Discerned and non-discerned particles in classical mechanics and convergence of quantum mechanics to classical mechanics

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ABSTRACT. We introduce into classical mechanics the concept of non-discerned particles for particles that are identical, non-interacting and prepared in the same way. The non-discerned particles correspond to an action and a density which satisfy the statistical Hamilton-Jacobi equations and allow to explain the Gibbs paradox. On the other hand, a discerned particle corresponds to a particular action that satisfies the special Hamilton-Jacobi equations. We then study the convergence of quantum mechanics to classical mechanics when $\hbar$ tends to 0 by considering two cases : the convergence to non-discerned classical particles and the convergence to a classical discerned particle. Based on these convergences, we propose an updated interpretation of quantum mechanics.

RÉSUMÉ. Nous introduisons en mécanique classique le concept de particules indiscernées pour des particules qui sont identiques, sans interaction et préparées de la même façon. Ces particules indiscernées correspondent à une action et une densité qui satisfassent les équations d'Hamilton-Jacobi statistiques et qui permettent d'expliquer le paradoxe de Gibbs. Par ailleurs, une particule discernée correspond à une action particulière qui satisfait les équations spéciales d'Hamilton-Jacobi. L'étude de la convergence de la mécanique quantique vers la mécnique classique quand $\hbar \to 0$ se fait alors simplement en considérant deux cas : la convergence vers des particules classiques indiscernées et la convergence vers des particules classiques discernées. Nous déduisons de ces convergences une interprétation renouvelée de la mécanique quantique.

1 Introduction

The indiscernability concept, which is very relevant in quantum and statistical physics, is not well-defined in the literature. In particular, it

is the origin of the Gibbs paradox. Indeed, *when one calculates the entropy of two mixed gases, the classical result for distinguishable particles is double the expected result. If the particles are considered indistinguishable, the correct result is recovered because of the indiscernability factor.* This paradox identified by Gibbs [1] in 1889, was solved only by means of quantum mechanics 35 years later by using the indiscernability postulate for quantum particles. Indeed it was Einstein who, in 1924, introduced the indiscernability of perfect gas molecules at the same time as Bose-Einstein statistics. In his homage to Einstein for the centenary of his birth in 1979, Alfred Kastler pointed out that [2] : " *the distinction between distinguishable and indistinguishable entities and the difference of statistical behavior between those two types of entities remains obscure. Boltzmann treated those 'molecules' as distinguishable entities, which has yielded the so-called Boltzmann statistics. On the contrary, Planck implicitly dealt with the "energy elements" he introduced as indistinguishable particles, which led to a probabilistic counting of a macroscopic state different from the Boltzmann one. In 1909, Einstein rightly criticized this lack of rigor.*" But as noted by Henri Bacry [3] p.129, *"the historical progression could have been very different. Indeed, logically, one could postulate the non-discernability principle in order to solve the Gibbs paradox. But this principle can be applied to all the principles of quantum mechanics or to those of classical mechanics.*" This same observation has been made by a large number of other authors. In 1965 Landé [4] demonstrated that this indiscernability postulate of classical particles is sufficient and necessary in order to explain why entropy vanished. In 1977, Leinaas and Myrheim [5] used it for the foundation of their identical classical and quantum particles theory. Moreover, as noted by Greiner, in addition to the Gibbs paradox, several cases where it is needed to consider indistinguishable particles in classical mechanics and distinguishable particles in quantum mechanics can be found [6] p.134 : "*Hence, the Gibbs factor $\frac{1}{N!}$ is indeed the correct recipe for avoiding the Gibbs paradox. From now on we will therefore always take into account the Gibbs correction factor for indistinguishable states when we count the microstates. However, we want to emphasize that this factor is no more than a recipe to avoid the contradictions of classical statistical mechanics. In the case of distinguishable objects (e.g., atoms which are localized at certain grid points), the Gibbs factor must **not** be added. In classical theory the particles remain distinguishable. We will meet this inconsistency more frequently in classical statistical mechanics.*" But nowadays, most of the textbooks contain

definitions such as :"*in classical mechanics, two particles in a system are always distinguishable*" *and "in quantum mechanics, two particles are always indistinguishable*" [7] p.328-329, do not answer the concrete problems exposed by Greiner in both classical statistics mechanics and quantum statistical mechanics. In this article we propose an accurate definition of both discernability and indiscernability in classical mechanics and a way to avoid ambiguities and paradoxes. These definitions yield an understandable interpretation both of the action in classical mechanics and the wave function in quantum mechanics. We only consider the case of a single particle or a system of identical particles without interactions and prepared in the same way. The case of identical particles with interactions will be presented in a future paper. In paragraph 2, we introduce the discerned and non-discerned particles concepts in classical mechanics through the Hamilton-Jacobi equations. In the following paragraphs, we study the convergence of quantum mechanics to classical mechanics when $\hbar$ tends to 0 by considering two cases : the first corresponds to the convergence to non-discerned classical particles, and the second corresponds to the convergence to a classical discerned particle. Based on these convergences, we propose an updated interpretation of quantum mechanics.

2   Discerned and non-discerned particles in classical mechanics

Let us consider in classical mechanics a system of identical particles without interactions.

