline{u}_i \leq u_i(t) \leq \bar{u}_i \quad \forall t \in [0, \underline{t_Z}], \forall i \in I \\ \underline{v}_i \leq v_i(t) \leq \bar{v}_i \quad \forall t \in [0, \underline{t_Z}], \forall i \in I \\ x_i(0) = x_i^0, \quad v_i(0) = v_i^0 \quad \forall i \in I \\ x_i(\underline{t_Z}) = x_i^{\underline{t_Z}}, \quad v_i(\underline{t_Z}) = v_i^{\underline{t_Z}} \quad \forall i \in I \end{array} \right.$$

Remark 4 *The only difference between problem ($\mathcal{P}_{\text{postzone}}$) and problem ($\mathcal{P}_{\text{prezone}}$) is that the final position has now to be fixed: $x_i(\underline{t_Z}) = x_i^{\underline{t_Z}}$.*

Remark 5 *The fixed values of the velocity at times 0 and $\underline{t_Z}$, respectively v_i^0 and $v_i^{\underline{t_Z}}$, have to be in the interval $[\underline{v}_i, \bar{v}_i]$, otherwise it would not exist any point satisfying the constraints of problem ($\mathcal{P}_{\text{prezone}}$).*

In the following, we suppose that the values of v_i^0 and $v_i^{\underline{t_Z}}$ are in the interval $[\underline{v}_i, \bar{v}_i]$.

In the first part of this section the bound constraints on the velocity are not

taken into account, then the problem ($\mathcal{P}_{\text{prezone}}$) is relaxed providing:

$$(\mathcal{P}_{\text{R-prezone}}) \left\{ \begin{array}{l} \min_{u,v,x} \sum_{i \in I} \int_0^{\underline{t_Z}} u_i^2(t) dt \\ s.t. \\ \dot{v}_i(t) = u_i(t) \quad \forall t \in [t_0, \underline{t_Z}], \forall i \in I \\ \dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [t_0, \underline{t_Z}], \forall i \in I \\ \underline{u}_i \leq u_i(t) \leq \bar{u}_i \quad \forall t \in [t_0, \underline{t_Z}], \forall i \in I \\ x_i(0) = x_i^0, \quad v_i(0) = v_i^0 \quad \forall i \in I \\ x_i(\underline{t_Z}) = x_i^{\underline{t_Z}}, \quad v_i(\underline{t_Z}) = v_i^{\underline{t_Z}} \quad \forall i \in I \end{array} \right.$$

The way to consider the bound constraints on the velocity will be discussed below to derive a new formulation of problem (\mathcal{P}).

The following two propositions provide some properties about the solution of problem ($\mathcal{P}_{\text{R-prezone}}$).

Proposition 3 (A solution of Problem ($\mathcal{P}_{\text{R-prezone}}$)) *If*

$$\underline{u}_i \leq \frac{6 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^2} - \frac{2(v_i^{\underline{t_Z}} + 2v_i^0)}{\underline{t_Z}} \leq \bar{u}_i$$

and

$$\underline{u}_i \leq \frac{-6 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^2} + \frac{4(v_i^{\underline{t_Z}} + v_i^0)}{\underline{t_Z}} \leq \bar{u}_i$$

Then one has

$$\begin{aligned} u_i^*(t) &= - \left(\frac{12 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^3} - \frac{6(v_i^{\underline{t_Z}} + v_i^0)}{\underline{t_Z}^2} \right) t + \frac{6 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^2} - \frac{2(v_i^{\underline{t_Z}} + 2v_i^0)}{\underline{t_Z}}, \\ v_i^*(t) &= - \left(\frac{6 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^3} - \frac{3(v_i^{\underline{t_Z}} + v_i^0)}{\underline{t_Z}^2} \right) t^2 + \left(\frac{6 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^2} - \frac{2(v_i^{\underline{t_Z}} + 2v_i^0)}{\underline{t_Z}} \right) t + v_i^0, \\ (x_i^X)^*(t) &= - \left(\frac{2 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^3} - \frac{(v_i^{\underline{t_Z}} + v_i^0)}{\underline{t_Z}^2} \right) d_i^X t^3 + \left(\frac{3 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^2} - \frac{(v_i^{\underline{t_Z}} + 2v_i^0)}{\underline{t_Z}} \right) d_i^X t^2 \\ &\quad + v_i^0 d_i^X t + (x_i^0)^X, \\ (x_i^Y)^*(t) &= - \left(\frac{2 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^3} - \frac{(v_i^{\underline{t_Z}} + v_i^0)}{\underline{t_Z}^2} \right) d_i^Y t^3 + \left(\frac{3 \left((x_i^{\underline{t_Z}})^D - (x_i^0)^D \right)}{d_i^D \underline{t_Z}^2} - \frac{(v_i^{\underline{t_Z}} + 2v_i^0)}{\underline{t_Z}} \right) d_i^Y t^2 \\ &\quad + v_i^0 d_i^Y t + (x_i^0)^Y, \end{aligned}$$

where $d_i^D = d_i^X$ if $d_i^X \neq 0$ else $d_i^D = d_i^Y$ (note that $d_i = (d_i^X, d_i^Y)$ is the cartesian direction of aircraft i and cannot be the null vector).

The optimal cost is:

$$J^* := J(u^*) = \sum_{i \in I} \left(- \left(\frac{6 \left((x_i^{t_Z})^D - (x_i^0)^D \right)}{d_i^D t_Z^3} - \frac{3(v_i^{t_Z} + v_i^0)}{t_Z^2} \right) t_f^2 + \left(\frac{6 \left((x_i^{t_Z})^D - (x_i^0)^D \right)}{d_i^D t_Z^2} - \frac{2(v_i^{t_Z} + 2v_i^0)}{t_Z} \right) t_f \right).$$

Proof 4 Let us first relax problem $(\mathcal{P}_{R_prezone})$ by discarding bound constraints on u : $\underline{u}_i \leq u_i(t) \leq \bar{u}_i, \quad \forall t \in [t_0, t_Z] \forall i \in I$.

Note that problem $(\mathcal{P}_{R_prezone})$ corresponds to the sum of n independent sub-problems, one for each $i \in I$, as follows:

$$(\mathcal{P}_{prezone}^i) \left\{ \begin{array}{l} \min_{u_i, v_i, x_i} \int_0^{t_Z} u_i^2(t) dt \\ \text{s.t.} \\ \dot{v}_i(t) = u_i(t) \quad \forall t \in [t_0, t_Z], \\ \dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [t_0, t_Z], \\ x_i(0) = x_i^0, \quad v_i(0) = v_i^0 \\ x_i(t_Z) = x_i^{t_Z}, \quad v_i(t_Z) = v_i^{t_Z} \end{array} \right.$$

As mentioned in Remark 4, problem $(\mathcal{P}_{postzone}^i)$ and problem $(\mathcal{P}_{prezone}^i)$ only differ because the final position has now to be fixed. Thus, the Hamiltonian of this new system is the same as for problem $(\mathcal{P}_{postzone}^i)$.

