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BIOGRAPHY

Christophe Macabiau was born in 1968 in Moissac, France. He graduated in 1992 as an electronics engineer at the Ecole Nationale de l'Aviation Civile (ENAC) in Toulouse, France. He is specialized in signal processing and in radionavigation electronics. After working in 1993 for the MLS Project Office in Ottawa, Canada, he became a Ph.D. candidate at the Laboratoire de Traitement du Signal et des Télécommunications of the ENAC in 1994. He is working on the application of precise GPS positioning techniques to aeronautics.

Abdelahad Benhallam obtained his Ph.D. in communications from the Institut National Polytechnique of Toulouse in 1988. His areas of research include satellite communications, radionavigation and nonstationary signal processing. He is currently responsible of the Laboratoire de Traitement du Signal et des Télécommunications (LTST) activities, at the Ecole Nationale de l'Aviation Civile (ENAC).

ABSTRACT

Ambiguity resolution 'on-the-fly' procedures are designed to deliver in real time the integer biases that give full access to the accurate pseudorange information contained within the GPS carrier phase measurements. The key performance parameters of these procedures are their time of convergence and error rate, usually estimated from experience but rarely analytically determined. Using results derived for a multiple hypotheses sequential test called the M-ary Sequential Probability Ratio Test (MSPRT), this paper presents a theoretical analysis of the performance parameters of the Maximum A Posteriori Ambiguity Search (MAPAS) procedure. In particular, expressions of bounds and asymptotic values of the expected stopping time and error probability of MAPAS are determined as functions of the decision threshold, thus providing a means to control the performances. This study shows the influence of the number of satellites as well as the importance of the mode of selection of the primary satellites. The figures obtained are checked against observed values, showing the validity of the determined bounds and the consistency of the asymptotic values, although they lack accuracy when the number of satellites is low.

1. INTRODUCTION

A high level of positioning accuracy can be obtained through the use of the pseudorange information contained within the GPS signal carrier phase measurements. However, this pseudorange information is biased because of the ambiguous nature of the carrier phase measurements. Full access to the accurate value of the pseudorange requires the resolution of that bias, called the phase measurement ambiguity.

This can be done using one of the numerous 'on-the-fly' ambiguity resolution procedures developed over these past 10 years, either performing an ambiguity search like the Ambiguity Function Method (AFM), described by Remondi (Remondi, 1991) and Mader (Mader, 1992), the Least Squares Ambiguity Search (LSAST), presented by Hatch (Hatch, 1991) and Lachapelle et al. (Lachapelle et al., 1992), the Maximum A Posteriori Ambiguity Search (MAPAS) method, presented by Macabiau (Macabiau, 1995), or an integer ambiguity estimation like the Fast Ambiguity Resolution Approach (FARA) described by Frei and Beutler (Frei and Beutler, 1990), the Fast Ambiguity Search Filter (FASF) described by Chen (Chen, 1993), the optimized Cholesky decomposition method, described by Landau and Euler (Landau and Euler, 1992), the ambiguity transform method presented by Teunissen (Teunissen, 1994) and the Direct Integer Ambiguity Search (DIAS), presented by Ming and Schwarz (Ming and Schwarz, 1995).

They all can solve ambiguities in seconds in average operating conditions, although they are likely to fail raising the ambiguities in due time, or may even provide the user with incorrect ambiguities. Thus, in order to determine the suitability of such a procedure for a particular application, it is desirable to know its time of convergence and failure rate, which are directly related to its

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availability and integrity. This paper is intended to give theoretical expressions of bounds and asymptotic values of the average stopping time and error probability of the MAPAS procedure.

The paper first recalls the theoretical principles of the MAPAS procedure, then presents the MAPAS method as a particular application of a more general test called the MSPRT, allowing us to determine theoretical expressions of bounds and asymptotic values of its time of convergence and error probability, and to check their computed value against observed ones.

2. THE MAPAS METHOD

The carrier phase measurements delivered to an ambiguity resolution procedure are affected by all kinds of errors such as satellite clock dither through Selective Availability, atmospheric propagation perturbations, satellite and receiver clock offsets, multipath and thermal noise. The use of data collected by a receiver of known location and of all the available satellite observables enables the elimination of common errors such as clock errors and atmospheric perturbations if the receivers are not too far away. Thus, double differences of phase are usually formed between the moving receiver and a reference station, using a selected base satellite. These double differences can be linearized around a position estimate $\hat{X}(k)$ provided by the use of DGPS for example. If the receivers are located less than approximately 20 km apart, a first order model of the linearized double differences of phase at epoch k is as follows:

$$\Phi(k) = -C(k)\delta X(k) - N + B(k) \qquad (1)$$

where

- $\Phi(k)$ is the $(n_k - 1) \times 1$ vector of the linearized double differences of phase, n_k being the number of visible satellites
- $\delta X(k)$ is the 3×1 vector of the position estimation error: $\delta X(k) = \hat{X}(k) - X(k)$
- N is the $(n_k - 1) \times 1$ vector of the double differenced ambiguities, $N \in \mathbb{Z}^{n_k - 1}$
- $C(k)$ is the $(n_k - 1) \times 3$ vector of the difference between the direction cosines of the base satellite and the other satellites at $\hat{X}(k)$, normalized by the wavelength
- $B(k)$ is the $(n_k - 1) \times 1$ vector of the phase measurement noise

The double differenced measurement noise $B(k)$ is composed of the original phase measurement noise $b_i(k)$, of the multipath error $\varepsilon_{mult_i}(k)$ and of all the residual noises arising from the formation of the single and double differences. The phase measurement noise $b_i(k)$ is assumed to be a discrete white gaussian noise, having zero mean and variance σ^2. The carrier phase tracking error $\varepsilon_{mult_i}(k)$ induced by multipath can cause loss of lock and may confuse the ambiguity resolution procedure, as this error is usually left unmodeled.