Definition 1 - *A classical particle is* **potentially discerned** *if its initial position $\boldsymbol{x}_o$ and its initial velocity $\boldsymbol{v}_0$ are known.*

Let us note that there is an abuse of language when one talks about a classical particle. One should rather speak of a particle that is studied in the framework of classical mechanics.

We now consider a particle within a stationary beam of classical identical particles such as electronic, atomic or molecular beams (CO_2 or C_{60}). For such particle, one only knows, initially, the probability density $\rho_0(\mathbf{x})$ and the velocity field $\mathbf{v}_0(\mathbf{x})$ through the action $S_0(\mathbf{x})$; this action is known to within a constant from the equation $\mathbf{v}_0(\mathbf{x}) = \frac{\nabla S_0(\mathbf{x})}{m}$ where m is the particle mass. This yields the following definition :

Definition 2 - *A classical particle, of which at initial time only the density of its initial position $\rho_0(\boldsymbol{x})$ and initial action $S_0(\boldsymbol{x})$, is referred to as* **potentially non-discerned***.*

This notion is intrinsic to a particle. It gives the initial conditions, which means the way it has been prepared. Therefore, it is an indiscernability on the initial particle position. It doesn't depend on the observer but on the effective modeling scale of the phenomenon.

In this article, we are only interested in the case where the **N particles are prepared in the same way** with the same initial density $\rho_0(\mathbf{x})$ and the same initial action $S_0(\mathbf{x})$ evolving in the same potential $V(x)$ and which can have independent behaviors. It is the case of classical identical particles **without interactions** and prepared in the same way, such as C_{60} or neutral molecules. It is still the case for instance for electrons prepared in the same way, and although they are able to interact with each other, they will have independent behaviors because they are generated one by one in the system. The general case of interacting identical particles which are not prepared in the same way will be presented in a future paper.

Definition 3 - *N identical particles, prepared in the same way, with the same initial density $\rho_0(\boldsymbol{x})$, the same initial action $S_0(\boldsymbol{x})$, and evolving in the same potential $V(\boldsymbol{x})$ are called* **non-discerned***.*

We have named those particles non-discerned and not indistinguishable because, if their initial positions are known, their trajectories will be known as well. Nevertheless, when one counts them, they will have the same properties as the indistinguishable ones. Thus, if the initial density $\rho_0(\mathbf{x})$ is given, and one randomly chooses N particles, the N! permutations are strictly equivalent and correspond to the same configuration as for indistinguishable particles. This means that if X is the coordinate space of a non-discerned particle, the true configuration space of N non-discerned particles is not X^N but rather X^N/S_N where S_N is the permutation group.

2.1   Non-discerned particles and statistical Hamilton-Jacobi equations

For non-discerned particles, we have the following theorem :

THEOREM 1 - *The probability density $\rho\left(\mathbf{x},t\right)$ and the action $S\left(\mathbf{x},t\right)$ of classical particles prepared in the same way, with initial density $\rho_0(\boldsymbol{x})$, with the same initial action $S_0(\boldsymbol{x})$, and evolving in the same potential $V(\boldsymbol{x})$, are solutions to the **statistical Hamilton-Jacobi equations** :*

$$\frac{\partial S\left(\boldsymbol{x},t\right)}{\partial t} + \frac{1}{2m}\left(\nabla S(\boldsymbol{x},t)\right)^2 + V(\boldsymbol{x}) = 0 \qquad \forall\left(\boldsymbol{x},t\right) \in \mathbb{R}^3 \times \mathbb{R}^+ \quad (1)$$

$$S(\boldsymbol{x},0) = S_0(\boldsymbol{x}) \qquad \forall\ \boldsymbol{x} \in \mathbb{R}^3. \tag{2}$$

$$\frac{\partial \rho\left(\boldsymbol{x},t\right)}{\partial t} + div\left(\rho\left(\boldsymbol{x},t\right)\frac{\nabla S\left(\boldsymbol{x},t\right)}{m}\right) = 0 \qquad \forall\left(\boldsymbol{x},t\right) \in \mathbb{R}^3 \times \mathbb{R}^+ \quad (3)$$

$$\rho(\mathbf{x},0) = \rho_0(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^3. \tag{4}$$

Let us recall that the velocity field is $\mathbf{v}(\mathbf{x},t) = \frac{\nabla S(\mathbf{x},t)}{m}$ and that the Hamilton-Jacobi equation (1) is not coupled to the continuity equation (3). The difference between discerned particles and non-discerned particles thus explains why the "recipes" proposed in some classical statistical mechanics books are useful. But as has been demonstrated above, it is not a principle which can be added. The nature of the discernability of the particles depends strongly on the experimental conditions determined by the modeling scale.

2.2 Discerned particles and special Hamilton-Jacobi equations

One can ask if it is possible to define an action for a potentially discerned particle in a potential field $V(\mathbf{x})$? Such an action should depend only on the starting point $\mathbf{x}_0$, the initial velocity $\mathbf{v}_0$ and the potential $V(\mathbf{x})$.