Hence, the maximum of the Hamiltonian with respect to u_i , (by fixing the value $z_{0,i}$ to -1) is the same as for problem $(\mathcal{P}_{postzone})$: $u_i^*(t) = \operatorname{argmax}_{u_i} H_i(\cdot) = \frac{z_{3,i}(t)}{2}$.

This result also proves that u^* is the unique maximum for problem $(\mathcal{P}_{prezone}^i)$.

By replacing $u_i(t)$ in the differential equations by $\frac{z_{3,i}(t)}{2}$, this yields a system with 6 equations, 6 unknowns, 3 initial conditions and 3 final conditions. Note that we have now two final conditions (coming from the final position at time t_Z). Therefore, the following system of differential equations has to be solved in a different way with respect to what has been done for the postzone, because $\dot{z}_{3,i}$ is no

more equal to zero and then $z_{3,i}$ and the control u_i^* is no more constant in $[0, t_Z]$:

$$\begin{aligned}
\dot{x}_i^X(t) &= \frac{\partial H_i(\cdot)}{\partial z_{1,i}} = v_i(t) d_i^X, \\
\dot{x}_i^Y(t) &= \frac{\partial H_i(\cdot)}{\partial z_{2,i}} = v_i(t) d_i^Y, \\
\dot{v}_i(t) &= \frac{\partial H_i(\cdot)}{\partial z_{3,i}} = \frac{z_{3,i}(t)}{2}, \\
\dot{z}_{1,i}(t) &= -\frac{\partial H_i(\cdot)}{\partial x_i^X} = 0, \\
\dot{z}_{2,i}(t) &= -\frac{\partial H_i(\cdot)}{\partial x_i^Y} = 0, \\
\dot{z}_{3,i}(t) &= -\frac{\partial H_i(\cdot)}{\partial v_i} = -z_{1,i}(t) d_i^X - z_{2,i}(t) d_i^Y.
\end{aligned}$$

The fourth and fifth equations yields that $z_{1,i}^*(t) = A$ and $z_{2,i}^*(t) = B$, where A and B are two real constants. Hence, the sixth equation gives $\dot{z}_{3,i}(t) = -A d_i^X - B d_i^Y$ and then

$$z_{3,i}(t) = (-A d_i^X - B d_i^Y) t + C,$$

where C is a new real constant. Note that this provides an optimal control u^* which is now affine:

$$u_i^*(t) = \frac{(-A d_i^X - B d_i^Y) t + C}{2}.$$

The third equation yields the optimal velocity v^* :

$$v_i^*(t) = -\frac{(A d_i^X + B d_i^Y)}{4} t^2 + \frac{C}{2} t + D,$$

where D is a new real constant (note that this constant is obviously distinct to the separation distance D and it is just only used in this proof).

The first equation $\dot{x}_i^X(t) = -\left(\frac{(A d_i^X + B d_i^Y)}{4} t^2 + \frac{C}{2} t + D\right) d_i^X$ yields the optimal abscissa position:

$$(x_i^X)^*(t) = -\frac{(A (d_i^X)^2 + B d_i^X d_i^Y)}{12} t^3 + \frac{C}{4} d_i^X t^2 + D d_i^X t + E,$$

where E is a new real constant.

Moreover, by introducing a new real constant F , and using the second equation, one obtains in the same way the optimal ordinate position:

$$(x_i^Y)^*(t) = -\frac{(A d_i^X d_i^Y + B (d_i^Y)^2)}{12} t^3 + \frac{C}{4} d_i^Y t^2 + D d_i^Y t + F.$$

From the 6 initial and final conditions, one can determine the six constant A, B, C, D, E, F . Directly, from the initial conditions one obtains $D = v_i^0$, $E = (x_i^0)^X$ and $F = (x_i^0)^Y$. From the final condition, it remains to solve a linear system of 3 equations and 3 variables A, B, C . By defining $Q := \frac{(A d_i^X + B d_i^Y)}{4}$, one obtains:

$$\begin{aligned} -Qt_Z^2 + \frac{C}{2} t_Z + v_i^0 &= v_i^{t_Z}, \\ -\frac{1}{3} Q d_i^D t_Z^3 + \frac{C}{4} d_i^D t_Z^2 + v_i^0 d_i^D t_Z + (x_i^0)^D &= (x_i^{t_Z})^D. \end{aligned}$$

Where d_i^D is equal to d_i^X or d_i^Y if and only if they are not null (they cannot be both equal to zero).

Hence, one obtains:

$$C = \frac{12 \left((x_i^{t_Z})^D - (x_i^0)^D \right)}{d_i^D t_Z^2} - \frac{4(v_i^{t_Z} + 2v_i^0)}{t_Z},$$

and

$$Q = \frac{6 \left((x_i^{t_Z})^D - (x_i^0)^D \right)}{d_i^D t_Z^3} - \frac{3(v_i^{t_Z} + v_i^0)}{t_Z^2}.$$

Only these five coefficients are necessary to find all the optimal controls and the associate state variables u^*, v^*, x^* and also the objective value of the cost function J^* .

Let us consider the bound constraints on u : $\underline{u}_i \leq u_i(t) \leq \bar{u}_i$ for all $i \in I$ and $t \in [0, t_Z]$:

$$\underline{u}_i \leq u_i^*(t) = -2Qt + \frac{C}{2} \leq \bar{u}_i, \forall t \in [0, t_Z]$$

As $u^*(t)$ is affine, one has just to check the constraint at time $t = 0$ and $t = t_Z$ providing: $2\underline{u}_i \leq C \leq 2\bar{u}_i$ and $2\underline{u}_i \leq -4Qt_Z + C \leq 2\bar{u}_i$. Then one obtains the two following conditions:

$$\underline{u}_i \leq \frac{C}{2} = \frac{6 \left((x_i^{t_Z})^D - (x_i^0)^D \right)}{d_i^D t_Z^2} - \frac{2(v_i^{t_Z} + 2v_i^0)}{t_Z} \leq \bar{u}_i, \quad (3)$$

and

$$\underline{u}_i \leq -2Qt_Z + \frac{C}{2} = \frac{-6 \left((x_i^{t_Z})^D - (x_i^0)^D \right)}{d_i^D t_Z^2} + \frac{4(v_i^{t_Z} + v_i^0)}{t_Z} \leq \bar{u}_i. \quad (4)$$

Hence, the unique solution verifying the optimality conditions derived from the PMP is u^*, v^*, x^* if the above conditions (3) and (4) on the control are satisfied. Note that $u_i^*(t)$ can be also constant in the prezone as in the postzone, if and only if $Q = 0$.

The optimal cost function is computed from u^* :

$$J(u^*) = \sum_{i \in I} \int_0^{t_f} (u^*)^2 dt = \sum_{i \in I} \int_0^{t_f} \left(-2Qt + \frac{C}{2} \right)^2 dt = \sum_{i \in I} \left(-Qt_f^2 + \frac{C}{2} t_f \right).$$

Therefore, we proved that from the relaxed problem ($\mathcal{P}_{R_prezone}$), the optimal control u^* can be provided using the PMP; the velocity, the position and the final cost can then be derived. However, such a solution exists if the conditions (3) and (4) are satisfied.