In the following developments, it is assumed that $\varepsilon_{mult_i}(k) = 0$.

The determination of the position is conditioned on the resolution of the double differenced ambiguity vector N. The principle of the MAPAS procedure is to test for thousands of possible values of N, as described in (Macabiau, 1995). More precisely, as only three of the double differenced ambiguities are independent in the noise-free model derived from (1), it looks for the best three-integer combination, in a search set $\mathcal{N}(k)$ of M_k elements, to be affected to the double differenced phase ambiguities of four particular satellites, called the primary satellites. MAPAS works on the same principle as LSAST, as presented in (Macabiau, 1996).

Once the primary satellites are selected and the search set defined, the model (1) can be split into 2 systems of equations:

$$\Phi_P(k) = -C_P(k)\delta X(k) - N_P + B_P(k) \qquad (2)$$

$$\Phi_S(k) = -C_S(k)\delta X(k) - N_S + B_S(k) \qquad (3)$$

where the first system is the system of the primary satellites, and the other system is the system of the non primary satellites, called secondary satellites.

Thus, for each candidate $N_{P_{abc}} = [a\, b\, c]^\top$ in the set, a position estimate is computed:

$$\hat{X}_{P_{abc}}(k) = S_P(k)\Phi_P(k) + S_P(k)N_{P_{abc}} \qquad (4)$$

where $S_P(k)$ is the pseudo-inverse of the primary system (2).

Then, the corresponding secondary ambiguities are estimated as follows:

$$\hat{N}_{S_{abc_i}}(k) = Round\left[\sum_{j=k_i}^{k-1} \tilde{N}_{S_{abc_i}}(j)\right] \qquad (5)$$

where k_i is the first epoch of lock on the signal transmitted by satellite i and

$$\tilde{N}_{S_{abc}}(k) = -\Phi_S(k) - C_S(k)\delta\hat{X}_{P_{abc}}(k) \qquad (6)$$

This enables the calculation of a predicted double differenced observation:

$$\hat{\Phi}_{S_{abc}}(k) = -C_S(k)\delta\hat{X}_{P_{abc}}(k) - \hat{N}_{S_{abc}}(k)$$

Then, the a posteriori probability of this candidate is computed conditionally on the associated prediction error $z_{S_{abc}}(k) = \hat{\Phi}_S(k) - \Phi_{S_{abc}}(k)$ using Bayes' rule:

$$P\left[N_P = N_{P_{abc}} \mid z_{S_{abc}}{}_1^k\right] = \frac{f\left(z_{S_{abc}}{}_1^k | N_P = N_{P_{abc}}\right)}{\sum_{abc \in \mathcal{N}_k} f\left(z_{S_{abc}}{}_1^k \mid N_P = N_{P_{abc}}\right)} \qquad (7)$$

Presented at ION GPS 95, Palm Springs, 1995

where

$$f\left(z_{S_{abc}}(k) \mid N_P = N_{P_{abc}}\right) = \frac{1}{2\pi^{\frac{n_k-4}{2}}\sqrt{det(\Sigma(k))}} \times exp\left(-\frac{1}{2} z_{S_{abc}}^{\top}(k)\Sigma^{-1} z_{S_{abc}}(k)\right)$$

and

$$\begin{aligned}\Sigma(k) = C_S(k)S_P(k)\Sigma_{PP}(k)S_P(k)^{\top}C_S(k)^{\top} \\ +\Sigma_{SS}(k) - C_S(k)S_P(k)\Sigma_{PS}(k) \\ -\Sigma_{PS}(k)^{\top}S_P(k)^{\top}C_S(k)^{\top}\end{aligned}$$

$\Sigma_{PP}(k)$ and $\Sigma_{SS}(k)$ are the covariance matrices of the primary and the secondary observations.

If this probability is lower than a predefined acceptance threshold P_{min}, then the candidate is eliminated from the search set. If it is larger than a predefined upper threshold P_0, then this combination is elected as the true one.

3. THE MAPAS METHOD AS AN MSPRT

The MAPAS method is a multiple hypotheses test that sorts between thousands hypotheses represented by their associated three-integer vector:

$$H_{abc} = \{[a\ b\ c]^T : Np = [a\ b\ c]^T\} \tag{8}$$

The decision is taken using the data

$$\Phi_1^n = [\Phi(1) \ldots \Phi(n)] \tag{9}$$

The test is a mapping g that associates to the observation data Φ_1^n a particular hypothesis H_{abc} :

$$g\left(\Phi_1^n\right) = H_{abc}$$

The decision is taken at the epoch n when a preset decision condition is satisfied. Thus the size of the sample Φ_1^n is not known before the test is performed, and a compromise must be made between the delay in making the decision and the accuracy of that decision by specifying the decision condition. This kind of test is called a sequential test.

The important sets of parameters used to assess the quality of a sequential test are the set of the error probabilities and the set of the Average Sample Numbers (ASNs).