THEOREM 2 - *If $\xi(t)$ is the classical trajectory in the field $V(\boldsymbol{x})$ of a particle with the initial position $\boldsymbol{x}_0$ and with initial velocity $\boldsymbol{v}_0$, then the function*

$$S^\xi\left(\mathbf{x},t\right) = m\frac{d\xi(t)}{dt}\cdot\boldsymbol{x} + g(t) \tag{5}$$

*where $\frac{dg(t)}{dt} = -\frac{1}{2}m(\frac{d\xi(t)}{dt})^2 - V(\xi(t)) - m\frac{d^2\xi(t)}{dt^2}\cdot\xi(t)$, is called **local action**, and is solution to special Hamilton-Jacobi equations.*

$$\frac{\partial S^\xi\left(\boldsymbol{x},t\right)}{\partial t}\big|_{\boldsymbol{x}=\xi(t)} + \frac{1}{2m}\left(\nabla S^\xi(\boldsymbol{x},t)\right)^2\big|_{\boldsymbol{x}=\xi(t)} + V(\boldsymbol{x})\big|_{\boldsymbol{x}=\xi(t)} = 0 \qquad \forall t \in \mathbb{R}^+$$

$$(6)$$

$$\frac{d\xi(t)}{dt} = \frac{\nabla S^\xi(\xi(t),t)}{m} \qquad \forall t \in \mathbb{R}^+ \qquad (7)$$

$$S^\xi(\boldsymbol{x},0) = m\boldsymbol{v}_0\boldsymbol{x} \quad and \quad \xi(0) = \boldsymbol{x}_0. \qquad (8)$$

The local action satisfies the Hamilton-Jacobi equations only along the trajectory $\xi(t)$. The introduction of such an action linked to a trajectory appears as strange and devoid of any effective interest other than a theoretical one by proposing a framework for defining discerned particles. This action will take on a meaning in paragraph 4 when we show that it corresponds to the convergence of coherent state when $\hbar$ tends to 0. We have defined two kinds of actions, a global one $S(\mathbf{x},t)$ and a local one $S^\xi(\mathbf{x},t)$. The global action $S(\mathbf{x},t)$ is a field defined for all $\mathbf{x}$ independently of the starting point $\mathbf{x}_0$. But the local one $S^\xi(\mathbf{x},t)$ depends on the trajectory $\xi(t)$ and the starting point $\mathbf{x}_0$. **The least action principle is valid only for the global action and not for the local one.** This difference provides an answer to the precautions emitted by some physicists bothered by the use of the least action principle. In particular, Henri Poincaré who wrote in "La science et l'hypothèse" : [8]

"The statement of the least action principle is somehow shocking for the mind. To move from one point to another, a material molecule, removed from the action of any force, but subject to moving on a surface, will move through the geodesic line, which means the shortest path. This molecule seems to know the point one wishes to guide it to, to predict the time it needs to reach it by choosing one path or another and to choose the most suitable one. The statement thus presents the particle as a free and animated being. It is clear that it would be better to replace it with a less shocking statement and where, as the philosophers would say, the final causes would not seem to be taking the place of efficient causes." This paradox can be solved if one remarks that the least action principle can only be applied to a global action and not to a local one because this former one depends on the starting or final point.

3   Convergence to non-discerned particles when $\hbar \to 0$.

Let us consider the wave function solution to the Schrödinger equation $\Psi(\mathbf{x},t)$:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\triangle\Psi + V(\mathbf{x})\Psi \qquad \forall(\mathbf{x},t) \in \mathbb{R}^3 \times \mathbb{R}^+ \qquad (9)$$

$$\Psi(\mathbf{x}, 0) = \Psi_0(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^3. \tag{10}$$

With the variable change $\Psi(\mathbf{x}, t) = \sqrt{\rho^\hbar(\mathbf{x}, t)} \exp(i \frac{S^\hbar(\mathbf{x}, t)}{\hbar})$, the density $\rho^\hbar(\mathbf{x}, t)$ and the action $S^\hbar(\mathbf{x}, t)$ depend on the parameter $\hbar$. The Schrödinger equation may be divided into Madelung equations [9] (1926) which correspond to :

$$\frac{\partial S^\hbar(\mathbf{x}, t)}{\partial t} + \frac{1}{2m}(\nabla S^\hbar(\mathbf{x}, t))^2 + V(\mathbf{x}) - \frac{\hbar^2}{2m} \frac{\triangle \sqrt{\rho^\hbar(\mathbf{x}, t)}}{\sqrt{\rho^\hbar(\mathbf{x}, t)}} = 0 \quad \forall(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}^+ \tag{11}$$

$$\frac{\partial \rho^\hbar(\mathbf{x}, t)}{\partial t} + \nabla \cdot (\rho^\hbar(\mathbf{x}, t) \frac{\nabla S^\hbar(\mathbf{x}, t)}{m}) = 0 \qquad \forall(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}^+ \tag{12}$$

with initial conditions

$$\rho^\hbar(\mathbf{x}, 0) = \rho_0^\hbar(\mathbf{x}) \qquad et \qquad S^\hbar(\mathbf{x}, 0) = S_0^\hbar(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^3. \tag{13}$$

In the two following paragraphs we study the convergence of the density $\rho^\hbar(\mathbf{x}, t)$ and the action $S^\hbar(\mathbf{x}, t)$ in the Madelung equations when $\hbar$ tends to 0. It is subtle and remains a difficult problem. For this reason, we only consider two typical cases, for which analytical solutions exist. The difference between the two examples is the fact that *the initial conditions are not the same due to a different preparation of the particles and initial conditions for the potential when $\hbar$ tends to 0.*

Definition 4 - *A quantum system is **non-discerned semi-classically** if it satisfies the two following conditions*

- its initial probability density $\rho_0^\hbar(\mathbf{x})$ and its initial action $S_0^\hbar(\mathbf{x})$ converge respectively, to regular functions $\rho_0(\mathbf{x})$ and $S_0(\mathbf{x})$ when $\hbar \to 0$.