Remark 6 Analogously to the case of the postzone, we can observe that the bounds on the control are in practice quite large, and so easily satisfied. It follows that in general, we have a solution of problem ($\mathcal{P}_{R_prezone}$).

Let us now take into account the bounds constraints on $v_i(t)$. In this case, we have to consider the following assumption on the shape of the optimal control, which follows directly from Proposition 3 and by the numerical solutions provided in Section 7:

Assumption 1 The optimal control on the time window $[0, t_Z]$ has the following shape, for each aircraft i : $u_i^*(t)$ is affine or $u_i^*(t)$ is affine from $t = 0$ until a critical time that we denote t_i^c , and it is null after. The critical time t_i^c occurs when the velocity of aircraft i reaches a bound constraint \underline{v}_i or \bar{v}_i . This yields a velocity $v_i(t)$ which is quadratic or quadratic and then constant at time t_i^c . Furthermore, let us suppose also that $u^*(t)$ remains in its bounds; this implies that the conditions of Proposition 3 are satisfied.

Remark 7 Note that if the velocity $v_i(t)$ stays in its bounds for all t in $[0, t_Z]$, then by fixing $t_i^c = t_Z$, one obtains just one shape for the control: affine and null at time t_i^c .

By decomposing the time interval into three parts $[0, \underline{t}_Z]$, $[\underline{t}_Z, \overline{t}_Z]$ and $[\overline{t}_Z, t_f]$, by using Propositions 1, 2, 3 and Assumption 1, one can reformulate problem (\mathcal{P}) starting from reformulation (\mathcal{P}_1) . This provides a new optimization problem with less variables and constraints than problems (\mathcal{P}) and (\mathcal{P}_1) .

Theorem 2 (Reformulation of problem (\mathcal{P})) *Under Assumption 1, another reformulation of problem (\mathcal{P}) , with $t_0 = 0$, is:*

$$\begin{array}{l}
 \left. \begin{array}{l}
 \min_{u, v, x,} \sum_{i \in I} \left(\frac{1}{3} A_i^2 (t_i^c)^3 + A_i B_i (t_i^c)^2 + B_i^2 t_i^c + \int_{\underline{t}_Z}^{\overline{t}_Z} u_i^2(t) dt + \frac{(v_i^f - v_i(\overline{t}_Z))^2}{t_f - \overline{t}_Z} \right) \\
 (A, B, t^c) \in \mathbb{R}^{3n} \\
 \text{s.t.} \\
 \dot{v}_i(t) = u_i(t) \quad \forall t \in [\underline{t}_Z, \overline{t}_Z], \forall i \in I \\
 \dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [\underline{t}_Z, \overline{t}_Z], \forall i \in I \\
 \underline{u}_i \leq u_i(t) \leq \overline{u}_i \quad \forall t \in [\underline{t}_Z, \overline{t}_Z], \forall i \in I \\
 \underline{v}_i \leq v_i(t) \leq \overline{v}_i \quad \forall t \in [\underline{t}_Z, \overline{t}_Z], \forall i \in I \\
 \underline{u}_i \leq \frac{v_i^f - v_i(\overline{t}_Z)}{t_f - \overline{t}_Z} \leq \overline{u}_i \quad \forall i \in I \\
 \underline{u}_i \leq B_i \leq \overline{u}_i \quad \forall i \in I \\
 \underline{u}_i \leq A_i t_i^c + B_i \leq \overline{u}_i \quad \forall i \in I \\
 v_i(\underline{t}_Z) = \frac{1}{2} A_i (t_i^c)^2 + B_i t_i^c + v_i^0 \quad \forall i \in I \\
 x_i(\underline{t}_Z) = v_i(\underline{t}_Z) d_i (\underline{t}_Z - t_i^c) + \left(\frac{1}{6} A_i (t_i^c)^3 + \frac{1}{2} B_i (t_i^c)^2 + v_i^0 t_i^c \right) d_i + x_i^0 \quad \forall i \in I \\
 \left[x_i(\underline{t}_Z) = \left(\frac{\underline{t}_Z}{2} - \frac{t_i^c}{3} \right) d_i A_i (t_i^c)^2 + \left(\underline{t}_Z - \frac{t_i^c}{2} \right) B_i d_i t_i^c + \underline{t}_Z d_i v_i^0 + x_i^0 \right] \quad \forall i \in I \\
 t_i^c = \min\{\underline{t}_Z\} \cup \left\{ t \in [0, \underline{t}_Z], \frac{1}{2} A_i t^2 + B_i t + C_i = \underline{v}_i \right\} \\
 \quad \cup \left\{ t \in [0, \underline{t}_Z], \frac{1}{2} A_i t^2 + B_i t + C_i = \overline{v}_i \right\} \quad \forall i \in I \\
 x_i(0) = x_i^0, \quad v_i(0) = v_i^0 \quad \forall i \in I \\
 x_i(t_f) \text{ free}, \quad v_i(t_f) = v_i^f \quad \forall i \in I \\
 \|x_i(t) - x_j(t)\|^2 \geq D^2 \quad \forall t \in [\underline{t}_Z, \overline{t}_Z], \\
 \quad \forall (i, j) \in I^2, i < j
 \end{array} \right\}
 \end{array}$$

Note that there are major differences between the problem formulations \mathcal{P}_1 and \mathcal{P}_2 . In the formulation \mathcal{P}_2 , for each aircraft i , new real optimization variables

A_i , B_i , and t_i^c are introduced. Furthermore, the cost function in \mathcal{P}_2 is no convex anymore.

Proof 5 The time window $[0, t_f]$ is now decomposed into three parts $[t_0, t_Z]$, $[t_Z, \overline{t_Z}]$ and $[\overline{t_Z}, t_f]$, hence problem (P) becomes almost separable. The links occur at time in t_Z and also at $\overline{t_Z}$. Indeed, the three subproblems can provide disconnected solutions of u , v and x , but only at times t_Z and $\overline{t_Z}$.

This new reformulation starts from reformulation (\mathcal{P}_1) yielding the last part of the objective function $\sum_{i \in I} \frac{(v_i^f - v_i(\overline{t_Z}))^2}{t_f - \overline{t_Z}}$ and the constraints $\underline{u}_i \leq \frac{v_i^f - v_i(\overline{t_Z})}{t_f - \overline{t_Z}} \leq \overline{u}_i, \forall i \in I$. Note that these are discussed in the proof of Theorem 1. Hence, we have just to demonstrate the first part of this new formulation, from $t = 0$ to $\overline{t_Z}$.