The set of the error probabilities is the set of the conditional probabilities

$$\alpha_{abc} = P\left[g\left(\Phi_0^n\right) \neq H_{abc} \mid H_{abc}\ \ true\right]$$

We can build the total weighted error probability as

$$\alpha = \sum_{abc \in \mathcal{N}} P\left[H_{abc}\ \ true\right] \alpha_{abc} \tag{10}$$

The set of the ASNs is the set of the conditional expectations :

$$ASN_{abc} = E\left[n \mid H_{abc}\ \ true\right]$$

further denoted $E_{f_{abc}}[n]$.

The MSPRT (M-ary Sequential Probability Ratio Test) is a more general multiple hypotheses sequential test designed by Baum and Veeravalli (Baum and Veeravalli, 1994) that they formulated in the following way.

Let $X_1, X_2, \ldots, X_n$ be an infinite sequence of random variables, independent and identically distributed (i.i.d.) with density f, and let H_j be the hypothesis that $f = f_j$ for $j = 0, 1, \ldots, M-1$. Assume that the prior probabilities of the hypotheses are known, and let π_j denote the prior probability of hypothesis H_j for each j. The stopping time of the MSPRT is

$$Na = \text{ first } n \geq 1 \text{ such that } p_n^k > \frac{1}{1+A_k} \tag{11}$$

for at least one k, and the final decision is δ such that

$$\delta = H_m \text{ , where } m = arg\ \max_j p_{Na}^j$$

where $p_n^k = P[H = H_j \mid X_1, X_2, \ldots, X_n]$ is the posterior probability of H_j.

The MSPRT is a generalization of the SPRT (Sequential Probability Ratio Test). Although the SPRT is optimal in the sense that it provides a minimal stopping time for a given error probabilities set, the MSPRT is an approximation of the Bayesian optimal solution. However, Baum and Veeravalli showed in (Baum and Veeravalli, 1995) that the MSPRT is asymptotically efficient as, for a given error probabilities set, it becomes the fastest decision making test when the threshold components A_k decrease towards 0.

Several theoretical results concerning this test are presented by Baum and Veeravalli (Baum and Veeravalli, 1994). In particular, expressions are given for bounds and asymptotic values of the ASN and error probability.

The MAPAS method can be viewed as a particular application of the MSPRT to the observation sequence formed by the secondary phase prediction errors. However, comparing (7) with (11) shows that the observation sequences $z_{S_{abc}}(k)$ used by MAPAS depends on the tested hypotthesis $[a\,b\,c]^T$, which is not true for the MSPRT. This problem can be solved by noting that the a posteriori probability of a candidate is independent of the three-integer vector used to compute the prediction error. To show this, we can write

$$\begin{aligned}z_{S_{abc}}(k) = \\ -C_S(k)S_P(k)[N_{P_{abc}}(k) - N_P + B_P(k)] \\ +N_S - \hat{N}_{S_{abc}}(k) - B_S(k)\end{aligned} \tag{12}$$

thus

$$\begin{aligned}E\left[z_{S_{abc}}(k) \mid N_p = [\alpha\,\beta\gamma]^T\right] = \\ -C_S(k)S_P(k)\left[N_{P_{abc}}(k) - N_{P_{\alpha\beta\gamma}}(k)\right] \\ +N_{S_{\alpha\beta\gamma}}(k) - \hat{N}_{S_{abc}}(k)\end{aligned}$$

Presented at ION GPS 95, Palm Springs, 1995

and the argument of the exponential in the gaussian probability density function is

$$z_{S_{abc}}(k) - E\left[z_{S_{abc}}(k) \mid N_p = [\alpha\,\beta\,\gamma]^T\right] = -C_S(k)S_P(k)\left[N_{P_{\alpha\beta\gamma}}(k) - N_P\right] + N_S - N_{S_{\alpha\beta\gamma}}(k)$$

which is independent on $[a\,b\,c]^\top$. Thus, we can write

$$f\left(z_{S_{abc}}(k) \mid N_{P_{abc}}\right) = f\left(z_{S_{\alpha\beta\gamma}}(k) \mid N_{P_{abc}}\right)$$

allowing us to reformulate MAPAS using the decision criterion

$$P\left[N_P = N_{P_{abc}} \mid z_{S_{\alpha\beta\gamma}}{}_1^k\right] = \frac{f\left(z_{S_{\alpha\beta\gamma}}{}_1^k | N_P = N_{P_{abc}}\right)}{\sum_{abc \in N_k} f\left(z_{S_{\alpha\beta\gamma}}{}_1^k \mid N_P = N_{P_{abc}}\right)}$$

which is identical to the decision criterion (11) used by the MSPRT, considering that the prior probabilities of each hypothesis are equal. Here, $[\alpha\,\beta\,\gamma]^T$ can be any fixed three-integer vector.

Several hypotheses have to be made for the MAPAS method to be called an MSPRT:

1. We must assume that the direction cosines of the satellites from the moving receiver's point of view are constant during the whole resolution process. This is necessary if we want to consider that the residuals $z_{S_{\alpha\beta\gamma}}(k)$ are identically distributed over time. This hypothesis is a pessimistic assumption, as the evolution of the satellite geometry, although slow for vehicles of classical dynamics, enhances the selectivity of the procedure.

2. We must suppose that the phase measurement noise $B_i(k)$ is an independent sequence over time. This is a quite strong assumption, as usually the double differenced noise has slowly varying components which are mainly due to the carrier phase tracking error induced by multipath. This hypothesis limits the range of the theoretical developments presented in this paper to the applications using measurements unaffected by low-frequency noise.