- its interaction with the potential field $V(\boldsymbol{x})$ can be described classically. The simplest case corresponds to particles in vacuum with only geometric constraints.

As previously described, this is the case of a set of non-interacting particles prepared in the same way : free particles beam in a linear potential, electronic or C_{60} beam in the Young's slits diffraction, atomic beam in Stern and Gerlach experiment.

3.1 *Convergence to statistical Hamilton-Jacobi equations*

If we consider the system with classical initial conditions

$$\rho_0^\hbar(\mathbf{x}) = \rho_0(\mathbf{x}) = (2\pi\sigma_0^2)^{-\frac{3}{2}} e^{-\frac{(\mathbf{x}-\zeta_0)^2}{2\sigma_0^2}} \qquad and \qquad S_0^\hbar(\mathbf{x}) = S_0(\mathbf{x}) = m\mathbf{v}_0\!\cdot\!\mathbf{x}. \tag{14}$$

in a linear potential field $V(\mathbf{x}) = -\mathbf{K}\cdot\mathbf{x}$. The density $\rho^\hbar(\mathbf{x},t)$ and the action $S^\hbar(\mathbf{x},t)$, solutions to the Madelung equations (11)(12)(13) with the initial condition (14), are respectively equal to [10] :

$$\rho^\hbar(\mathbf{x},t) = (2\pi\sigma_\hbar^2(t))^{-\frac{3}{2}} e^{-\frac{\left(\mathbf{x}-\zeta_0-\mathbf{v}_0 t-\mathbf{K}\frac{t^2}{2m}\right)^2}{2\sigma_\hbar^2(t)}} \tag{15}$$

$$S^\hbar(\mathbf{x},t) = -\frac{3\hbar}{2} tg^{-1}(\hbar t/2m\sigma_0^2) - \frac{1}{2}m\mathbf{v}_0^2 t + m\mathbf{v}_0\cdot\mathbf{x} + \mathbf{K}\cdot\mathbf{x}t$$
$$-\frac{1}{2}\mathbf{K}\cdot\mathbf{v}_0 t^2 - \frac{\mathbf{K}^2 t^3}{6m} + \frac{\left(\mathbf{x}-\zeta_0-\mathbf{v}_0 t-\mathbf{K}\frac{t^2}{2m}\right)^2 \hbar^2 t}{8m\sigma_0^2\sigma_\hbar^2(t)} \tag{16}$$

with

$$\sigma_\hbar(t) = \sigma_0\left(1 + \left(\hbar t/2m\sigma_0^2\right)^2\right)^{\frac{1}{2}}. \tag{17}$$

The constants σ_0, $\mathbf{v}_0$, ζ_0 and $\mathbf{K}$ are given and independent of $\hbar$; σ_0 for example corresponds to the hole width for preparing the particle beam.

When $\hbar \to 0$, $\sigma_\hbar(t)$ converges to σ_0 and one gets the following theorem :

THEOREM 3 - *When $\hbar \to 0$, the density $\rho^\hbar(\boldsymbol{x},t)$ and the action $S^\hbar(\boldsymbol{x},t)$ converge to*

$$\rho(\boldsymbol{x},t) = (2\pi\sigma_0^2)^{-\frac{3}{2}} e^{-\frac{\left(\boldsymbol{x}-\zeta_0-\boldsymbol{v}_0 t-\boldsymbol{K}\frac{t^2}{2m}\right)^2}{2\sigma_0^2}}$$
$$and \qquad S(\boldsymbol{x},t) = -\frac{1}{2}m\boldsymbol{v}_0^2 t + m\boldsymbol{v}_0\cdot\boldsymbol{x} + \boldsymbol{K}\cdot\boldsymbol{x}t - \frac{1}{2}\boldsymbol{K}\cdot\boldsymbol{v}_0 t^2 - \frac{\boldsymbol{K}^2 t^3}{6m}. \tag{18}$$

which are solutions to statistical Hamilton-Jacobi equations (1)(2)(3)(4).

Thus, when $\hbar \to 0$, for *semi-classical non-discerned particles*, the probability density $\rho^{\hbar}(\mathbf{x}, t)$ of the wave function tends to the probability density of a statistical set of classical particles $\rho(\mathbf{x}, t)$. We conjecture that this result in the case of a linear potential field can be generalized to *semi-classically discerned particles* for other potentials.

CONJECTURE - *For semi-classically non-discerned particles, when $\hbar \to 0$, for all $\boldsymbol{x}$ and t bounded, the density $\rho^{\hbar}(\boldsymbol{x}, t)$ and the action $S^{\hbar}(\boldsymbol{x}, t)$, which are solutions to Madelung equations(11)(12)(13), converge to $\rho(\boldsymbol{x}, t)$ et $S(\boldsymbol{x}, t)$, which are solutions to statistical Hamilton-Jacobi equations. (1)(2)(3)(4).*

This conjecture is verified for the convergence of the density $\rho^{\hbar}(\mathbf{x}, t)$ with an explicit calculation for the Stern-Gerlach experiment [11], for the EPR one [12], and by numerical simulation for the Young's slits experiment [13, 14].