From Assumption 1, we can directly derive the shape of the solution. Indeed, the control is affine from $t = 0$ to t_i^c :

$$u_i^*(t) = A_i t + B_i,$$

where A_i and B_i are real constants which are the new unknown of our new formulation. This provides $3 \times n$ new optimization variables: A_i, B_i and t_i^c . Thus, two new constraints have to be added in order to satisfy the conditions of Proposition 3:

$$\underline{u}_i \leq B_i \leq \overline{u}_i \quad \text{and} \quad \underline{u}_i \leq A_i t_i^c + B_i \leq \overline{u}_i, \quad \forall i \in I.$$

From the solution on the optimal control, one can derive the first part of the cost function J :

$$\int_0^{\overline{t_Z}} (u_i^*(t))^2 dt = \int_0^{t_i^c} (A_i t + B_i)^2 dt = \frac{A_i^2}{3} (t_i^c)^3 + A_i B_i (t_i^c)^2 + B_i^2 t_i^c.$$

Note that $u_i^*(t) = 0$ for all t in $[0, t_i^c]$.

From $u_i^*(t) = A_i t + B_i$ for all t in $[0, t_i^c]$ and zero after, one can provide the velocity:

$$v_i^*(t) = \begin{cases} \frac{A_i}{2} t^2 + B_i t + v_i^0, & \forall t \in [0, t_i^c] \\ \frac{A_i}{2} (t_i^c)^2 + B_i t_i^c + v_i^0, & \forall t \in [t_i^c, \overline{t_Z}] \end{cases}$$

This implies the link between the prezone and the zone at time t_Z :

$$v_i^*(\overline{t_Z}) = v_i^*(t_i^c) = \frac{A_i}{2} (t_i^c)^2 + B_i t_i^c + v_i^0, \quad \forall i \in I.$$

Now, from the optimal velocity, one can derive the position of aircraft i :

$$x_i^*(t) = t_i^c \begin{cases} \frac{A_i}{6} d_i t^3 + \frac{B_i}{2} d_i t^2 + v_i^0 d_i t + x_i^0, & \forall t \in [0, t_i^c] \\ \left(\frac{A_i}{2} (t_i^c)^2 + B_i t_i^c + v_i^0 \right) d_i (t - t_i^c) + \frac{A_i}{6} d_i (t_i^c)^3 + \frac{B_i}{2} d_i (t_i^c)^2 + v_i^0 d_i (t_i^c) + x_i^0, & \forall t \in [t_i^c, \overline{t_Z}] \end{cases}$$

Hence, the following constraint at time \underline{t}_Z on the position is:

$$x_i^*(\underline{t}_Z) = \left(\frac{A_i}{2}(t_i^c)^2 + B_i t_i^c + v_i^0 \right) d_i(\underline{t}_Z - t_i^c) + \frac{A_i}{6} d_i(t_i^c)^3 + \frac{B_i}{2} d_i(t_i^c)^2 + v_i^0 d_i(t_i^c) + x_i^0.$$

Now, it remains to explain the critical time t_i^c . It is defined as the time in $[0, \underline{t}_Z]$ when the velocity of aircraft i reaches for the first time a limit \underline{v}_i or \overline{v}_i . Note that if the velocity stays in $[\underline{v}_i, \overline{v}_i]$, then t_i^c has to be fixed to \underline{t}_Z . Therefore, the following condition involving t_i^c , A_i and B_i holds:

$$t_i^c = \min\{\underline{t}_Z\} \cup \left\{ t \in [0, \underline{t}_Z], \frac{1}{2} A_i t^2 + B_i t + C_i = \underline{v}_i \right\} \cup \left\{ t \in [0, \underline{t}_Z], \frac{1}{2} A_i t^2 + B_i t + C_i = \overline{v}_i \right\}$$

That concludes the proof.

Note that the control u can be discontinuous, because it is possible for an aircraft to roughly change its acceleration. Therefore, u_i can be discontinuous at time \overline{t}_Z but also at times t_i^c and \underline{t}_Z ; indeed, u_i has just to be integrable over $[0, t_i^c]$, $[t_i^c, \underline{t}_Z]$, $[\underline{t}_Z, \overline{t}_Z]$ and $[\overline{t}_Z, t_f]$. For the velocity v and the position x , one must keep the continuity of the solution at times t_i^c , \underline{t}_Z and \overline{t}_Z .

Remark 8 *The new formulation (\mathcal{P}_2) only provides solutions for all t in $[\underline{t}_Z, \overline{t}_Z]$ (if such an optimal solution exists). All the remaining solutions, for all time t in $[0, \underline{t}_Z]$, have been developed in the proof of Theorem 2. The last part of the solutions for a time t in $[\overline{t}_Z, t_f]$ are given in Proposition 2.*

6 Applying the direct shooting method

Let us now focus on the numerical solution of Problem (\mathcal{P}) and its two reformulations (\mathcal{P}_1) and (\mathcal{P}_2) . In all these three cases, we have to solve an optimal control

problem in a time window $[t_a, t_b]$, with $t_0 \leq t_a \leq t_b \leq t_f$, of the following type:

$$(\tilde{\mathcal{P}}) \left\{ \begin{array}{l} \min_{u,v,x} \sum_{i \in I} \int_{t_a}^{t_b} u_i^2(t) dt \\ s.t. \\ \dot{v}_i(t) = u_i(t) \quad \forall t \in [t_a, t_b], \forall i \in I \\ \dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [t_a, t_b], \forall i \in I \\ \underline{u}_i \leq u_i(t) \leq \overline{u}_i \quad \forall t \in [t_a, t_b], \forall i \in I \\ \underline{v}_i \leq v_i(t) \leq \overline{v}_i \quad \forall t \in [t_a, t_b], \forall i \in I \\ x_i(t_a) = x_i^a, \quad v_i(t_a) = v_i^a \quad \forall i \in I \\ x_i(t_b) \text{ free}, \quad v_i(t_b) = v_i^b \quad \forall i \in I \\ \|x_i(t) - x_j(t)\|^2 \geq D^2 \quad \forall t \in [t_a, t_b], \\ \forall (i,j) \in I^2, i < j \end{array} \right.$$

Problem $(\tilde{\mathcal{P}})$ corresponds to Problem (\mathcal{P}) when $t_a = t_0$ and $t_b = t_f$. In the case when we consider the problem decomposition based on the *postzone*, then $t_a = t_0$ and $t_b = \overline{t_Z}$; finally in the case when we consider the problem decomposition based on the *prezone*, then $t_a = \underline{t_Z}$ and $t_b = \overline{t_Z}$.

By applying the direct shooting method to the optimal control problem $(\tilde{\mathcal{P}})$, the ordinary differential equations are computationally treated by discretizing the variables appearing in the equations with respect to the time. Numerical integrators (for example, *Euler*-type integrators) are used to approximate the differential equations.