3. We need to consider that the rejection process of the MAPAS method, performed through the comparison of the posterior probabilities with the threshold P_{min}, has no influence on the structure of the test. That is, we must consider that the influence of the rejected combinations would have been negligible in the selection process if they had been kept in. Thus, we assume that all the M_0 hypotheses are considered at each measurement epoch. This hypothesis is optimistic for the error probability and pessimistic for the expected stopping time.

In the MAPAS case, all the A_k values are identical, and there is no need to distinguish between them. Also, this threshold component will be simply denoted A. Furthermore, we can note that

$$P_0 = \frac{1}{1+A} \text{ and } A = \frac{1-P_0}{P_0}$$

which means that $P_0 \approx 1 - A$ when A is small.

4. BOUNDS ON THE STOPPING TIME AND ERROR PROBABILITY

Baum and Veeravalli have determined bounds on the expected stopping time and error probability of the MSPRT. They can be applied to the MAPAS method as shown in this section.

Let N_a denote the stopping time, and δ the decision taken at time N_a. It can be proven that the ASN of the MAPAS method is finite by showing first that it is exponentially bounded, as the probability that t exceeds any N_a decreases exponentially with n. The demonstration, given for a general case in (Baum and Veeravalli, 1994), results for the MAPAS method in

$$P\left[N_a > n \mid H_{abc} \text{ true}\right] \leq \frac{(M_0 - 1)^{\frac{3}{2}}}{\sqrt{A}} \max_{[ijk] \neq [abc]} (\rho_{ijk})^n$$

where $\rho_{ijk} = E_{f_{abc}}\left[\sqrt{\frac{f\left(z_{S_{\alpha\beta\gamma}} | N_P = N_{P_{ijk}}\right)}{f\left(z_{S_{\alpha\beta\gamma}} | N_P = N_{P_{abc}}\right)}}\right]$.

By the Cauchy-Schwartz inequality, it can be shown that

$$\rho_{ijk} < 1 \text{ for } [ijk] \neq [abc]$$

Consequently, for an assumed correct ambiguity value the corresponding stopping time is exponentially bounded. Then, N_a is necessarily finite.

Let $P\left[N_P = [a\,b\,c]^T \mid [i\,j\,k]^T\right]$ be the probability that the candidate $[a\,b\,c]^T$ is accepted assuming $[i\,j\,k]^T$ is the correct ambiguity. Then, $P\left[N_P = [a\,b\,c]^T \mid [a\,b\,c]^T\right]$ is the probability to retain the correct ambiguity. If α denotes the total error probability introduced in (10), then we have

$$\alpha = 1 - \sum_{[ijk] \in \mathcal{N}} P\left[N_P = [i\,j\,k]^T\right] P\left[N_P = [i\,j\,k]^T \mid [i\,j\;\;k]^T\right]$$

and

$$P\left[N_P = [a\,b\,c]^T \mid [a\,b\,c]^T\right] = \sum_{n=0}^{N_a} P\left[N_P = [a\,b\,c]^T \mid [a\,b\,c]^T, N_a = n\right]$$

Due to the MSPRT formulation, this probability is shown to be bounded in (Baum and Veeravalli, 1994), as follows:

$$P\left[N_P = [a\,b\,c]^T \mid [a\,b\,c]^T\right] \geq \frac{1}{1+A} \sum_{[ijk]} P\left[N_P = [a\,b\,c]^T \mid [i\,j\,k]^T\right]$$

A summation over the vectors $[a,b,c]^T$ leads to

$$1-\alpha \geq \frac{1}{1+A}$$

that is

$$\alpha \leq \frac{A}{1+A} \tag{13}$$

The deduced upper bound of α depends only on the decision parameter A. Furthermore, it can be shown that

$$\alpha \leq A \tag{14}$$

which is equivalent to (13) for small values of A.

Thus, it can already be determined that, if the desired error probability is approximately 10^{-10}, like for aircraft landing for example, then by setting $A = 10^{-10}$, which is $P_0 = 1-10^{-10}$, this objective can theoretically be fulfilled.

5. EXPRESSION OF THE ASYMPTOTIC VALUES OF THE EXPECTED STOPPING TIME AND ERROR PROBABILITY

When the decision criterion P_0 is close to 1, that is when A is small, an expression of the value of the ASN and of the error probability is given by Baum and Veeravalli.

These asymptotic expressions all depend on the quality of the discrimination that can be made between the different hypotheses by observing the data. The level of separation is quantified by a parameter called the Kullback-Leibler information that represents the distance between two hypotheses among the erratic values of the random variable, characterized by its covariance matrix.

Denoting the Kullback-Leibler information between probability density functions f_{abc} and f_{ijk} as

$$D(f_{abc},f_{ijk}) = E_{f_{abc}}\left[\ln\frac{f_{abc}(Z_{S_{\alpha\beta\gamma}})}{f_{ijk}(Z_{S_{\alpha\beta\gamma}})}\right] \tag{15}$$

it can be shown that

$$E_{f_{abc}}[Na] \rightarrow \frac{-\ln(A)}{\min\limits_{[ijk]\neq[abc]} D(f_{abc},f_{ijk})} \text{ as } A\rightarrow 0 \tag{16}$$

Thus, as the separation between the hypotheses decreases, the number of measurements needed to identify clearly a combination increases. The vector $[i\,j\,k]^T$ minimizing $D(f_{abc},f_{ijk})$ is the integer combination for which one the secondary phase residuals are the most similar to those of the true hypothesis $[a\,b\,c]^T$.

This result can be applied to the MAPAS procedure by calculating $\min\limits_{[ijk]\neq[abc]} D(f_{abc},f_{ijk})$.