More generally, we can demonstrated this conjecture when the potential $V(\mathbf{x})$ is a quadratic function in $\mathbf{x}$ and when $\rho_0^{\hbar}(\mathbf{x}) = \rho_0(\mathbf{x})$ and $S_0^{\hbar}(\mathbf{x}) = S_0(\mathbf{x})$ are non-dependent functions of $\hbar$. In this case, the wave function $\Psi(\mathbf{x}, t)$ at time t is written as a function of the initial wave function $\Psi_0(\mathbf{x})$ by the Feynman formula[15] (p. 58) :

$$\Psi(\mathbf{x}, t) = \int F(t, \hbar) \exp(\frac{i}{\hbar} S_{cl}(\mathbf{x}, t; \mathbf{x}_0) \Psi_0(\mathbf{x}_0) d\mathbf{x}_0$$

$$= F(t, \hbar) \int \sqrt{\rho_0(\mathbf{x}_0)} \exp(\frac{i}{\hbar}(S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0)) d\mathbf{x}_0$$

where $F(t, \hbar)$ is an independent function of $\mathbf{x}$ and of $\mathbf{x}_0$ and where $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$ is the classical action $min_{\mathbf{u}(s)} \int_0^t L(\mathbf{x}(s), \mathbf{u}(s), s) ds$, the minimum is taken over all trajectories $\mathbf{x}(s)$ with velocity $\mathbf{u}(s)$ from $\mathbf{x}_0$ to $\mathbf{x}$ between 0 and t. The theorem of the stationary phase shows that, if $\hbar$ tends towards 0, we have

$$\Psi(\mathbf{x}, t) \sim \exp(\frac{i}{\hbar} min_{\mathbf{x}_0}(S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0))).$$

that is to say that the quantum action $S^h(\mathbf{x}, t)$ converges to the function

$$S(\mathbf{x}, t) = min_{\mathbf{x}_0}(S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0)). \tag{19}$$

However, $S(\mathbf{x}, t)$ given by (19) is the solution to the Hamilton-Jacobi equation (1) with the initial condition (2). This is a consequence of the principle of the least action and a fundamental property of the minplus analysis we have developed[16, 17] following Maslov[18].

Moreover, as the quantum density $\rho^h(\mathbf{x}, t)$ verifies the continuity equation (12) of the Madelung equations, we deduce, since $S^h(\mathbf{x}, t)$ tends towards $S(\mathbf{x}, t)$, that $\rho^h(x, t)$ converges to the classical density $\rho(\mathbf{x}, t)$, which satisfies the continuity equation (3) of the statistical Hamilton-Jacobi equations.

3.2   De Broglie-Bohm quantum trajectories

Those last convergence examples show that for *semi-classically non-discerned particles*, the Madelung equations converge to statistical Hamilton-Jacobi equations. The uncertainty of the position of a quantum particle corresponds in that case to an uncertainty of the position of a classical particle, only whose initial density has been defined. **In classical mechanics, this uncertainty is removed by giving the initial position of the particle. It would be illogical not to do the same in quantum mechanics.**

We assume that for *semi-classically non-discerned particles*, a quantum particle is not well described by its wave function. It is therefore necessary to add its initial position and it becomes natural to introduce the so-called de Broglie-Bohm trajectories. In this interpretation, its velocity is given by [19, 20] :

$$\mathbf{v}^\hbar(\mathbf{x}, t) = \frac{1}{m} \nabla S^\hbar(\mathbf{x}, t) \tag{20}$$

or by the alternative form [21, 22, 23, 24] :

$$\mathbf{v}^\hbar(\mathbf{x}, t) = \frac{1}{m} \nabla S^\hbar(\mathbf{x}, t) + \frac{\hbar}{2m} \nabla \ln \rho^\hbar(\mathbf{x}, t) \times \mathbf{k}, \tag{21}$$

where $\mathbf{k}$ is the unit vector parallel to the particle spin vector.

This spin current $\frac{\hbar}{2m} \nabla \rho^\hbar(\mathbf{x}, t) \times \mathbf{k}$ corresponds to Gordon's current when one changes from the Dirac equation to the Pauli equation and subsequently to the Schrodinger equation[23]. This current is very important because it allows us to return to quantum mechanics on small scales, in particular in relation to Compton's wavelength, as in the Foldy and Wouthuysen transformation [25].

We have the following classical property : if a system of particles with initial density $\rho_0(\mathbf{x})$ has de Broglie-Bohm-like trajectories defined by the velocity field $\mathbf{v}^\hbar(\mathbf{x}, t)$ from equations (20) or (21), then the probability density of those particles at time t is equal to $\rho^\hbar(\mathbf{x}, t)$, the square of the wave function magnitude. In the case of semi-classical non-discerned particles, this shows that the Broglie-Bohm interpretation reproduces the predictions of standard quantum mechanics.