By discretizing the time in $[t_a, t_b]$ into N steps ($h = \frac{t_b - t_a}{N}$), and then by discretizing the controls u_i and the state variables v_i, x_i (for all aircraft i), we obtain the following NLP problem:

$$(\tilde{P}^{disc}) \left\{ \begin{array}{l} \min_{(x,v,u)} \quad h \sum_{i \in I} \sum_{k \in K} (u_i^{(k)})^2 \\ s.t. \\ v_i^{(k+1)} = \text{NUM}_{v_i}(u_i^{(k)}) \quad \forall k \in K, \forall i \in I \\ x_i^{(k+1)} = \text{NUM}_{x_i}(v_i^{(k)}) \quad \forall k \in K, \forall i \in I \\ \underline{u}_i \leq u_i^{(k)} \leq \overline{u}_i \quad \forall k \in K, \forall i \in I \\ \underline{v}_i \leq v_i^{(k)} \leq \overline{v}_i \quad \forall k \in K, \forall i \in I \\ v_i^{(0)} = v_i^a \quad \forall i \in I \\ x_i^{(0)} = x_i^a \quad \forall i \in I \\ v_i^{(N)} = v_i^b \quad \forall i \in I \\ \|x_i^{(k)} - x_j^{(k)}\|^2 \geq D^2 \quad \forall k \in K, \forall i < j, (i, j) \in I^2 \end{array} \right.$$

where NUM is a numerical integrator depending on the state variables, and $u_i^{(k)}$, $v_i^{(k)}$ and $x_i^{(k)}$ denote the values of the acceleration, velocity and respectively position at time $t_k = kh$. Variables are indexed on the set $K = \{0, \dots, N-1\}$.

From problem (\mathcal{P}_1) , we obtain the following discretized problem:

$$(\tilde{P}_1^{disc}) \left\{ \begin{array}{l} \min_{(x,v,u)} \quad h \sum_{i \in I} \sum_{k \in K} (u_i^{(k)})^2 + \sum_{i \in I} \frac{(v_i^f - v_i^{(N)})^2}{t_f - t_Z} \\ s.t. \\ v_i^{(k+1)} = \text{NUM}_{v_i}(u_i^{(k)}) \quad \forall k \in K, \forall i \in I \\ x_i^{(k+1)} = \text{NUM}_{x_i}(v_i^{(k)}) \quad \forall k \in K, \forall i \in I \\ \underline{u}_i \leq u_i^{(k)} \leq \overline{u}_i \quad \forall k \in K, \forall i \in I \\ \underline{v}_i \leq v_i^{(k)} \leq \overline{v}_i \quad \forall k \in K, \forall i \in I \\ \underline{u}_i \leq \frac{v_i^f - v_i^{(N)}}{t_f - t_Z} \leq \overline{u}_i \quad \forall i \in I \\ v_i^{(0)} = v_i^0 \quad \forall i \in I \\ x_i^{(0)} = x_i^0 \quad \forall i \in I \\ v_i^{(N)} = v_i^{\overline{t_Z}} \quad \forall i \in I \\ \|x_i^{(k)} - x_j^{(k)}\|^2 \geq D^2 \quad \forall k \in K, \forall i < j, (i, j) \in I^2 \end{array} \right.$$

From problem (\mathcal{P}_2) , we obtain the following discretized problem:

$$\left. \begin{array}{l}
\min_{u, v, x,} \sum_{i \in I} \left(\frac{1}{3} A_i^2 (t_i^c)^3 + A_i B_i (t_i^c)^2 + B_i^2 t_i^c + h \sum_{k \in K} (u_i^{(k)})^2 + \frac{(v_i^f - v_i^{(N)})^2}{t_f - t_Z} \right) \\
(A, B, t^c) \in (\mathbb{R}^n)^3 \\
s.t. \\
v_i^{(k+1)} = \text{NUM}_{v_i}(u_i^{(k)}) \quad \forall k \in K, \forall i \in I \\
x_i^{(k+1)} = \text{NUM}_{x_i}(v_i^{(k)}) \quad \forall k \in K, \forall i \in I \\
\underline{u}_i \leq u_i^{(k)} \leq \overline{u}_i \quad \forall k \in K, \forall i \in I \\
\underline{v}_i \leq v_i^{(k)} \leq \overline{v}_i \quad \forall k \in K, \forall i \in I \\
\underline{u}_i \leq \frac{v_i^f - v_i^{(N)}}{t_f - t_Z} \leq \overline{u}_i \quad \forall i \in I \\
\underline{u}_i \leq B_i \leq \overline{u}_i \quad \forall i \in I \\
\underline{u}_i \leq A_i t_i^c + B_i \leq \overline{u}_i \quad \forall i \in I \\
t_i^c = \min\{t_Z\} \cup \left\{ t \in [0, t_Z], \frac{1}{2} A_i t^2 + B_i t + C_i = \underline{v}_i \right\} \\
\cup \left\{ t \in [0, t_Z], \frac{1}{2} A_i t^2 + B_i t + C_i = \overline{v}_i \right\} \quad \forall i \in I \\
v_i^{(0)} = \frac{1}{2} A_i (t_i^c)^2 + B_i t_i^c + v_i^0 \quad \forall i \in I \\
x_i^{(0)} = v_i(t_Z) d_i(t_Z - t_i^c) + \left(\frac{1}{6} A_i (t_i^c)^3 + \frac{1}{2} B_i (t_i^c)^2 + v_i^0 t_i^c \right) d_i + x_i^0 \quad \forall i \in I \\
v_i^{(N)} = v_i^{\overline{t_Z}} \quad \forall i \in I \\
\|x_i^{(k)} - x_j^{(k)}\|^2 \geq D^2 \quad \forall k \in K, \forall i < j, (i, j) \in I^2
\end{array} \right\} (\tilde{P}_2^{disc})$$

By considering h fixed, we have N discretization steps for (\tilde{P}^{disc}) with $t_a = t_0$ and $t_b = t_f$, and respectively N_1 and N_2 for problems (\tilde{P}_1^{disc}) and (\tilde{P}_2^{disc}) ; note that $N \geq N_1 \geq N_2$. Hence, for (\tilde{P}^{disc}) with $t_a = t_0$ and $t_b = t_f$, we have $4nN$ variables, $N \left(\frac{n(n-1)}{2} + 3n \right) + n$ equality and inequality constraints and $4nN$ bound constraints, for (\tilde{P}_1^{disc}) , $4nN_1$ variables, $N_1 \left(\frac{n(n-1)}{2} + 3n \right) + n$ equality and inequality constraints and $4nN_1 + n$ bound constraints and for (\tilde{P}_2^{disc}) , $4nN_2 + 3n$ variables, $N_1 \left(\frac{n(n-1)}{2} + 4n \right) + 3n$ equality and inequality constraints and $4nN_1 + 3n$ bound constraints.

Note that we can alternatively use a semi-direct shooting method on $(\tilde{\mathcal{P}})$ where

the only variables are the discretized controls $u_i^{(k)}$ and the discretized state variables are computed from $u_i^{(k)}$ through the equality constraints. This yields more compact formulations with nN variables, $N \binom{n(n-1)}{2} + n$ equality and inequality constraints and $4nN$ bound constraints. Nevertheless, we show in [12] that the direct shooting method is more efficient on the considered problem. Therefore in the following we use the direct shooting method for our computational tests.