The Kullback-Leibler information between the two multivariate normal distributions of the residuals $z_{S_{\alpha\beta\gamma}}$ representing hypotheses $[a\,b\,c]^T$ and $[i\,j\,k]^T$ is

$$D(f_{abc},f_{ijk}) = \frac{1}{2}\left(E_{f_{abc}}[z_{S_{\alpha\beta\gamma}}] - E_{f_{ijk}}[z_{S_{\alpha\beta\gamma}}]\right)^T \Sigma^{-1}\left(E_{f_{abc}}[z_{S_{\alpha\beta\gamma}}] - E_{f_{ijk}}[z_{S_{\alpha\beta\gamma}}]\right) \tag{17}$$

As we can see from (17), the Kullback-Leibler information can be interpreted as a signal-to-noise ratio representing the degree of distinction between the two probability density functions, as illustrated in figure 1.

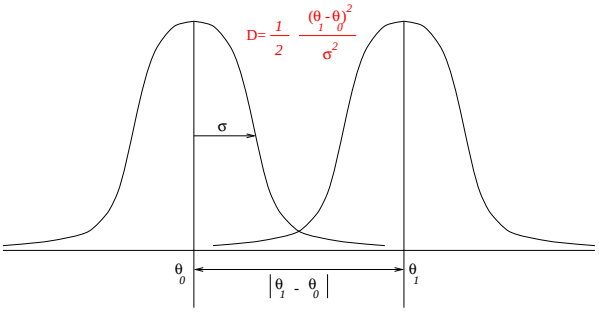

Figure 1: *Representation of the Kullback-Leibler information between two hypotheses in the scalar case.*

We can reformulate (12) using the following approximate expression of $\hat{N}_{S_{\alpha\beta\gamma}}(k)$ under low noise conditions:

$$\hat{N}_{S_{\alpha\beta\gamma}}(k) = Round\left[-\Phi_S(k) - C_S(k)\delta\hat{X}_{P_{\alpha\beta\gamma}}(k)\right] \tag{18}$$

Developing (6) using (12), (2) and (3) leads to

$$\hat{N}_{S_{\alpha\beta\gamma}}(k) = N_S + Round\left[-C_S S_P\left(N_{P_{\alpha\beta\gamma}} - N_P\right) - C_S S_P B_P - B_S\right] \tag{19}$$

Thus, (12) can be rewritten as follows

$$z_{S_{\alpha\beta\gamma}} = -C_S S_P\left(N_{P_{\alpha\beta\gamma}} - N_P + B_P\right) - B_S - Round\left[-C_S S_P\left(N_{P_{\alpha\beta\gamma}} - N_P + B_P\right) - B_S\right] \tag{20}$$

and we have

$$E_{f_{abc}}\left[z_{S_{\alpha\beta\gamma}}\right] = -C_S S_P\left(N_{P_{\alpha\beta\gamma}} - N_P\right) - Round\left[-C_S S_P\left(N_{P_{\alpha\beta\gamma}} - N_P\right)\right] \tag{21}$$

under the same low noise assumptions as previously.

Developing (17) for the multivariate normal distribution of the residuals $z_{S_{\alpha\beta\gamma}}$ leads to

$$D(f_{abc},f_{ijk}) = \frac{1}{2}\left(C_S S_P\delta N - Round\left[C_S S_P\delta N\right]\right)^T \Sigma^{-1}\left(C_S S_P\delta N - Round\left[C_S S_P\delta N\right]\right) \tag{22}$$

where $\delta N = [a\,b\,c]^T - [i\,j\,k]^T$.

The optimum δN value represents the ambiguity combination $[i\,j\,k]^T$ for which one the lines of constant

Presented at ION GPS 95, Palm Springs, 1995

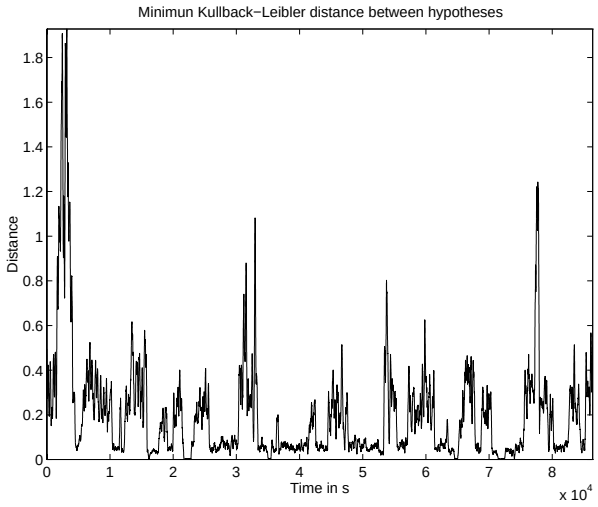

Figure 2: *Average minimum Kullback-Leibler distance between hypotheses (primary satellites with minimum PDOP). Mean = 0.20, std = 0.24, min* $= 1.4\times10^{-3}$*, max = 1.93.*

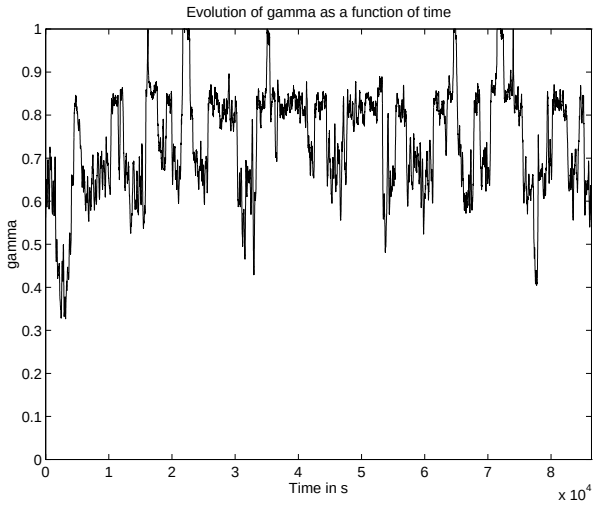

Figure 3: *Evolution of* γ *over 24h (primary satellites with minimum PDOP). Mean = 0.7, std = 0.12, min = 0.33, max = 1.*

phase intersect the most similarly as in $[a\,b\,c]^T$.