In one dimension, for the initial particle position $x_0 = \zeta_0 + \eta_0$ with initial velocity v_0, in a linear potential $V(x) = -Kx$ and with velocity (20), one recovers the Broglie-Bohm trajectory : $\xi_\hbar(t) = \zeta_0 + v_0 t - K\frac{t^2}{2m} + \eta_0\frac{\sigma_\hbar(t)}{\sigma_0}$ which converges to the classical trajectory $\xi(t) = \zeta_0 + \eta_0 + v_0 t - K\frac{t^2}{2m}$ when $\hbar \to 0$.

In three dimensions, for a particle initial position such as $\mathbf{x}_0 = \zeta_0 + \eta_0$ with an initial velocity $\mathbf{v}_0$, in a linear potential $V(\mathbf{x}) = -Kx_3$ and with the velocity (21), we have the Bohm-Broglie trajectory[13] : $\xi_{0,1}^\hbar(t) = \zeta_{0,1} + v_{0,1}t + \sqrt{\eta_{0,1}^2 + \eta_{0,2}^2}\frac{\sigma_\hbar(t)}{\sigma_0}\cos\varphi(t)$, $\xi_2^\hbar(t) = \zeta_{0,2} + v_{0,2}t + \sqrt{\eta_{0,1}^2 + \eta_{0,2}^2}\frac{\sigma_\hbar(t)}{\sigma_0}\sin\varphi(t)$, $\xi_3^\hbar(t) = \zeta_{0,3} + v_{0,3}t - K\frac{t^2}{2m} + \eta_{0,3}\frac{\sigma_\hbar(t)}{\sigma_0}$ avec $\varphi(t) = \arctan(\frac{\eta_{0,1}}{\eta_{0,2}}) - \arctan(\frac{\hbar t}{2m\sigma_0^2})$, which converges to the classical trajectory $\xi_{0,1}(t) = \zeta_{0,1} + \eta_{0,1} + v_{0,1}t$, $\xi_{0,2}(t) = \zeta_{0,2} + \eta_{0,2} + v_{0,2}t$, $\xi_{0,3}(t) = \zeta_{0,3} + \eta_{0,3} + v_{0,3}t - K\frac{t^2}{2m}$ when $\hbar \to 0$.

Generally, when $\hbar \to 0$, one deduces from conjecture that $\mathbf{v}^\hbar(\mathbf{x}, t)$ given from equations (20) or (21) converge to the classical velocity $\mathbf{v}(\mathbf{x}, t) = \frac{1}{m}\nabla S(\mathbf{x}, t)$. This leads to the fact that the Broglie-Bohm trajectories converge to the classical ones. We verify this conjecture with an explicit calculation for the Stern-Gerlach experiment [11] and by numerical simulation for the Young's slits experiment [13, 14].

4   Convergence to discerned particles when $\hbar \to 0$.

Definition 5 - *A quantum system is **discerned semi-classically** if it satisfies the two conditions*

- its initial probability density $\rho_0^\hbar(\mathbf{x})$ and its initial action $S_0^\hbar(\mathbf{x})$ converge respectively, when $\hbar \to 0$, to a Dirac distribution and an action $S_0(\mathbf{x})$.

- its interaction with the potential field $V(\boldsymbol{x})$ can be described classically.

This situation occurs when the wave packet corresponds to a quasi-classical coherent state which were introduced in 1926 by Schrödinger [26], and is of great importance in quantum optics since Glauber [27](1965). They have three properties : their gravity center follows a classical trajectory ; they verify a Heisenberg equality and not an inequality ; the wave packet shape doesn't change during motion (or at least it recovers its shape after a cycle). It still occurs when the wave packet corresponds to the periodic trajectories of a non-dispersive wave packet, which are eigenvectors of the Floquet operator. For the hydrogen atom, the existence of a localized wave packet on the classical trajectory (an old dream of Schrödinger) and which was predicted in 1994 by Bialynicki-Birula, Kalinski, Eberly, Buchleitner et Delande [28, 29, 30], has been discovered recently by Maeda and Gallagher [31] on Rydberg atoms.

4.1 Convergence of coherent states to the solutions to the special Hamilton-Jacobi equations

For the two dimensional harmonic oscillator, $V(\mathbf{x}) = \frac{1}{2}m\omega^2\mathbf{x}^2$, coherent states are built [10] from the initial wave function $\Psi_0(\mathbf{x})$ which corresponds to the density and initial action :

$$\rho_0^\hbar(\mathbf{x}) = (2\pi\sigma_\hbar^2)^{-1}e^{-\frac{(\mathbf{x}-\mathbf{x}_0)^2}{2\sigma_\hbar^2}} \qquad and \qquad S_0(\mathbf{x}) = S_0^\hbar(\mathbf{x}) = m\mathbf{v}_0 \cdot \mathbf{x} \quad (22)$$

with $\sigma_\hbar = \sqrt{\frac{\hbar}{2m\omega}}$. Here, $\mathbf{v}_0$ and $\mathbf{x}_0$ are still constant vectors and independent from $\hbar$, but $\sigma_\hbar$ will tend to 0 as $\hbar$.

