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Reduced RLT constraints for polynomial programming

SONIA CAFIERI
PIERRE HANSEN
LUCAS LÉTOCART
LEO LIBERTI
FRÉDÉRIC MESSINE

Abstract. An extension of the reduced Reformulation-Linearization Technique constraints from quadratic to general polynomial programming problems with linear equality constraints is presented and a strategy to improve the associated convex relaxation is proposed.

1. Introduction

Reduced RLT constraints (rRLT) are a special class of Reformulation-Linearization Technique (RLT) constraints, that apply to nonconvex (both continuous and mixed-integer) quadratic programming problems subject to linear equality constraints [2, 4, 3]. rRLT are obtained by replacing some of the quadratic terms with suitable linear constraints. These turn out to be a subset of the RLT constraints for quadratic programming [7].

We present an extension of the rRLT theory to the case of general polynomial programs. Then, we show a strategy to choose the basis of a matrix involved in the rRLT constraints generation so as to tighten the bound of the associated convex relaxation. This allows to improve the performance of a spatial Branch-and-Bound algorithm applied to nonconvex NLP and MINLP problems where such convex relaxation is computed at each node.

2. Extending rRLT to polynomial programs

Let $n$ be the number of variables, $q$ the degree of the polynomials in the targeted problem and $\mathcal{N} = \{1, \ldots, n\}$, $Q = \{2, \ldots, q\}$. For each monomial $x_{j_1} \cdots x_{j_p}$, $p \in Q$, appearing in the problem, we define a finite sequence $J = (j_1, \ldots, j_p)$ and consider defining constraints of the following form:

$$w_J = \prod_{\ell \leq |J|} x_{j_{\ell}} \quad (2.1)$$

(for $|J| = 1$, i.e. $J = (j)$, we also define $w_J = x_j$). For all $p \in Q$, $J \in \mathcal{N}^p$ and any permutation $\pi$ in the symmetric group $S_p$ we have that $w_J = w_{\pi J}$ by commutativity. We therefore define an equivalence relation $\sim$ on $\mathcal{N}^p$ stating that for $J, K \in \mathcal{N}^p$, $J \sim K$ only if $\exists \pi \in S_p$ such that $J = \pi K$. We then consider the index tuple set $\mathcal{N}^p = \mathcal{N}^p/\sim$ to quantify over when indexing variables $w_J$.

We multiply the original linear constraints $Ax = b$ by all monomials $\prod_{\ell \leq p-1} x_{j\ell}$ and replace them by the corresponding added variables $w_{(J', j)}$, where $J' \in \mathcal{N}^{p-1}$. This yields the following rRLTS:

$$\forall p \in Q, J' \in \mathcal{N}^{p-1} \quad A w_{J'} = b w_{J'} \quad (2.2)$$

where $w_{J'} = (w_{(J',1)}, \ldots, w_{(J',n)})$. We then consider the companion system:

$$\forall p \in Q, J' \in \mathcal{N}^{p-1} \quad A z_{J'} = 0. \quad (2.3)$$

Since (2.3) is a linear homogeneous system, there is a matrix $M$ such that the companion system is equivalent to $M z = 0$, the columns of which are indexed by sequences in $\mathcal{N}^p$. We let $B \subseteq \mathcal{N}^p$

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and $N \subseteq N^p$ be index sets for basic and nonbasic columns of $M$. We define the following sets:

\[
C = \{(x, w) | Ax = b \land \forall p \in Q, J \in N^p(w_j = \prod_{t \leq |J|} x_{j_t})\}
\]

\[
R_N = \{(x, w) | Ax = b \land \forall p \in Q, J' \in N^{p-1}(A w_{j'} = bw_{j'}) \land \forall J \in N(w_J = \prod_{t \leq |J|} x_{j_t})\}.
\]

**Theorem 1.** For each partition $B, N$ into basic and nonbasic column indices for the companion system $Mz = 0$, we have $C = R_N$.

3. Tightening the convex relaxation

Replacing $C$ with $R_N$ for some nonbasis $N$ effectively replaces some monomial terms with linear constraints, and therefore contributes to simplify the problem. A convex relaxation for the reformulated problem is readily obtained by applying monomial convexification methods in the literature [5, 6, 1]. We observe that for any given linear system there is in general more than one way to partition the variables in basics and nonbasics. Hence the set $B$ can be chosen in such a way as to decrease the discrepancy between the feasible region and its convex relaxation. Given $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ and the sets $S = \{(x, w) | w = f(x)\}$ and $\bar{S} = \{(x, w) | f(x) \leq w \leq \bar{f}(x)\}$, where $f(x), \bar{f}(x)$ are respectively a convex lower and a concave upper bounding function for $f$ (and hence $\bar{S}$ is a convex relaxation of $S$), the convexity gap between $S$ and $\bar{S}$ can be defined as the volume $V(S)$ of the set $S$. Explicit expressions of $V(S)$ can be derived for a quadratic term $x_j^2$, for a bilinear term $x_i x_j$ using the Cayley-Menger formula in 3 dimensions, and for a general monomial, exploiting associativity recursively to rewrite it as product of lower degree monomials and using the preceding results.

Let $B, N$ be the basic/nonbasic sets of column indices of the companion system, which we can write as $M_B z_B + M_N z_N = 0$. The elements of $B, N$ are sequences $J \in J$. For $S \subseteq J$ and $p \in Q$ we define $V^{S, p} = \sum_{J \in S} V_J$ and $V^S = \sum_{p \in Q} V^{S, p}$. If, for all $p \in Q$, $V^{N, p} < V^{B, p}$ then the total convexity gap of $R_N$ is smaller than that of $C$. Thus, we aim to find $N$ such that $V^{N, p}$ is minimized, or equivalently, to find $B$ such that $V^{B, p}$ is maximized for all $p \in Q$. This yields the multi-objective problem:

\[
\begin{align*}
\forall p \in Q & \quad \max V^{B, p} \\
M_B & \text{ is a basis of } (2.3)
\end{align*}
\]

It can be shown that (3.1) is equivalent to a single-objective problem: any solution $B$ of (3.1) maximizing $V^B$ also maximizes $V^{B, p}$ for all $p \in Q$. In this way, we have derived a technique to choose a good basis for the companion system so as to improve the chances of tightening the lower bound of the convex relaxation associated to rRLT.

Preliminary computational experiments carried out on a set of randomly generated instances of the convex Quadratic Knapsack Problem (cQKP) show that the proposed strategy is promising in improving performances of a spatial Branch-and-Bound algorithm.

**Bibliography**