Simulations have been run to compute the minimum value of $D(f_{abc}, f_{ijk})$ for $\delta N \in \mathbb{Z}^3 / \{0\}$ such that $[i\,j\,k]^T \in \mathcal{N}$. It is useful to note that this minimum value is apparently independent of $[a\,b\,c]^T$, except for the fact that we must have $[i\,j\,k]^T \in \mathcal{N}$. Thus, a rigorous determination of this optimum value requires the search of the combination $[i\,j\,k]^T$ yielding the minimum value of (22) for each combination $[a\,b\,c]^T$. This is a very heavy calculation requiring a high computation power that we could not perform in all the cases. To simplify this problem, the optimization was done using an extensive search algorithm making no distinction between the different $[a\,b\,c]^T$ in the set, assuming that the resulting combination $[i\,j\,k]^T$ is in $\mathcal{N}$. This hypothesis has the tendancy to lower the minimum Kullback-Leibler distance, and represents a worst case assumption for the performances of the procedure.

Similarly, an expression of the asymptotic value of the error probability can be derived from (Baum and Veeravalli, 1994), as they showed that

$$\alpha \to A\gamma \text{ when } A \to 0 \tag{23}$$

where γ is a coefficient such as $0 < \gamma < 1$, calculated following (Woodroofe, 1982) depending on the minimum Kullback-Leibler information computed previously. This asymptotic value provides a closer approximation to the error probability than equation (13).

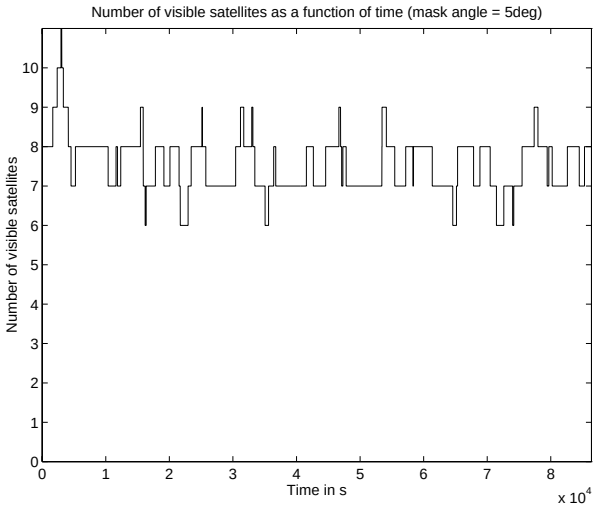

Figure 4: *Total number of visible satellites during the 24h.*

6. COMPARISON BETWEEN THEORETICAL AND OBSERVED VALUES

The theoretical expressions (16) and (23) were used to compute the predicted values of the expected stopping time and error probability for a point located at the beginning of the landing path over Toulouse-Blagnac airport. The values were calculated each second over 24h, representing the predicted performance of the MAPAS procedure that would be initiated at the corresponding time. These values were computed with various configurations of the MAPAS procedure. Then, we compared these figures against the observed values obtained for simulations of the whole landing procedure at the same date and time. All the computations and simulations were performed assuming a phase measurement

noise with standard deviation $\sigma = 1$ cm.

The calculation of the asymptotic values of the performance parameters of the MAPAS procedure is based on the determination of the minimum Kullback-Leibler distance between hypotheses, as well as on the computation of γ. This requires the selection of the primary satellites used by MAPAS. The primary satellites are selected according to their elevation angle and PDOP factor. The values of the Kullback-Leibler distance and of γ were computed for primary satellites with a minimum elevation angle of 10^o and for a minimum ideal PDOP, as well as for an objective PDOP of 7.5.

We first determined the value of this minimum Kullback-Leibler distance over 24 hours for a fixed point in the approach path over Toulouse-Blagnac airport. The calculation was done using the simplification described in the previous section. The evolution of this distance is shown in figure 2.

Similarly, the evolution of γ over 24h is plotted in figure 3.

The influence of the number of visible satellites is obvious from the comparison of figure 4 with figures 2 and 3. This comes from the fact that the separation between the hypotheses is easier when more observed data per epoch is used to check their consistency. Thus, the Kullback-Leibler distance increases with the number of satellites.

These computed values correspond to the expected stopping times plotted in figure 5, for $A = 10^{-10}$. The corresponding asymptotic value of the error probability is plotted in figure 6.