For this harmonic oscillator, the density $\rho^\hbar(\mathbf{x}, t)$ and the action $S^\hbar(\mathbf{x}, t)$, solutions to Madelung equations (11)(12)(13) with initial conditions (22), are equal to [10] :

$$\rho^\hbar(\mathbf{x}, t) = \left(2\pi\sigma_\hbar^2\right)^{-1} e^{-\frac{(\mathbf{x}-\xi(t))^2}{2\sigma_\hbar^2}} \quad and \quad S^\hbar(\mathbf{x}, t) = +m\frac{d\xi(t)}{dt}\cdot\mathbf{x}+g(t)-\hbar\omega t \tag{23}$$

where $\xi(t)$ is the trajectory of a classical particle evolving in the potential $V(\mathbf{x}) = \frac{1}{2}m\omega^2\mathbf{x}^2$, with $\mathbf{x}_0$ and $\mathbf{v}_0$ as initial position and velocity where $g(t) = \int_0^t(-\frac{1}{2}m(\frac{d\xi(s)}{ds})^2 + \frac{1}{2}m\omega^2\xi(s)^2)ds$. Because we have $2V(\xi(s)) = m\frac{d^2\xi(s)}{ds^2} \cdot \xi(s)$, it yields the following theorem :

THEOREM 4 - *When $\hbar \to 0$, for all $\boldsymbol{x}$ and t bounded, the density $\rho^\hbar(\boldsymbol{x}, t)$ and the action $S^\hbar(\boldsymbol{x}, t)$ converge respectively to $\rho^\xi(\boldsymbol{x}, t) = \delta(\boldsymbol{x}-$*

$\xi(t))$ *and* $S^\xi(\boldsymbol{x},t) = m\frac{d\xi(t)}{dt} \cdot \boldsymbol{v}_0 + g(t)$ *where* $S^\xi(\boldsymbol{x},t)$ *and the trajectory* $\xi(t)$ *are solutions to the special Hamilton-Jacobi equations (6)(7)(8).*

Therefore, the kinematic of the wave packet converges to the single harmonic oscillator described by $\xi(t)$. Because this classical particle is completely defined by its initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$, it can be considered as a discerned particle.

When $\hbar \to 0$, for all $\mathbf{x}$ and t bounded, the "quantum potential" $Q^\hbar(\mathbf{x},t) = -\frac{\hbar^2}{2m}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}} = \hbar\omega - \frac{1}{2}m\omega^2(\mathbf{x} - \xi(t))^2$ tends to $Q(\mathbf{x},t) = -\frac{1}{2}m\omega^2(\mathbf{x} - \xi(t))^2$. It is then zero on the trajectory ($\mathbf{x} = \xi(t)$).

More generally, let us consider semi-classically non-discerned particles where $\rho^\hbar(\mathbf{x},t)$ converge to Dirac distribution $\rho^\xi(\mathbf{x},t) = \delta(\mathbf{x}-\xi(t))$. Mathematically, one needs, as proposed by Kazandjian [32], to study the convergence of Madelung equations in the least square approach. This yields :

$$\int \rho_\hbar(\mathbf{x},t)[\frac{\partial S^\hbar(\mathbf{x},t)}{\partial t} + \frac{1}{2m}(\nabla S^\hbar(\mathbf{x},t))^2 + V(\mathbf{x}) + Q^\hbar(\mathbf{x},t)]^2 d\mathbf{x} = 0. \quad (24)$$

For $\hbar \neq 0$, the equation (24) yields the Madelung equations (11) ; in the limit $\hbar \to 0$, but, $\rho_\hbar(\mathbf{x},t)$ tends to 0 for all $\mathbf{x} \neq \xi(t)$, and converges to a Dirac distribution centered on $\xi(t)$; In this limit, the equation (24) can be written as

$$\frac{\partial S(\xi(t),t)}{\partial t} + \frac{1}{2m}(\nabla S(\xi(t),t))^2 + V(\xi(t)) + Q(\xi(t),t) = 0.$$

And because $Q(\xi(t),t) = 0$, this yields

$$\frac{\partial S(\xi(t),t)}{\partial t} + \frac{1}{2m}(\nabla S(\xi(t),t))^2 + V(\xi(t)) = 0 \quad (25)$$

which corresponds to the special Hamilton-Jacobi equations (6). Thus, in the general case of semi-classically non-discerned particles, the wave function kinematics converges to the motion of a discerned classical particle $\xi(t)$ which is completely defined by its initial position $\mathbf{x}_0$ and its initial velocity $\mathbf{v}_0$.

It is then possible to consider, unlike in the semi-classically non-discerned case, that the wave function can be seen as a single quantum particle. The *semi-classically discerned case* is in agreement with the Copenhagen interpretation of the wave function, which contains all the information on the particle.

4.2   Interpretation for the semi-classically discerned particles

In the *semi-classically discerned case*, the Broglie-Bohm interpretation is not relevant mathematically, unlike the semi-classically non-discerned case. Other assumptions are possible. A natural interpretation is the one proposed by Schrödinger [26] in 1926 for the coherent states of the harmonic oscillator. In the **Schrödinger interpretation**, the quantum particle in the semi-classically discerned case is a spatially extended particle, represented by a wave packet whose center follows the classical trajectory. For the coherent states of the harmonic oscillator in two dimensions, the velocity field (21) at time t and at point $\mathbf{x}$ is then equal to :

$$\mathbf{v}^{\hbar}(\mathbf{x}, t) = \mathbf{v}(t) + \Omega \times (\mathbf{x} - \xi(t)) \qquad (26)$$

with $\Omega = \omega \mathbf{k}$. They behave as extended particles which have the same evolution as spinning particles in two dimensions. But this cannot be generalized easily in three dimensions. It seems that it is not possible to consider in three dimensions the particle as a solid in motion. This is the main difficulty in the Schrödinger interpretation : does the particle exist within the wave packet ? We think that this reality can only be defined on the scale where the Schrödinger equation is the effective equation. Some solutions are nevertheless possible on smaller scales [33, 34], where the quantum particle is not represented by a point but is a sort of elastic string whose gravity center follows the classical trajectory $\xi(t)$.

