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Reduced RLT constraints for polynomial programming

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Abstract. An extension of the reduced Reformulation-Linearization Technique constraints from quadratic to general polynomial programming problems with linear equality constraints is presented and a strategy to improve the associated convex relaxation is proposed.

1. Introduction

Reduced RLT constraints (rRLT) are a special class of Reformulation-Linearization Technique (RLT) constraints, that apply to nonconvex (both continuous and mixed-integer) quadratic programming problems subject to linear equality constraints [2, 4, 3]. rRLT are obtained by replacing some of the quadratic terms with suitable linear constraints. These turn out to be a subset of the RLT constraints for quadratic programming [7].

We present an extension of the rRLT theory to the case of general polynomial programs. Then, we show a strategy to choose the basis of a matrix involved in the rRLT constraints generation so as to tighten the bound of the associated convex relaxation. This allows to improve the performance of a spatial Branch-and-Bound algorithm applied to nonconvex NLP and MINLP problems where such convex relaxation is computed at each node.

2. Extending rRLT to polynomial programs

Let $n$ be the number of variables, $q$ the degree of the polynomials in the targeted problem and $\mathcal{N} = \{1, \ldots, n\}$, $Q = \{2, \ldots, q\}$. For each monomial $x_{j_1} \cdots x_{j_p}$, $p \in Q$, appearing in the problem, we define a finite sequence $J = (j_1, \ldots, j_p)$ and consider defining constraints of the following form:

$$w_J = \prod_{\ell \leq |J|} x_{j_\ell}$$

(for $|J| = 1$, i.e. $J = (j)$, we also define $w_J = x_j$). For all $p \in Q$, $J \in \mathcal{N}^p$ and any permutation $\pi$ in the symmetric group $S_p$ we have that $w_J = w_{\pi J}$ by commutativity. We therefore define an equivalence relation $\sim$ on $\mathcal{N}^p$ stating that for $J, K \in \mathcal{N}^p$, $J \sim K$ only if $\exists \pi \in S_p$ such that $J = \pi K$. We then consider the index tuple set $\bar{\mathcal{N}}^p = \mathcal{N}^p/\sim$ to quantify over when indexing variables $w_J$.

We multiply the original linear constraints $Ax = b$ by all monomials $\prod_{\ell \leq p-1} x_{j_\ell}$ and replace them by the corresponding added variables $w_{(J', j)}$, where $J' \in \bar{\mathcal{N}}^{p-1}$. This yields the following rRLTs:

$$\forall p \in Q, J' \in \bar{\mathcal{N}}^{p-1} \quad A w_{J'} = b w_{J'},$$

where $w_{J'} = (w_{(J', 1)}, \ldots, w_{(J', n)})$. We then consider the companion system:

$$\forall p \in Q, J' \in \bar{\mathcal{N}}^{p-1} \quad A z_{J'} = 0.$$  

Since (2.3) is a linear homogeneous system, there is a matrix $M$ such that the companion system is equivalent to $Mz = 0$, the columns of which are indexed by sequences in $\bar{\mathcal{N}}^p$. We let $B \subseteq \bar{\mathcal{N}}^p$

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and \( N \subseteq N^p \) be index sets for basic and nonbasic columns of \( M \). We define the following sets:

\[
C = \{ (x, w) \mid Ax = b \land \forall p \in Q, J \in \bar{N}^p (w_j = \prod_{\ell \leq |J|} x_{j_\ell}) \}
\]

\[
R_N = \{ (x, w) \mid Ax = b \land \forall p \in Q, J' \in \bar{N}^{p-1} (A w_{j'} = b w_{j'}) \land \forall J \in N (w_j = \prod_{\ell \leq |J|} x_{j_\ell}) \}.
\]

**Theorem 1.** For each partition \( B, N \) into basic and nonbasic column indices for the companion system \( Mz = 0 \), we have \( C = R_N \).

### 3. Tightening the convex relaxation

Replacing \( C \) with \( R_N \) for some nonbasis \( N \) effectively replaces some monomial terms with linear constraints, and therefore contributes to simplify the problem. A convex relaxation for the reformulated problem is readily obtained by applying monomial convexification methods in the ear constraints, and therefore contributes to simplify the problem. A convex relaxation for the

For each partition \( B, N \):

\[
\max \{ \text{volume} \} = \max \{ \text{volume} \} \text{ such that } \max x_i \text{ for all } p \in Q, V^{N,p} < V^{B,p} \text{ then the total convexity gap of } R_N \text{ is smaller than that of } C. \]

Thus, we aim to find \( N \) such that \( V^{N,p} \) is minimized, or equivalently, to find \( B \) such that \( V^{B,p} \) is maximized for all \( p \in Q \). This yields the multi-objective problem:

\[
\begin{align*}
\forall p \in Q & \quad \max V^{B,p} \\
M_B \text{ is a basis of } (2.3) \end{align*}
\]

(3.1)

It can be shown that (3.1) is equivalent to a single-objective problem: any solution \( B \) of (3.1) maximizing \( V^B \) also maximizes \( V^{B,p} \) for all \( p \in Q \). In this way, we have derived a technique to choose a good basis for the companion system so as to improve the chances of tightening the lower bound of the convex relaxation associated to rRLT.

Preliminary computational experiments carried out on a set of randomly generated instances of the convex Quadratic Knapsack Problem (cQKP) show that the proposed strategy is promising in improving performances of a spatial Branch-and-Bound algorithm.

### Bibliography


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