7 Computational Results

In this section, we discuss numerical results obtained by applying the proposed optimal control-based approaches to solve the aircraft conflict avoidance problem. Computational experiments were carried out using the AMPL [18] modelling environment and the Interior Point method-based solver IPOPT [25] for large-scale nonlinear problems. We implemented our approach implying the application of the PMP in the prezone in C programming language, as the computation of t^c induces logical expressions to be verified, that makes an AMPL implementation (easily applied in the two cases other than the prezone) not much performing. All the numerical tests were performed on a 2.66GHz Intel Xeon (octo-core) processor with 32GB of RAM and Linux Operating System.

Test problem instances are generated in the following way. Aircraft are placed on a circle, flying with a heading pointing toward the center of the circle but deviated with an angle belonging to $[-\pi/6, +\pi/6]$, see Figure 5 for an illustration. The aircraft trajectories are represented as straight lines in \mathbb{R}^2 . The horizontal separation norm is 5NM. For each aircraft i , its initial (nominal) velocity $v_i^{t_0}$ is either 400 or 447NM/h, or belongs to the interval $[400; 450]$. These values correspond to common aircraft speeds. The tight bounds imposed on velocity values are due to operational reasons, and follow the directives of the ERASMUS project [4] to perform a *subliminal* velocity control. Namely, we consider:

- $[v_i^{t_0} - 6\%v_i^{t_0}, v_i^{t_0} + 3\%v_i^{t_0}]$ for the ‘weak’ ERASMUS regulation;
- $[v_i^{t_0} - 12\%v_i^{t_0}, v_i^{t_0} + 6\%v_i^{t_0}]$ for the ‘strong’ ERASMUS regulation;
- $[v_i^{t_0} - 10\%v_i^{t_0}, v_i^{t_0} + 10\%v_i^{t_0}]$ for a symmetric-range regulation.

The bounds imposed on aircraft accelerations are based on the Eurocontrol’s Base of Aircraft Data (BADA) [17], and have values $\bar{u}_i = -\underline{u}_i = 4000\text{NM}/\text{h}^2$.

A time window $[t_0, t_f]$ with $t_0 = 0$ and $t_f = 1\text{h}$ is considered to issue separation maneuvers (acting on velocities), ensuring separation throughout all the time window. At final time t_f , aircraft are expected to return to their nominal velocities $v_i^{t_0}$ (terminal conditions in our optimal control model).

Table 1 summarizes the characteristics of the considered problem instances: number of aircraft n , number of initial *potential conflicts*, number of initial conflicts actually occurring unless speed changes are performed (*potential concourses*), percentage of time spent in the *zone* (see Section 3), nominal velocities (in NM/h), allowed percentage of speed change (*e.g.*, with the weak ERASMUS regulation, from -6% to $+3\%$), and sum of all initial potential conflict times.

In our numerical experiments, the time discretization step is equal to $h = 15''$. The *starting points* are deduced from the trivial controls, in the sense that all the state variables (velocities and positions) are computed from command (acceleration) values equal to 0. A complete descriptions of these 30 instances can be found in [11].

In Tables 2, 3 and 4, we present the results obtained applying the proposed optimal control approaches, presented in Sections 4, 5 and 6. In these tables, for each problem instance, we report the number of variables, the number of constraints, the total number of iterations and the number of function evaluations carried out by the NLP solver (IPOPT), the computed objective function value and the computing time.

Table 2 reports the results obtained applying the direct shooting method on the whole time window $[0, t_f]$, while the results in Table 3 and the results in Table 4 are obtained applying such method respectively only in the time interval $[0, \overline{t_Z}]$ (*i.e.*, from the initial time to the end of the zone), then computing the PMP-based solution in the postzone, and only in the time interval $[\underline{t_Z}, \overline{t_Z}]$ (*i.e.*, in the zone), then computing the PMP-based solution in the prezone as well as in the postzone.

Let us first consider Table 2. We are able to compute an optimal solution for 29 problem instances out of 30 (pb16 for which the maximum number of iterations is reached). As expected, the computing time increases with the size of the problem and its complexity, that is mainly related to the number of potential conflicts and concourses. Applying the direct method on the whole time window yields in two cases to computing times that exceed 300 seconds.

For the sake of illustration, Figure 5 represents the \mathbb{R}^2 -trajectory configuration for the 13-aircraft conflict avoidance problem pb29. For the same problem instance, Figure 6 displays the optimal velocity solution curves.

Combining the direct method to the PMP in the postzone (Table 3), computing times are drastically reduced. As expected, the benefit is directly related to the size of the time window during which the direct method is applied, and thereby to the size of the zone: the more the zone is small, the more the PMP is applied on a larger time interval, thus yielding a reduction in the total computing time. More specifically, the size of the zone is related to the number of variables and constraints

pb. ID	nb. of aircr.	nb. of confl.	nb. of conc.	ZONE time % w.r.t. 1h	nominal velocity (NM/h)			velocity bound percents			conflict times sum (seconds)
					400	447	[400; 450]	[-6; 3]	[-12; 6]	[-10; 10]	
pb01	3	2	3	13	✓					✓	61.0
pb02	3	2	3	16	✓					✓	50.5
pb03	3	2	2	11	✓			✓			73.3
pb04	4	3	3	16			✓			✓	150.0
pb05	4	3	3	16			✓		✓		102.2
pb06	5	4	5	26		✓				✓	206.1
pb07	5	4	5	26			✓			✓	225.0
pb08	6	4	6	41	✓			✓			315.7
pb09	6	4	5	31			✓		✓		207.9
pb10	6	4	5	15			✓	✓			225.0
pb11	7	4	11	41	✓				✓		416.4
pb12	7	4	11	36		✓			✓		372.6
pb13	7	4	9	26			✓			✓	291.6
pb14	8	4	7	41	✓				✓		346.2
pb15	8	4	5	53	✓			✓			236.6
pb16	8	6	10	16	✓			✓			411.2
pb17	9	5	13	35	✓				✓		357.0
pb18	9	5	13	31		✓			✓		319.5
pb19	9	5	13	31		✓				✓	319.5
pb20	10	5	5	31		✓			✓		232.0
pb21	10	5	5	31		✓				✓	232.0
pb22	10	5	5	35	✓				✓		259.2
pb23	11	6	17	45	✓			✓			521.1
pb24	11	5	7	54	✓			✓			259.1
pb25	11	6	15	51			✓		✓		320.4
pb26	12	6	12	32	✓			✓			643.0
pb27	12	6	15	46		✓		✓			472.2
pb28	12	6	12	29		✓		✓			575.4
pb29	13	9	14	67			✓	✓			1294.0
pb30	13	6	14	59		✓		✓			418.3

Table 1: Data instances: problem names, number of aircraft, number of initial *potential conflicts*, number of initial *potential concourses*, time spent in the *zone*, nominal velocity, percentages of speed changes, and sum of durations of initial *potential conflict*.