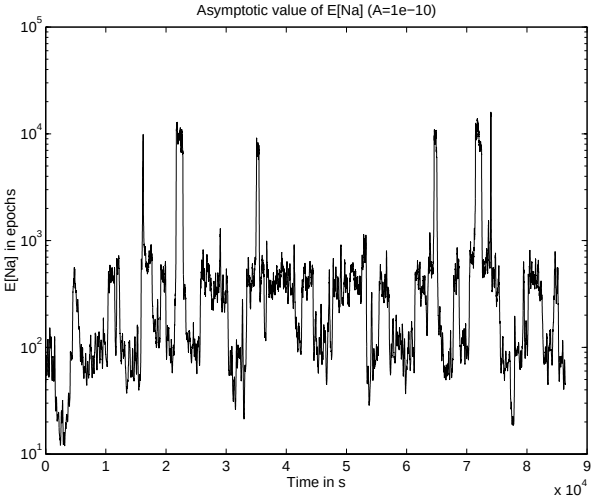

Figure 5: *Asymptotic value of the expected stopping time for primary satellites with minimum PDOP. Mean = 594, std = 1687, min = 11.9, max* = 1.6× 10^4.

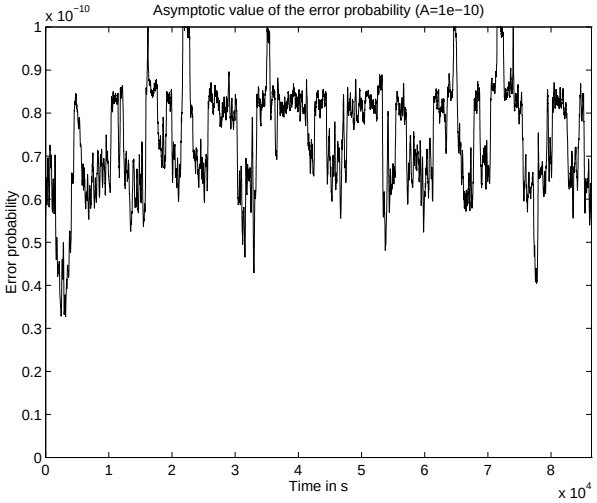

Figure 6: *Asymptotic value of the error probability for primary satellites with minimal PDOP. Mean* = 7.4×10^{-11}, *std* = 1.2×10^{-11}, *min* = 3.3×10^{-11}, *max* = 1.0×10^{-10}.

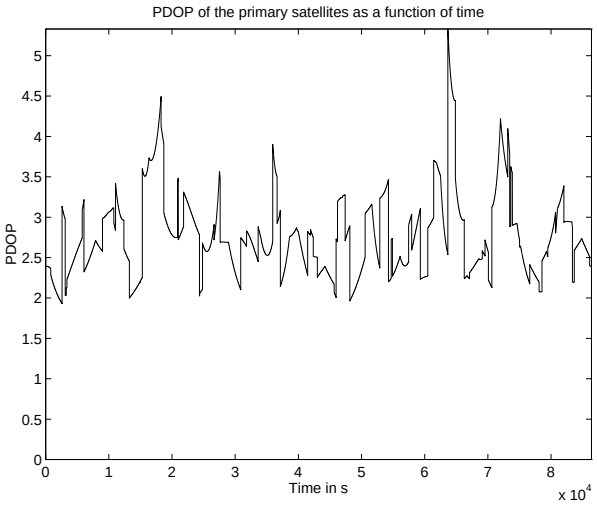

Figure 7: *PDOP of the primary satellites (first case).*

The influence of the PDOP of the primary satellites, plotted in figure 7 for this first case, can be emphasized by the comparison of the figures 2, 5 and 6 with figures 9, 10 and 11, when the PDOP is now as in figure 8.

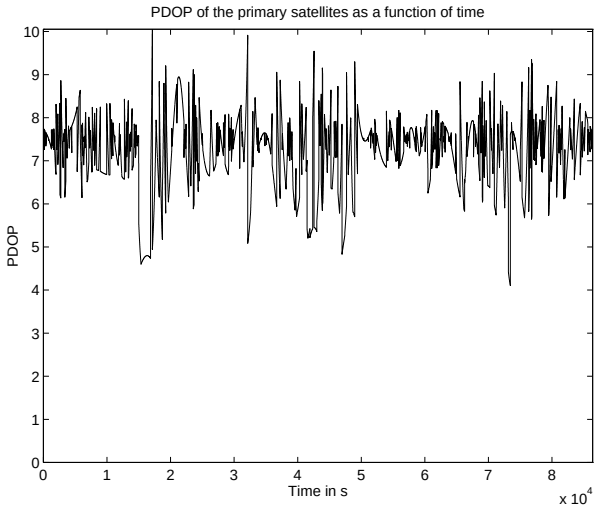

Figure 8: *PDOP of the primary satellites (second case).*

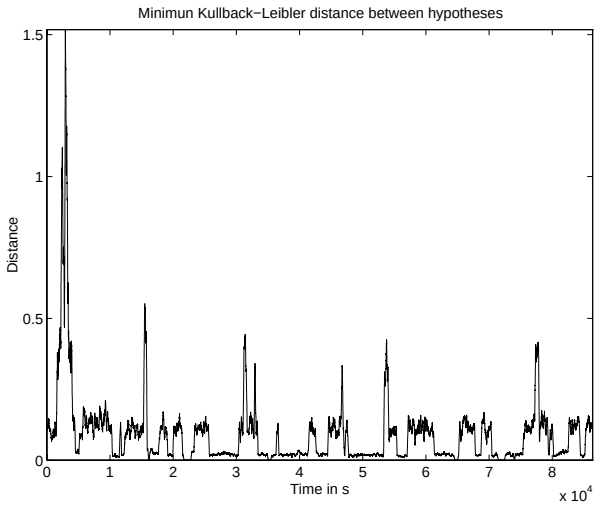

Figure 9: *Average minimum distance between 2 hypotheses over 24h (primary satellites with PDOP close to 7.5). Mean = 0.09, std = 0.13, min =* 5.0×10^{-4}*, max = 1.52.*