Another possible interpretation for the semi-classical discerned particles is the Bohr model of the atom (1913) found again by de Broglie [36] in 1924 with conditions of resonance between the wave and the particle. In the Bohr-deBroglie interpretation, the quantum particle is a point (in relation to the wave packet size) which follows a trajectory in resonance with its internal vibration in the wave.

The principle of an interpretation that depends on the particle preparation conditions is not really new. It has already been figured out by Einstein and de Broglie. For Louis de Broglie, its real interpretation was the double solution theory introduced in 1927 in which the pilot-wave is just a low-level product : "*I introduced as a 'double solution theory' the idea that it was necessary to distinguish two different solutions but both linked to the wave equation, one that I called wave u which was a real*

physical wave but not normalizable having a local anomaly defining the particle and represented by a singularity, the other one as the Schrödinger Ψ of wave, which is normalizable without singularities and being a probability representation."

We consider as interesting L. de Broglie's idea of the existence of a statistical wave, Ψ and of a soliton wave u; however, it is not a double solution which appears here but a double interpretation of the wave function according to the initial conditions.

Einstein's point of view is well summed up in one of his final papers (1953), "*Elementary reflections concerning the foundation of quantum mechanics*" in homage to Max Born :

"*The fact that the Schrödinger equation associated to the Born interpretation does not lead to a description of the "real states" of an individual system, naturally incites one to find a theory that is not subjected to this limitation. Up to now, the two attempts have in common that they conserve the Schrödinger equation and abandon the Born interpretation. The first one, which marks a de Broglie's return, was continued by Bohm.... The second one, which aimed to get a "real description" of an individual system and which might be based on the Schrödinger equation, is very late and is from Schrödinger himself. The general idea is briefly the following : the function ψ represents in itself the reality and it is not necessary to add Born's statistical interpretation.[...] From previous considerations, it results that the only acceptable interpretation of the Schrödinger equation is the statistical interpretation given by Born. Nevertheless, this interpretation doesn't give the 'real description' of an individual system, it just gives statistical statements of a set of systems.*"

Thus, it is because de Broglie and Schrödinger keep the Schrödinger equation that Einstein, who considers it as fundamentally statistical, refused each of their interpretations.

Finally, there exist situations where the **Broglie-Bohm interpretation of the Schrödinger wave function is probably wrong**. It is in particular the case of state transitions for a hydrogen atom. Indeed, since Delmelt'experiment [35] in 1986, the physical reality of individuals quantum jumps has been fully validated. The semi-classical approximation, where the interaction with the potential field can be described classically, is no longer possible and one must use electromagnetic field quantization since the exchanges occur photon by photon. Einstein thought that it is not possible to find an individual deterministic behavior from the Schrö-

dinger equation. It is the same for Heisenberg who developed matrix mechanics and the second quantization from this example.

This doesn't mean that one has to renounce to determinism and realism, but rather that Schrödinger's statistical wave function does not permit, in that case, to discover an individual behavior.

5   Conclusion

The introduction into classical mechanics of the concepts of **non-discerned particles** and **discerned particles** respectively verifying the **statistical Hamilton-Jacobi equations** and the **local Hamilton-Jacobi equations** gives a simple answer to some paradoxes in classical statistical mechanics and allows to have a better understanding of the least action principle.

When one studies the convergence of the Madelung equations when $\hbar \to 0$, we obtain the following results :

- In the ***semi-classically non-discerned case*** the quantum particles converge to classical non-discerned ones, verifying the statistical Hamilton-Jacobi equations. The wave function is not sufficient to represent the quantum particles. One needs to add it the initial positions, as for classical particles, in order to describe them completely. Thus, ***the Broglie-Bohm interpretation is relevant***.

- In the ***semi-classically discerned case*** the quantum particles converge to classical discerned ones, verifying the local Hamilton-Jacobi equations. ***The Broglie-Bohm interpretation is not imperative*** because the wave function is sufficient to represent the particles as in the Copenhagen interpretation. However, one can make some realistic and deterministic assumptions such as the **Schrödinger** and the **Bohr-deBroglie interpretations**.

- In the case where ***the semi-classical approximation is no longer valid*** , as in the transition states in the hydrogen atom, the two interpretations are wrong as claimed by ***Heisenberg. Consequently, Born's statistical interpretation is the only possible interpretation of the Schrödinger equation.*** This doesn't mean that it is necessary to give up to determinism and realism, but rather that the Schrödinger wave function doesn't allow, in that case, to reveal the individual behavior of a particle. An individual interpretation needs to use creation and annhilation operators of quantum Field Theory.

Therefore, as Einstein said, the situation is much more complex than what de Broglie-Bohm interpretation suggests.

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