pb. ID	nb. of var.	nb. of constr.	nb. of it.	nb. of eval.	obj. value	cpu time (seconds)
pb01	2898	5781	19	31	78.212	0.892
pb02	2898	5781	37	234	145.547	1.828
pb03	2898	5781	23	137	912.703	0.976
pb04	3864	8190	33	42	1073.489	1.860
pb05	3864	8190	97	551	648.361	5.220
pb06	4830	10840	138	957	3161.899	10.888
pb07	4830	10840	45	121	948.502	3.720
pb08	5796	13731	136	158	1271.155	18.804
pb09	5796	13731	31	116	335.791	3.840
pb10	5796	13731	63	79	1675.380	6.960
pb11	6762	16863	41	75	601.908	6.864
pb12	6762	16863	52	99	796.172	8.968
pb13	6762	16863	26	141	714.660	4.664
pb14	7728	20236	68	139	931.898	13.232
pb15	7728	20236	41	158	1857.248	9.760
pb16	7728	20236	3000	4238	118446.731	592.500
pb17	8694	23850	26	105	646.542	8.192
pb18	8694	23850	95	581	814.456	30.272
pb19	8694	23850	48	212	814.456	15.332
pb20	9660	27705	847	8941	6130.617	339.148
pb21	9660	27705	112	974	6130.617	43.396
pb22	9660	27705	119	1040	4921.333	43.676
pb23	10626	31801	105	220	5885.763	64.260
pb24	10626	31801	57	240	1612.852	33.076
pb25	10626	31801	51	135	2675.255	31.464
pb26	11592	36138	161	841	2412.280	122.644
pb27	11592	36138	152	1107	6134.474	116.588
pb28	11592	36138	137	329	3065.079	105.488
pb29	12558	40716	78	337	4930.398	79.176
pb30	12558	40716	136	749	3025.378	137.604

Table 2: Numerical results obtained by applying the direct solution method on the whole time window: number of variables, number of constraints, number of iterations, number of function evaluations, objective value, computing time.

of the NLP problem to be solved with the direct method. The computed solution is the same in most of the cases, except for two problem instances: pb16 which is again not solved and pb08. Remark that computed solutions are local optima and so, different minima can be provided.

Applying the PMP also in the prezone (Table 4) requires a larger computational effort compared to the application of the PMP in the postzone only, yielding larger computing times. This behaviour is related to the increased complexity of the problem to be solved when the PMP is applied in the prezone, as discussed in Section 5. As a consequence, we are not able, in general, to reduce furtherly the computing time with respect to the case of the PMP on the postzone only, and computing times in some cases are even larger than those required by the application of the direct method on the whole time window (Table 2). On the other hand, we remark a better quality of the solution in most cases, as expected. Remark also that pb16 is now solved to optimality. Furthermore, our assumptions of the form of the solution are confirmed by the experimental results.

In order to more clearly compare the results obtained for our three proposed approaches, we make use of the performance profiles of Dolan and Moré [15]. We recall that when using performance profiles, for $\tau \geq 0$ on the x-axis, one can read on the y-axis the fraction $\rho_s(\tau)$ of problems for which the performance of a given algorithm is within a factor 2^τ of the best one. In our case, Figure 3 displays a performance profile comparing the results in terms of objective function values, and Figure 4 displays a performance profile comparing the results in terms of computing time. Figure 3 shows that the approach based on the PMP applied in both the prezone and the postzone ensures the most accurate computation of the solutions. Figure 4 clearly highlights the benefit, in terms of CPU time, of using the approach based on the PMP applied in the postzone only.

To summarize, the case of PMP in the postzone appears to offer the best tradeoff between ease of implementation and quality of solution, while being very efficient in terms of computing time.

8 Conclusion

In this paper, we consider the problem of aircraft conflict avoidance using only (subliminal) velocity regulation, keeping the \mathbb{R}^3 -trajectories unchanged. We propose a decomposition tailored to this problem into specific *zones* with respect to the pairwise aircraft separation constraints. We show that into two of these zones conflicts cannot occur, so separation constraints can be neglected, and we can provide analytical solutions by using the PMP, where the criterion is the sum of squared accelerations. This leads to two new formulations, $(\mathcal{P}_{\text{postzone}})$ and $(\mathcal{P}_{\text{prezone}})$,

pb. ID	nb. of var.	nb. of constr.	nb. of it.	nb. of eval.	obj. value	cpu time (seconds)
pb01	1698	3057	45	48	78.212	0.924
pb02	1674	3033	27	59	145.547	0.540
pb03	1686	3018	29	44	912.703	0.556
pb04	2280	4208	45	47	1073.489	1.232
pb05	2200	4068	60	90	648.361	1.524
pb06	2990	5845	89	276	3161.899	3.520
pb07	2990	5845	59	98	948.502	2.428
pb08	3492	7593	77	96	2839.976	4.548
pb09	3996	8100	56	103	335.791	3.676
pb10	3156	6060	94	102	1675.380	3.956
pb11	5138	11074	67	99	601.908	7.236
pb12	4578	9863	90	141	796.172	8.244
pb13	4130	8533	63	242	714.660	4.940
pb14	6320	13844	114	172	931.898	16.160
pb15	6384	14768	118	208	1857.248	17.888
pb16	4592	9112	3000	3907	123212.425	195.508
pb17	5598	12843	47	65	646.542	6.408
pb18	5022	11511	50	76	814.456	6.128
pb19	4914	11286	45	81	814.456	5.536
pb20	5820	13640	54	150	6130.617	7.720
pb21	5700	13385	37	52	6130.617	4.748
pb22	6500	15235	49	69	4921.333	7.292
pb23	7062	18348	210	875	5885.763	62.988
pb24	8426	22000	197	1259	1612.852	79.524
pb25	7678	20251	99	197	2675.255	36.728
pb26	7224	17856	109	194	2412.280	33.808
pb27	7176	19884	146	808	6134.474	57.076
pb28	6456	15918	133	247	3065.079	36.372
pb29	10478	31057	136	235	4930.398	103.036
pb30	9438	27755	146	651	3025.378	94.232

Table 3: Numerical results obtained by applying the direct solution method from t_0 to the end of the zone, followed by the PMP-based solution in the postzone: number of variables, number of constraints, number of iterations, number of function evaluations, objective value, computing time.