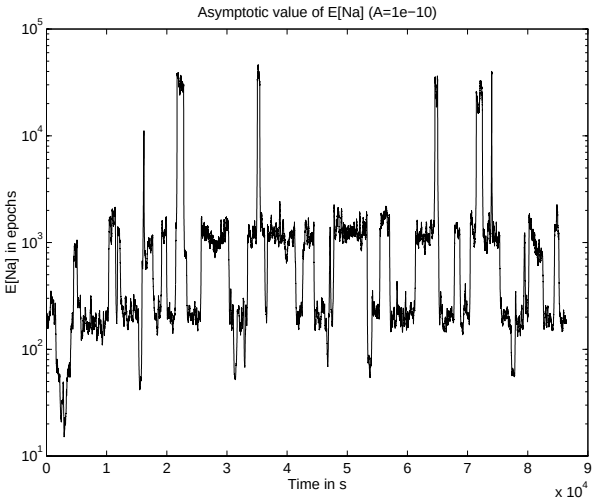

Figure 10: *Asymptotic value of the expected stopping time over 24h (primary satellites with PDOP close to 7.5). Mean = 1674, std =* 5.3×10^{3}*, min = 15.2, max =* 4.6×10^{4}*.*

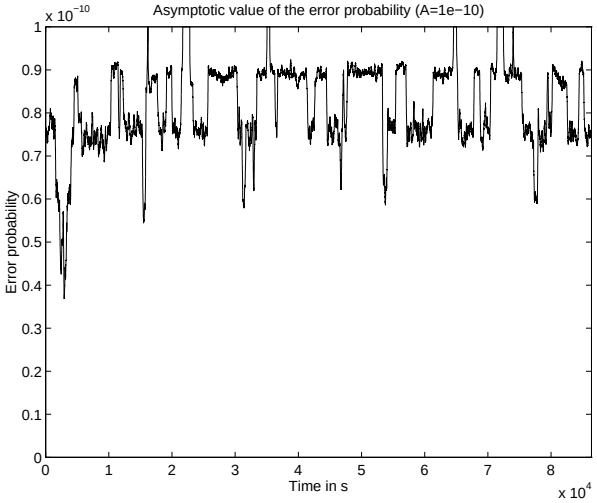

Figure 11: *Asymptotic value of the error probability for* $A = 10^{-10}$ *(primary satellites with PDOP close to 7.5). Mean =* 8.2×10^{-11}*, std =* 9.6×10^{-12}*, min =* 3.7×10^{-11}*, max =* 1.0×10^{-10}*.*

To determine the accuracy of the asymptotic values, we ran the calculation of the minimum Kullback-Leibler distance at four distinct GPS times and compared the obtained results against observed ones. This was done for both primary satellites selection modes, as shown in tables 1 and 2. In order to obtain observable values of error probabilities, the threshold component A has been set to the relatively high value of 10^{-2}.

As we can see from tables 1 and 2, the observed error probability appears to be lower than the asymptotic one, and in every case the lower bound (13) is satisfied. This major result enables to determine the value of the design threshold P_0, using the desired error probability.

Furthermore, the accuracy of the computed asymptotic values improves with the number of visible satellites, as the computed distance seems to be more stable.

7. CONCLUSION

The MAPAS procedure has been modeled as an M-ary Sequential Probability Ratio Test (MSPRT), so that general MSPRT results are applicable. Thus, the time of convergence of the MAPAS procedure has been shown to be finite, and an upper bound of the error probability has been given as a function of the decision threshold. Furthermore, asymptotic values of these two performance parameters have been given.

Comparison of the theoretical and observed values shows that the upper bound of the error probability seems to be satisfied in every case.

The relative evolution of the asymptotic values of the error probabilities and expected stopping time shows the influence of the number of visible satellites and of the PDOP of the primary satellites. Although these theoretical values are not very accurate when there are few visible satellites, a good prediction of the performances of the procedure can be obtained when the number of satellites is larger than 7.

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GPS time	194800	202800	172800	175750
PDOP of primaries	3.3	2.3	2.4	3.0
# sat.	6	7	8	11
Asymptotic $E[N_a]$	1690	74.4	9.3	2.5
Asymptotic α	9.6×10^{-3}	8.1×10^{-3}	5.6×10^{-3}	3.4×10^{-3}
Observed $E[N_a]$	172.4	104.2	51.9	11.1
Observed α	7.7×10^{-5}	1.2×10^{-4}	2.3×10^{-4}	2.1×10^{-4}
# trials	13000	13000	13000	13000

Table 1: *Comparison between computed asymptotic and observed values when the primary satellites are the ones with the lowest PDOP.*

GPS time	194800	202800	172800	175750
PDOP of primaries	8.1	7.5	7.6	7.5
# sat.	6	7	8	11
Asymptotic $E[N_a]$	6514	303	35.6	3.1
Asymptotic α	9.8×10^{-3}	9.0×10^{-3}	7.4×10^{-3}	3.7×10^{-3}
Observed $E[N_a]$	171.2	103.0	51.9	14.0
Observed α	5.5×10^{-4}	4.1×10^{-3}	6.2×10^{-4}	2.9×10^{-4}
# trials	23000	23000	23000	23000

Table 2: *Comparison between computed asymptotic and observed values when the primary satellites have the closest PDOP to 7.5.*

Presented at ION GPS 95, Palm Springs, 1995