pb. ID	nb. of var.	nb. of constr.	nb. of it.	nb. of eval.	obj. value	cpu time (seconds)
pb01	105	204	35	40	77.373	0.184
pb02	123	240	24	29	143.983	0.180
pb03	87	168	35	53	903.409	0.128
pb04	164	398	42	61	1060.880	0.512
pb05	164	398	30	43	641.733	0.336
pb06	325	955	48	70	3122.111	2.616
pb07	325	955	33	34	938.711	1.696
pb08	612	2112	64	91	1756.963	15.348
pb09	462	1587	85	120	331.512	11.968
pb10	234	789	110	247	1656.187	3.068
pb11	714	2814	50	58	597.113	19.208
pb12	637	2506	80	100	788.753	26.520
pb13	455	1778	67	84	706.879	8.716
pb14	816	3616	121	174	1210.623	63.448
pb15	1048	4660	111	158	1833.433	131.428
pb16	328	1420	239	543	3238.583	10.300
pb17	783	3843	99	130	639.962	52.548
pb18	702	3438	94	114	805.225	36.196
pb19	693	3393	86	104	805.356	32.976
pb20	790	4255	86	119	6056.946	44.472
pb21	780	4200	94	158	6057.702	49.548
pb22	880	4750	91	140	4938.707	61.796
pb23	1221	7216	356	1495	5775.604	604.440
pb24	1474	8734	128	174	1586.150	408.272
pb25	1386	8206	84	130	2636.618	200.364
pb26	972	6186	109	143	2162.484	112.396
pb27	1356	8682	164	232	5961.284	437.956
pb28	864	5484	114	153	3022.051	82.972
pb29	2145	14859	208	311	4805.068	2137.156
pb30	1898	13130	137	200	2994.086	988.284

Table 4: Numerical results obtained by applying the direct solution method only in the zone, and computing the PMP-based solution in the prezone as well as in the postzone: number of variables, number of constraints, number of iterations, number of function evaluations, objective value, computing time.

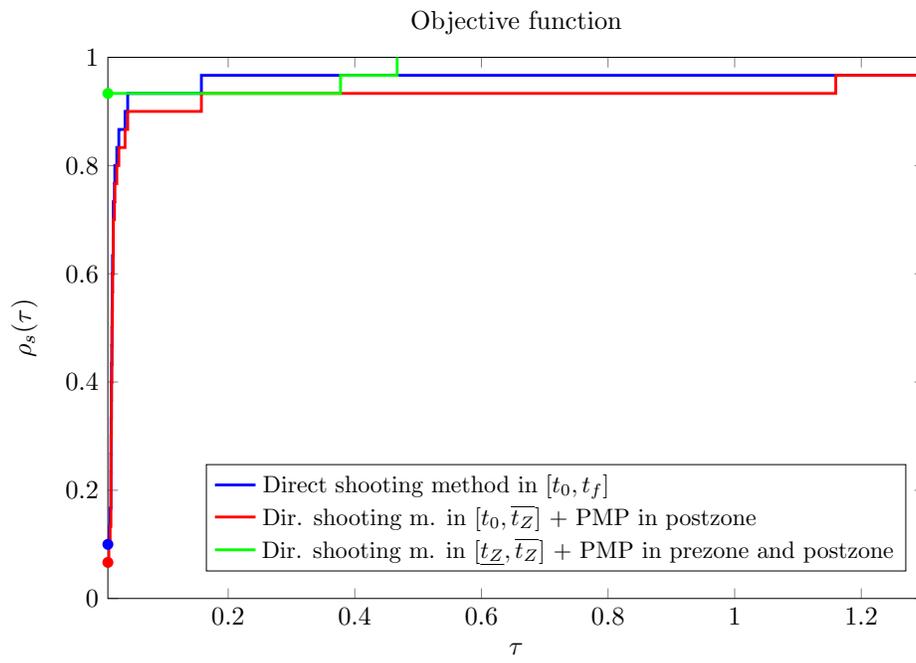


Figure 3: Performance profile on the value of the objective function.

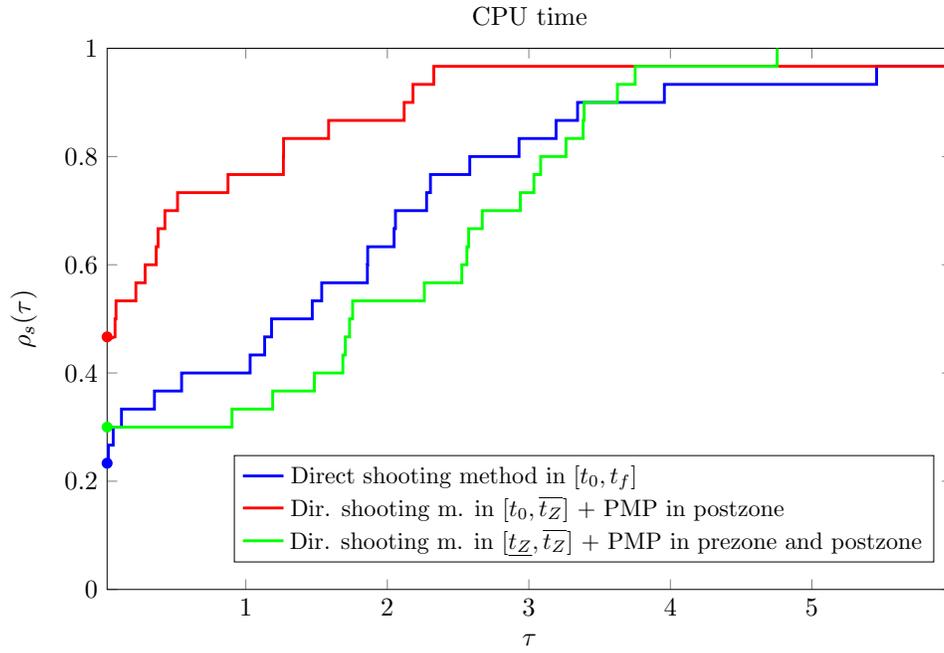


Figure 4: Performance profile on the computing time.

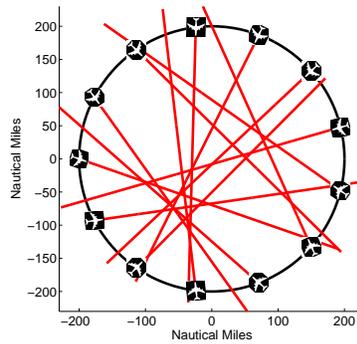


Figure 5: \mathbb{R}^2 -trajectory configuration for the 13-aircraft conflict avoidance problem pb29.

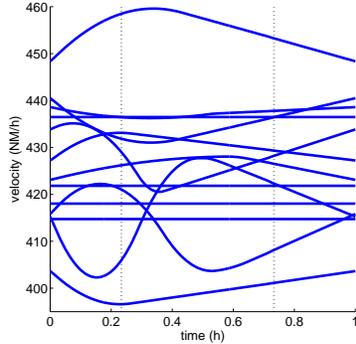


Figure 6: Optimal velocity solution curves for the **13**-aircraft conflict avoidance problem pb29.

which are solved by using numerical direct shooting methods. Numerical tests on thirty problem instances show that formulation ($\mathcal{P}_{\text{postzone}}$) provides the best results in terms of computing time, while formulation ($\mathcal{P}_{\text{prezone}}$) provides a solution on all instances, that are also the most smooth solutions.

The main advantages of our approaches are the continuity of the velocity solutions, the subliminal velocity variations as suggested in the European ERASMUS project, the accuracy on the state informations (velocities and positions, along the whole time window), and the reasonable computing time (for small number of aircraft involved, *e.g.*, less than 0.5 seconds for a problem involving 4 aircraft along a one-hour time horizon, using the formulation ($\mathcal{P}_{\text{prezone}}$)).

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